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Strongly weak idempotent nil-clean rings

Biadiglign Asmare[†], **Tilahun Abebaw**[‡], **Kolluru Venkateswarlu**[§*]

[†]*Department of Mathematics, College of Natural and Computational Science, Addis Ababa University, Addis Ababa, Ethiopia*
[‡]*Department of Mathematics, College of Natural and Computational Science, Addis Ababa University, Addis Ababa, Ethiopia*
[§]*Department of Computer Science and Systems Engineering, College of Engineering, Andhra University, Visakhapatnam, Andhra Pradesh, India*
Emails: biadiglign.asmare@aau.edu.et, tilahun.abebaw@aau.edu.et, drkvenkateswarlu@gmail.com

Abstract. We introduce the concept of strongly weak idempotent nil-clean rings which is a generalization of strongly weakly nil clean rings. We characterize strongly weak idempotent nil-clean rings in terms of the set of nilpotent elements, homomorphic images, and Jacobson radicals. We prove that a ring R is strongly weak idempotent nil-clean if and only if for any $a \in R$, $a - a^3$ is nilpotent if and only if $Nil(R)$ forms an ideal and $R/Nil(R)$ is reduced weak idempotent nil-clean if and only if R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and $a^2 - a^4$ is nilpotent. Moreover, we prove that a strongly weak idempotent nil-clean ring R with $2 \in J(R)$ satisfies nil-involution property.

Keywords: Strongly weakly nil-clean rings, Strongly weak idempotent nil-clean rings, Strongly π-regular rings, Strongly clean rings, Nil-involution.
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1 Introduction

Throughout this paper, R stands for associative ring with unity. We denote the set of all nilpotents, the set of all idempotents, the set of weak idempotents, the group of units and the Jacobson radicals of a ring R by $Nil(R)$, $Id(R)$, $w(R)$, $U(R)$ and $J(R)$, respectively.
We recall the following definitions from [1], [3] and [4]. A ring R is said to be

(1) strongly nil-clean if for every $r \in R$ there exists a nilpotent n and an idempotent $e^2 = e \in R$ such that $r = n + e$ and $ne = en$.

*Corresponding author
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(2) strongly weakly nil-clean if every element in R is the sum or difference of a nilpotent and an idempotent that commute.

(3) strongly clean if every element in R can be written as the sum of a unit and an idempotent that commute.

(4) periodic if for any $a \in R$, there exist distinct $m, k \in \mathbb{N}$ such that $a^m = a^k$.

(5) strongly π-regular if for every element a in a ring R, there exists $r \in R$ and a positive integer k such that $a^k = a^{k+1}r$.

A ring is said to satisfy the nil-involution property if every element is a sum of a unit and an involution (i.e., an element whose square is 1). In other words, a ring R satisfies the nil-involution property if, for every $a \in R$, $a = u + v$ where $u \in Nil(R) \pm 1$ and $v^2 = 1$. The following hold:
Strongly nil-clean rings $\implies$ Strongly weakly nil-clean rings $\implies$ Strongly clean rings.
In this paper, we introduce a new class of rings which generalizes the notion of strongly weakly nil-clean rings (for short strongly win-clean rings) and is a super class of strongly weakly nil-clean rings and a subclass of strongly clean rings. We obtain the necessary and sufficient conditions for strongly weak idempotent nil-clean rings in relation to periodic rings, strongly π-regular rings and strongly clean rings. In the next section, we look at the definition, examples and basic properties of strongly weak idempotent nil-clean rings. Next, we prove some results related to homomorphic images such as $Nil(R)$ of a strongly weak idempotent nil-clean ring R forms an ideal and strongly weak idempotent nil-clean rings do not have homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ and vice versa. Finally, we characterize elementwise strongly weak idempotent nil-clean rings, strongly π-regular rings and strongly clean rings.

2 Main results

2.1 Examples and properties

We begin with the following

Definition 1. Let R be a ring. Then

(1) an element w in R is said to be weak idempotent if $w^2 = w^4$.

(2) an element $a \in R$ is called strongly weak idempotent nil-clean if there exists a nilpotent n and a weak idempotent w such that $a = n + w$ and $nw = wn$. The ring is said to be strongly weak idempotent nil-clean if each element of the ring is strongly weak idempotent nil-clean.

Example 1. Consider the set of 2×2 upper triangular matrices over integer modulo 3, $\mathbb{Z}_3$,

$$T_2(\mathbb{Z}_3) = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{Z}_3 \right\}.$$

Then $T_2(\mathbb{Z}_3)$ is strongly win-clean.

Example 2. The ring

$$M_2(\mathbb{Z}_3)=\Bigg\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}2&0\\0&2\end{pmatrix},\begin{pmatrix}0&2\\2&0\end{pmatrix},\begin{pmatrix}1&2\\0&2\end{pmatrix},$$
$$\begin{pmatrix}0&2\\1&2\end{pmatrix},\begin{pmatrix}2&0\\2&1\end{pmatrix},\begin{pmatrix}1&0\\1&2\end{pmatrix},\begin{pmatrix}0&1\\2&1\end{pmatrix},\begin{pmatrix}1&2\\1&0\end{pmatrix},\begin{pmatrix}2&1\\0&1\end{pmatrix},$$
$$\begin{pmatrix}2&2\\0&0\end{pmatrix},\begin{pmatrix}0&0\\2&2\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&1\end{pmatrix},\begin{pmatrix}0&2\\0&1\end{pmatrix},\begin{pmatrix}2&0\\1&0\end{pmatrix},$$
$$\begin{pmatrix}0&1\\0&2\end{pmatrix},\begin{pmatrix}1&0\\2&0\end{pmatrix},\begin{pmatrix}1&1\\2&2\end{pmatrix},\begin{pmatrix}2&2\\1&1\end{pmatrix},\begin{pmatrix}1&1\\1&1\end{pmatrix},\begin{pmatrix}2&2\\2&2\end{pmatrix},$$
$$\begin{pmatrix}1&2\\2&1\end{pmatrix},\begin{pmatrix}2&1\\1&2\end{pmatrix},\begin{pmatrix}2&1\\2&0\end{pmatrix}\Bigg\}$$

is strongly win-clean ring but not strongly weakly nil-clean.

Remark 1. Every weak idempotent and nilpotent element are strongly win-clean.

Observe that every strongly weakly nil-clean ring is strongly win-clean, but the converse is not true.

Proposition 1. *If x is strongly win-clean element in a ring R, then so is x^m for a positive integer m.*

Proof. Let x be a strongly win-clean element. Then $x = n + w$ and $nw = wn$ where $n \in Nil(R)$ and $w \in wi(R)$. Now we prove by induction on m.
For $m = 2$, $x^2 = (n + w)^2 = w^2 + 2nw + n^2 = w^2 + n(2w + n)$ where $w^2 \in wi(R)$ and $(2w + n)n \in Nil(R)$.
Assume that it is true for $m-1$. Then $x^{m-1} = (n+w)^{m-1} = \sum_{k=0}^{m-1}\binom{m-1}{k}n^kw^{m-1-k} = \binom{m-1}{0}w^{m-1}+\binom{m-1}{1}nw^{m-2}+\binom{m-1}{2}n^2w^{m-3}+\cdots+\binom{m-1}{m-1}n^{m-1}$, and

$$\begin{aligned}
x^m =&(n+w)^m = \sum_{k=0}^{m}\binom{m}{k}n^kw^{m-k} = w^m + \binom{m}{1}nw^{m-1} + \binom{m}{2}n^2w^{m-2}\\
&+\cdots+\binom{m}{m-1}n^{m-1}w + n^m\\
=&w^m + \left[\binom{m-1}{0}+\binom{m-1}{1}\right]nw^{m-1}+\cdots\\
&+\left[\binom{m-1}{m-2}+\binom{m-1}{m-1}\right]n^{m-1}w + n^m\\
=&\Big[\binom{m-1}{0}nw^{m-1}+\cdots+\binom{m-1}{m-2}n^{m-1}w + n^m\Big]\\
&+\Big[w^m + \binom{m-1}{1}nw^{m-1}+\cdots+\binom{m-1}{m-1}n^{m-1}w\Big]
\end{aligned}$$

$$
\begin{aligned}
&=\Big[w^{m-1}+\binom{m-1}{1}nw^{m-2}+\cdots+n^{m-1}\Big]n\\
&\quad+\Big[w^{m-1}+\binom{m-1}{1}nw^{m-2}+\cdots+\binom{m-1}{m-1}n^{m-1}\Big]w\\
&=x^{m-1}n+x^{m-1}w \text{ by induction assumption.}\\
&=(n'+w')n+(n'+w')w \text{ where } x^{m-1}=n'+w'\\
&=(n'n+w'n+n'w)+w'w\\
&=n''+w''.
\end{aligned}
$$

As $n'' \in Nil(R)$ and $w'' \in wi(R)$, x^m is strongly win-clean element. □

Proposition 2. *Let R be a ring. Then the following statements are equivalent.*

(1) R is strongly win-clean.

(2) For any $a \in R$, $a - a^3$ is nilpotent.

(3) For any $a \in R$, there exists a weak idempotent $w \in \mathbb{Z}[a]$ (a polynomial in a with integer coefficients) such that $a - w$ is nilpotent.

Proof. (1) $\implies$ (2) Suppose R is strongly win-clean ring and $a \in R$. Then $a = n + w$ with $nw = wn$, where $n \in Nil(R)$ and $w \in wi(R)$. Then $a^3 = (n+w)^3 = n^3 + 3n^2w + 3nw^2 + w^3$, and hence $a - a^3 = n(1 - n^2 - 3nw - 3w^2) + (w - w^3)$ is nilpotent since $n(1 - n^2 - 3nw - 3w^2)$ and $(w - w^3)$ are nilpotents.
(2) $\implies$ (3) Assume that $a - a^3$ is nilpotent for any $a \in R$. Then $a - a^3 \in Nil(R)$, and hence $(a - a^3)^k = 0$ for some natural number k. Let $f(a) = \sum_{i=0}^{k} \binom{2k}{i} a^{2k-i}(1-a)^i = a^{2k} + \binom{2k}{1}a^{2k-1}(1-a) + \cdots + \binom{2k}{k}a^k(1-a)^k$.
$f(a) = a^k\Big[a^k + \binom{2k}{1}a^{k-1}(1-a) + \cdots + \binom{2k}{k}(1-a)^k\Big]$. So $f(a) \equiv 0(mod\ a^k)$ and hence $f^2(a) \equiv 0(mod\ a^k)$. Now $f(a) + \sum_{i=k+1}^{2k}\binom{2k}{i}(a)^{2k-i}(1-a)^i = (a + (1-a))^{2k} = 1$. But $\sum_{i=k+1}^{2k}\binom{2k}{i}a^{2k-i}(1-a)^i = (1-a)^k\sum_{i=k+j,1\le j\le k}^{2k}\binom{2k}{i}a^{k-j}(1-a)^j$. Thus $f(a) \equiv 1(mod\ (1-a)^k)$ implies $f^2(a) \equiv 1(mod\ (1-a)^k)$ and hence $1 - f^2(a) \equiv 0(mod\ (1-a)^k)$. Therefore, $f(a)^2(1 - f^2(a)) \equiv 0(mod\ a^k(1-a)^k)$. Let $w = f(a)$. Then $w^2(1-w^2) = 0$ and so $w \in \mathbb{Z}[a]$ is a weak idempotent. So,

$$
\begin{aligned}
w-a =&f(a)-a\\
=&(a^{2k}-a)+\Big[\binom{2k}{1}a^{2k-1}(1-a)+\cdots+\binom{2k}{k-1}(a^2)^{k+1}(1-a)^{k-1}\\
&+\binom{2k}{k}(a)^k(1-a)^k\Big]\\
=&\Big[(a^{2k}-a^{2k-3})+\cdots+(a^3-a)\Big]+\Big[\binom{2k}{1}a^{2k-2}(a-a^3)+\cdots\\
&+\binom{2k}{k}(a)^k(1-a)^{k-1}(a-a^3)\Big]
\end{aligned}
$$

$$=\Big(a^{2k-3}+a^{2k-5}+\cdots+a^2+1\Big)(a^3-a)+g(a)(a^3-a)$$
$$=h(a)(a-a^3)\ for\ some\ h(t), g(t)\in\mathbb{Z}[t].$$

Since $(a-a^3)^k=0$ and $h(a)(a-a^3)=(a-a^3)h(a)$, we have $(w-a)^k=0$. So $a-w\in Nil(R)$, i.e., $a=n+w$ for some $n\in Nil(R)$.
(3) $\implies$ (1) Suppose for any $a\in R$, there exists $a\in R$ such that $aw=af(a)=f(a)a=wa$ implies that $nw=aw-w^2=wa-w^2=wn$. □

Corollary 1. *Every subring of a strongly win-clean ring is strongly win-clean.*

Proof. It follows from Proposition 2. □

Corollary 2. *Let R be a strongly win-clean ring. Then w^2Rw^2 is strongly win-clean for $w\in wi(R)$.*

Proof. Since w^2Rw^2 is a subring of R, the Corollary follows. □

Proposition 3. *If R be strongly win-clean and S be strongly weakly nil-clean, then $R\oplus S$ is strongly win-clean.*

Proof. Let $A=R\oplus S$. Let $(a,b)\in A$. Then $a-a^3\in Nil(R)$ and $b\pm b^2\in Nil(S)$ [[1], Theorem 2.1]. Thus, $b-b^3)=b-b^3+b^2-b^2=(b-b^2)+b(b-b^2)$. By Lemma 4.2 of [1], $Nil(S)$ forms an ideal and hence $b-b^3\in Nil(S)$. So $(a,b)-(a,b)^3=(a-a^3,b-b^3)\in Nil(R\oplus S)$. Therefore, $R\oplus S$ is strongly win-clean by Proposition 2. □

For $a\in R$, the commutant and the double commutant of a in R are defined by $comm_R(a)=\{x\in R: ax=xa\}$ and $comm^2_R(a)=\{x\in R: xy=yx\ \forall y\in comm_R(a)\}$
Notation: If R is a ring, we denote $comm(a)$ and $comm^2(a)$ instead of $comm_R(a)$ and $comm^2_R(a)$ respectively.

$$R^{qnil}=\{a: 1+xa\in U(R)\ \forall x\in comm(a)\}$$

Definition 2 ([8]). 1. An element $a\in R$ is said to be quasinilpotent if $a\in R^{qnil}$.

2. An element $a\in R$ is said to be quasipolar if there exists $p^2=p\in comm^2(a)$ such that $a+p\in U(R)$ and $ap\in R^{qnil}$.

Proposition 4. *Every strongly win-clean ring is quasipolar.*

Proof. Let R be strongly win-clean and $a\in R$. By Proposition 2, there exists a weak idempotent $w\in\mathbb{Z}[a]$ such that $a-w\in Nil(R)$. If $n=a-w$, then $a=n+w=(n+w-1+w^2)+(1-w^2)$ where $1-w^2\in comm^2(a)$, $n+w-1+w^2\in U(R)$ and $(1-w^2)a=(1-w^2)n+(w-w^3)\in R^{qnil}$. Hence, R is quasipolar. □

2.2 Homomorphic images

Proposition 5. *Let R and $\{R_i : i \in I\}$ be a family of rings. Then the following are true.*

(1) If R is strongly win-clean then every homomorphic image of R is strongly win-clean.

(2) Let I be a nil ideal of R. Then R is strongly win-clean if and only if R/I is strongly win-clean.

(3) The direct product $\prod R_i$ is strongly win-clean if and only if each R_i is strongly win-clean.

Proof. (1) We know that the homomorphic image of nilpotent and weak idempotent are nilpotent and weak idempotent respectively. Also, the homomorphic image of two commuting elements is commuting. Thus every homomorphic image of strongly win-clean ring is strongly win-clean.

(2) ($\Longrightarrow$) It is obvious.
($\Longleftarrow$) Let $a \in R$. Then $\bar{a} \in \bar{R} = R/I$. Thus there exists a positive integer k such that $\overline{(a - a^3)^k} = \bar{0}$ by Proposition 2. Since I is nil ideal, $(a - a^3)^{km} = 0$ for some positive integer m. So $a - a^3 \in Nil(R)$. Hence, R is strongly win-clean by Proposition 2.

(3) It is obvious.

□

We recall the following Lemma which we use in the sequel.

Lemma 1 ([7]). *Let R be a ring, and let $k \geq 2$. Then the following are equivalent:*

(1) $Nil(R)$ forms an ideal whenever $a - a^k \in Nil(R)$ for all $a \in R$.

(2) $k \not\equiv 1(mod\ 3)$ and $k \not\equiv 1(mod\ 8)$.

Proposition 6. *The following are equivalent for a ring R.*

(1) R is strongly win-clean.

(2) 6 is nilpotent in R and $R/6R$ is strongly win-clean.

(3) $Nil(R)$ forms an ideal of R and $R/Nil(R)$ is reduced win-clean.

Proof. (1) $\Longrightarrow$ (2) Suppose R is strongly win-clean ring. Then $2^3 - 2 \in Nil(R)$ by Proposition 2. Thus $6 \in Nil(R)$ and hence $6R$ is nil ideal of R. So $R/6R$ is strongly win-clean as a homomorphic image of R.
(2) $\Longrightarrow$ (1) Since 6 is nilpotent, $6R$ is nil ideal. As $R/6R$ is strongly win-clean, R is strongly win-clean by Proposition 5.
(1) $\Longrightarrow$ (3) Let $a \in R$. Then $a - a^3 \in Nil(R)$. By Lemma 1, $Nil(R)$ forms an ideal of R.
(3) $\Longrightarrow$ (1) By Proposition 5, R is strongly win-clean. □

Corollary 3. *The following are equivalent for a ring R.*

(1) R is strongly win-clean.

(2) $R/Nil(R)$ is strongly win-clean and $Nil(R)$ forms an ideal of R.

(3) $R/J(R)$ is strongly win-clean and $J(R)$ is nil.

Proof. (1) $\iff$ (2) follows from Proposition 6.
(2) $\implies$ (3) Assume that $R/Nil(R)$ is strongly win-clean and $Nil(R)$ forms an ideal of R. Then R is strongly win-clean by Proposition 5(2). Let $a \in J(R)$. Then $a - a^3$ is nilpotent and also $a - a^3 \in J(R)$. So $J(R)$ is nil. Hence, $R/J(R)$ is strongly win-clean by Proposition 2(2).
(3) $\implies$ (1) Let $a \in R$. Then $\bar{a} \in \bar{R} = R/J(R)$ and hence $\bar{a} - \bar{a}^3 \in \bar{R}$. So $a - a^3 \in J(R)$. Since $J(R)$ is nil, $a - a^3$ is nilpotent. Hence, by Proposition 2, R is strongly win-clean. □

Proposition 7. *Let R be a strongly winc ring, then $J(R) \subseteq Nil(R)$. In particular, if R is strongly win-clean, then $J(R)$ is nil.*

Proof. Let $a \in J(R)$. Then $a = n + w$ and $nw = wn$, where $n \in Nil(R)$ and $w \in wi(R)$. Thus $(a - w)^k = 0$ for some $k \in \mathbb{N}$. So $(w - a)^k \in J(R)$. Next we show that $w \in J(R)$.

$$\begin{aligned}(w-a)^k =& w^k - \binom{k}{1} w^{k-1} a + \binom{k}{2} w^{k-2} a^2 - \binom{k}{3} w^{k-3} a^3 + \cdots \\ &+ \binom{k}{k-1} (-1)^{k-1} w a^{k-1} + \binom{k}{k} (-1)^k a^k \\ =& w^k - a\left[\binom{k}{1} w^{k-1} - \binom{k}{2} w^{k-2} a + \binom{k}{3} w^{k-3} a^2 - \cdots \right. \\ &\left. + \binom{k}{k-1} (-1)^{k-2} w a^{k-2} + \binom{k}{k} (-1)^{k-1} a^{k-1}\right] \\ =& w^k - a\left[\sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} w^{k-i} a^{i-1}\right] \\ =& w^k - as \textit{ where } s = \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} w^{k-i} a^{i-1}.\end{aligned}$$

Now $w^k - as \in J(R)$. Thus $w^k = (w^k - as) + as \in J(R)$ and hence $w^k \in J(R) \cap wi(R)$. Since $J(R)$ does not contain units and non-zero idempotents, w must be nilpotent. Now $a - w, w \in Nil(R)$ which in turn implies that $a \in Nil(R)$. Hence, $J(R) \subseteq Nil(R)$. □

Proposition 8. *Let R be a ring. Then R is strongly win-clean if and only if $J(R)$ is nil and $R/J(R)$ is isomorphic to a Boolean ring or $\mathbb{Z}_3$ or the product of two such rings.*

Proof. Suppose R is strongly win-clean. Then 6 is nilpotent by Proposition 6. So $6^k = 0$ for some positive integer k. Thus $2^k R \cap 3^k R = 0$ and $2^k R + 3^k R = R$. Hence, $R = R/2^k R \oplus R/3^k R$ by Chinese Remainder Theorem. By Proposition 5, $R_1 = R/2^k R$ and $R_2 = R/3^k R$ are strongly win-clean rings. Then R_1 is Boolean since $2 \in J(R_1) = 2^k R$ and $J(R_1)$ is nil by Proposition 7. If $R_2 \neq 0$, then $3 \in J(R_2) = 3^k R$ and also $2 = 3-1$ is unit in R_2. Thus, R_2 is commutative division ring such that $char(R_2) = 3$. Hence, $R_2 \cong \mathbb{Z}_3$ and $J(R_2)$ is nil by Proposition 7. Therefore, we finish the proof by Proposition 5(3). The converse is obvious. $\square$

Corollary 4. *A ring R is a strongly win-clean if and only if $R \cong R_1, R_2$ or $R_1 \oplus R_2$, where $R_1/J(R_1)$ is Boolean with $J(R_1)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$ with $J(R_2)$ is nil.*

Proof. It follows from Proposition 8. $\square$

The following Proposition is about a periodic ring and it can be found in ([4]).

Proposition 9. *Let I be a nil ideal of a ring R. Then R is periodic if and only if R/I is periodic. In particular, R is periodic if and only if $J(R)$ is nil and $R/J(R)$ is periodic.*

Proposition 10. *Let R be a ring. Then R is strongly win-clean if and only if R is periodic and $R/J(R)$ is strongly win-clean.*

Proof. ($\Longrightarrow$) Suppose R is a strongly win-clean and let $a \in R$. Then $a - a^3 \in Nil(R)$ by Proposition 2 (2). Thus $(a - a^3)^k = 0$ for some positive integer k. Now $a^k = a^{k+1}f(a)$, where $f(a) = \sum_{i=1}^{k} \binom{k}{i}(-1)^{i-1}a^{2i-1}$. Therefore, R is periodic by Chacron's Herstein Theorem ([2], Proposition 2) .
($\Longleftarrow$) Suppose R is periodic and $R/J(R)$ is strongly win-clean ring. By Proposition 9, $J(R)$ is nil. Then R is strongly win-clean by Corollary 3. $\square$

Proposition 11 ([7])**.** *Let R be a ring. For every $x \in R$, $x - x^2 \in Nil(R)$ if and only if $Nil(R)$ is an ideal and $R/Nil(R)$ is a Boolean ring.*

Proposition 12. *Let R be a ring. Then R is strongly win-clean if and only if*

(1) R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;

(2) For any $a \in R$, there exists k(depend on a) such that $a^k - a^{k+2} \in Nil(R)$.

Proof. ($\Longrightarrow$) (1) Assume that R has homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Let $f : R \to \mathbb{Z}_3 \oplus \mathbb{Z}_3$ be an epimorphism. Then $R \cong S$ where S is a subring of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. But the subrings of $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ are 0, $\mathbb{Z}_3 \oplus 0$, $0 \oplus \mathbb{Z}_3$, and $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. If $R \cong 0$ or $\mathbb{Z}_3 \oplus 0$ or $0 \oplus \mathbb{Z}_3$, then we are done. Assume that $R \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. But this is impossible by Proposition 4. Hence, R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.
(2) Let $a \in R$. Then $a - a^3 \in Nil(R)$ by Proposition 2. Choose $k = 1$, as required.
($\Longleftarrow$) Let $a \in R$. Then there exists k (depending on a) such that $a^k - a^{k+2} \in Nil(R)$. Thus $a^k(1 - a^2) \in Nil(R)$. Now $a^{2k}(1 - a^2)^k(1 - a^2)a^2 \in Nil(R)$ implies that $(a^2 - a^4)^{k+1} \in Nil(R)$. So $a^2 - a^4 \in Nil(R)$. Thus $2^2 - 2^4 = -2^2 \oplus 3 \in Nil(R)$ since $6 \in Nil(R)$. By Proposition 4, we have $R \cong R_1 \oplus R_2$, where $2 \in Nil(R_1)$, $3 \in Nil(R_2)$, $J(R_1)$ and $J(R_2)$ are nil.
For $x \in R_1$, since $\bar{R_1} = R_1/J(R_1)$ is Boolean, $\bar{x} - \bar{x}^2 \in Nil(\bar{R_1})$ by Proposition 11. Then $x - x^2 \in Nil(R_1)$ and hence $x(x - x^2) = x - x^3 \in Nil(R_1)$. By Proposition 2, R_1 is strongly nil clean.
Let $y \in R_2$. Then $y^k - y^{k+2} \in Nil(R_2)$. Thus $y^k(1 - y^2) \in Nil(R_2)$ and hence $y^k(1 - y^2)^k(y - y^3) \in Nil(R_2)$ which implies that $y - y^3 \in Nil(R_2)$. By Proposition 1, $Nil(R_2)$ forms an ideal of R_2. Then $J(R_2) = Nil(R_2)$ is nil. Thus $\bar{y} = \bar{y}^3$ in $R_2/J(R_2)$. Let $M \in Max(R_2)$. Then $R_2/M \cong$

$R_2/J(R_2)/M/J(R_2)$. So for any $\bar{d} \in R_2/M$, we have $\bar{d} = \bar{d}^3$. Hence, R_2/M is a commutative simple ring by [[7], Theorem 1] and also it is a field with 2 invertible. Hence, $R_2/M \cong \mathbb{Z}_3$. Construct a ring morphism $\phi : R_2/J(R_2) \to \prod_{M \in Max(R_2)} R_2/M, x + J(R_2) \mapsto (x + M_i)$. Then ϕ is injective. Since $\mathbb{Z}_3$ is simple, it follows that $R_2/J(R_2) \cong \prod_{M_i \in Max(R_2), i \in I} R_2/M_i$. By hypothesis, R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$, and hence $R_2/J(R_2)$ is not isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. So, $|I| = 1$. Therefore, $R_2/J(R_2) \cong \mathbb{Z}_3$. By Proposition 4, R is strongly win-clean. □

Corollary 5. *Let R be a ring. Then R is strongly win-clean if and only if*

(1) R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;

(2) for any $a \in R$, $a^2 - a^4 \in Nil(R)$.

Proof. ($\Longrightarrow$) (1) is obvious by Proposition 12.
(2) Let $a \in R$. Then $a - a^3 \in Nil(R)$. Thus $a^2 - a^4 = a(a - a^3) \in Nil(R)$, since $Nil(R)$ forms an ideal of R.
($\Longleftarrow$) The result follows from Proposition 12 by putting $k = 2$. □

Corollary 6. *Let R be a ring. Then R is strongly win-clean if and only if*

(1) R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;

(2) Every element of R is the sum of a weak idempotent element and two nilpotent elements that commute.

Proof. ($\Longrightarrow$) (1) is obtained from Proposition 12. (2) Let $a \in R$. Then there exist a weak idempotent $w \in wi(R)$ and a nilpotent $n \in Nil(R)$ such that $a - 6 = n + w$. Hence, $a = 6 + n + w$, as desired.
($\Longleftarrow$) Let $a \in R$. By hypothesis, $a = n + b + w$ for some $n, b \in Nil(R)$ and $w \in wi(R)$. Then $a^2 = (n+b+w)^2 = (n+b)^2 + 2w(n+b) + w^2$, $a^4 = (n+b+w)^4 = (n+b)^4 + 4(n+b)^3 w + 6(n+b)^2 w^2 + 4(n+b)w^3 + w^4$ and hence $a^2 - a^4 = (b+n)[(n+b) + 2w - (n+b)^3 - 4(b+n)^2 w - 6(b+n)w^2 - 4w^3] \in Nil(R)$. Therefore, R is strongly win-clean by Corollary 5. □

Corollary 7. *Let R be a ring. Then R is strongly win-clean if and only if*

(1) R has no homomorphic image $\mathbb{Z}_3 \oplus \mathbb{Z}_3$;

(2) For any $a \in R$, $a^2 \in R$ is strongly nil-clean.

Proof. ($\Longrightarrow$) Suppose R is a strongly win-clean ring. Then (1) is obtained from Proposition 12.
(2) Let $a \in R$. Then there exist a weak idempotent $w \in R$ and a nilpotent $n \in Nil(R)$ such that $a = n + w$ and $nw = wn$. Thus $a^2 = (w + n)^2 = w^2 + 2wn + n^2 = w^2 + n(2w + n)$, as required.
($\Longleftarrow$) Suppose (1) and (2) are true. By hypothesis, $a^2 = e + n$ and $ne = en$ for some $e \in Id(R)$ and $n \in Nil(R)$. Now, $a^4 = (n + e)^2 = n^2 + 2ne + e$. So $a^2 - a^4 = (e + n) - (e + 2ne + n^2) = n(1 - 2e - n) \in Nil(R)$. Hence, R is strongly win-clean by Corollary 5. □

2.3 Strongly π-regular and strongly clean rings

In this section, we investigate the various rings such as strongly π-regular rings, strongly clean rings and nil-involution.

Proposition 13. *Let R be a ring. An element $a \in R$ is a strongly win-clean if and only if a is strongly clean in R and $a - a^3$ is nilpotent.*

Proof. ($\Longrightarrow$) Suppose an element $a \in R$ is strongly win-clean. Let $a = n + w$ be a strongly win-clean decomposition in R. Then $a = (n + w - 1 + w^2) + (1 - w^2)$ is a strongly clean decomposition in R. Moreover, $a^3 = (n + w)^3 = n^3 + 3n^2w + 3nw^2 + w^3$ and so $a - a^3 = (n + w) - (n^3 + 3n^2w + 3nw^2 + w^3) = n(1 - n^2 - 3nw - 3w^2) + (w - w^3)$. Hence, $a - a^3$ is nilpotent.

($\Longleftarrow$) Suppose $a \in R$ is strongly clean element and $a-a^3$ is nilpotent. Let $a = u+e$ be a strongly clean decomposition in R and $a - a^3$ be a nilpotent. Then $a^3 = (u + e)^3 = u^3 + 3u^2e + 3ue + e$ and hence $a - a^3 = (u + e) - (u^3 + 3u^2e + 3ue + e) = u(1 - u^2 - 3ue - 3e)$. It follows that $1 - u^2 - 3ue - 3e$ is nilpotent. Take $u^2 + u + 4e + 3ue - 1$ as weak idempotent element. Now, we claim that these satisfy a clean decomposition of a.

$$\begin{aligned}
n + w - 1 + w^2 =&(1 - 3e - 3ue - u^2) + (u^2 + u + 4e + 3ue - 1) - 1\\
&+ (u^2 + u + 4e + 3ue - 1)^2\\
=&(u + e - 1) + u^4 + 2u^3 - u^2 - 2u + 1 + 6u^3e + 23u^2e\\
&+ 24ue + 8e\\
=&u^4 + 2u^3 - u^2 - u + 6u^3e + 23u^2e + 26ue + 9e\\
1 - w^2 =&1 - [u^2 + u + 4e + 3ue - 1]^2\\
=&- u^4 - 2u^3 + u^2 + 2u - 1 - 6u^3e - 23u^2e - 26ue - 8e.
\end{aligned}$$

Now,

$$\begin{aligned}
a =&(n + w - 1 + w^2) + (1 - w^2)\\
=&(u^4 + 2u^3 - u^2 - u + 6u^3e + 23u^2e + 26ue + 9e)\\
&+ (-u^4 - 2u^3 + u^2 + 2u - 1 - 6u^3e - 23u^2e - 26ue - 8e)\\
=&u + e.
\end{aligned}$$

□

Proposition 14 ([5]). *Let R be a ring. An element $a \in R$ is strongly π-regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = u+e$, $ae = ea$ and eae is nilpotent. The element a is strongly regular if and only if there is an idempotent $e \in R$ and a unit $u \in R$ such that $a = u + e$, $ae = ea$ and eae is zero.*

Proposition 15. *Any strongly win-clean endomorphism is strongly π-regular. Any strongly win-clean element is strongly π-regular.*

Proof. It can be proved elementwise as follows. If n is nilpotent and w is a weak idempotent such that $nw = wn$ and $a = n + w$, then $a = (n + w - 1 + w^2) + (1 - w^2)$ is a strongly π-regular decomposition of a. $\square$

Proposition 16 ([5]). *Let R be a ring and $a \in R$ is a strongly π-regular element with strongly π-regular decomposition $a = e + u$. If $a = f + v$ is another strongly π-regular decomposition of a, then $e = f$ and $u = v$.*

Proposition 17. *If an element of a ring is strongly win-clean, then it has precisely one strongly win-clean decomposition.*

Proof. By Proposition 15, any strongly win-clean decomposition is automatically strongly π-regular decomposition. So, the result holds by Proposition 16. $\square$

Proposition 18. *Let R be a ring and $a \in R$. Suppose that a is strongly π-regular with strongly π-regular decomposition $a = u + e$. Then a is strongly win-clean if and only if $1 - u^2 - 3ue - 3e$ is nilpotent.*

Proof. Suppose a is strongly win-clean ring. Then $a = n + w$, $nw = wn$ where $n \in Nil(R)$ and $w \in wi(R)$. Thus $a = (n + w - 1 + w^2) + (1 - w^2)$ is strongly π-regular decomposition. By Proposition 16, $u = n + w - 1 + w^2$ and $e = 1 - w^2$. Then

$$\begin{aligned}1 - u^2 - 3ue - 3e =& 1 - (n + w - 1 + w^2)^2 - 3(n + w - 1 + w^2)(1 - w^2) \\ & - 3(1 - w^2) \\ =& 1 - n^2 + 2n - 2nw - 2nw^2 + 2w - 1 - 2w^3 - 3n \\ & + 3nw^2 - 3w + 3 + 3w^3 - 3w^2 - 3 + 3w^2 \\ =& n(w^2 - n - 1 - 2w) + (w^3 - w).\end{aligned}$$

The converse follows directly from the proof of Proposition 13. $\square$

The following Corollary follows from Proposition 18

Corollary 8. *Let R be a ring. A unit $u \in R$ is strongly win-clean if and only if it is a square root of unipotent.*

Proof. Suppose $u \in U(R)$ is strongly win-clean. Then $u = 0 + u$ is strongly π-regular decomposition of u. By Proposition 18, $1 - u^2$ is nilpotent. So, $1 - u^2 = n$ where n is nilpotent. Hence, $u^2 = -n+1$. Conversely, Assume that $a \in U(R)$ and $a^2 = n+1$ is unipotent. Then $1-a^2 = -n$. By Proposition 18, we have $e = 0$ and hence $a = 0 + a$ is strongly π-regular decomposition of a. Therefore, a is strongly win-clean. $\square$

Proposition 19. *Every strongly win-clean ring R is a strongly π-regular and also strongly clean.*

Proof. Its proof follows from Proposition 13 and Proposition 15. $\square$

Corollary 9. *Let R be a ring. A unit u is strongly win-clean if and only if R is strongly π-regular and every unit of R is a square root of unipotent.*

Proof. The forward direction holds by Proposition 15 and Corollary 8. For the reverse direction, let $a \in R$. By hypothesis, we have a strongly π-regular decomposition $a = u + e$. We claim that $-u^2 - 3ue - 3e$ is a square root of a unit, i.e., $-u^2 - 3ue - 3e = a^2$ where a is a unit.
We consider the Peirce decomposition with respect to e with which $-u^2 - 3ue - 3e$ commutes. Then $(-u^2 - 3ue - 3e)(1 - e) = -u^2(1 - e) - 3ue(1 - e) = -u^2(1 - e)$ is a unit in $(1 - e)R(1 - e)$ and hence $-u^2 - 3ue - 3e$ is a square of a unit u in R. Moreover,

$$\begin{aligned}
e(-u^2 - 3ue - 3e) =& e[(-u^2 - 3ue) - 3e] = e[-u(u + 3e) - 3e] \\
=& e[-u(a + 2e) - 3e] = -eu(a + 2e) - 3e \\
=& - u(ea) - 2ue - 3e = -u(ea) + e(-2u - 3) \\
=& - u(ea) + e(-2(u + 1) - 1).
\end{aligned}$$

Thus $-u(ea)+e(-2(u+1)-1)$ is unipotent in eRe. Since $-u^2-3ue-3e$ is unit, $1-u^2-3ue-3e$ must be nilpotent. Hence, by Proposition 18, a is strongly win-clean. □

The next Corollary follows from Proposition 13 and 20.

Corollary 10. *Let $a \in R$. The following are equivalent:*

(1) a is strongly win-clean.

(2) a is strongly π-regular and $a - a^3$ is a nilpotent.

(3) a is uniquely strongly clean and $a - a^3$ is a nilpotent.

Proof. (1) $\implies$ (2) Assume that a is strongly win-clean. Then a is strongly π-regular by Proposition 19 and $a - a^3$ is nilpotent by Proposition 2.
(2) $\implies$ (3) Suppose a is strongly π-regular and $a - a^3$ is a nilpotent. Then by Proposition 16 and 17, a is uniquely strongly clean.
(3) $\implies$ (1) Let $a = u + e$ and $ue = eu$ where $u \in U(R)$ and $e \in Id(R)$. Then $a - a^3 = (u + e) - (u^3 + 3u^2e + 3ue + e) = u(1 - u^2 - 3ue - 3e)$ is nilpotent. So $1 - u^2 - 3ue - 3e$ is nilpotent. Hence, by Proposition 18, a is strongly win-clean. □

Corollary 11. *A strongly win-clean ring is uniquely strongly clean, i.e., every element is uniquely strongly clean.*

Proposition 20. *Let a be a strongly win-clean element of R. Then*

(1) a has a unique strongly win-clean decomposition in R.

(2) a is a strongly π-regular element of R.

(3) a is a uniquely strongly clean element of R.

Proof. (1) Let $a = n_1 + w_1$ and $a = n_2 + w_2$ be two strongly win-clean decomposition in R. Then $w_1 = (n_2 - n_1) + w_2$ but this is impossible because $n_2 - n_1$ not necessarily nilpotent.

(2) Suppose a is strongly win-clean element of a ring R. Then $a = n + w$ and $nw = wn$ where $n \in Nil(R)$ and $w \in wi(R)$. Now $a = n + w = (n - 1 + w + w^2) + (1 - w^2)$. Let $u = n - 1 + w + w^2$ and $e = 1 - w^2$. Then,

$$ae = (n + w)(1 - w^2) = n(1 - w^2) + w(1 - w^2) = n - nw^2 + w - w^3,$$

and

$$ea = (1 - w^2)(n + w) = (1 - w^2)n + (1 - w^2)w = n - nw^2 + w - w^3.$$

Thus, $ae = ea$. So $eae = ea = (1 - w^2)(n + w) = (1 - w^2)n + (w - w^3)$ is nilpotent. By Proposition 14, a is strongly π-regular element. Also, a is strongly clean. Therefore, R is strongly π-regular and hence strongly clean.

(3) By Proposition 13, a is strongly clean element of R and $a - a^3$ is nilpotent. By proof of Proposition 13, two different idempotents which give strongly clean decompositions of a must yield two different idempotents which give strongly win-clean decompositions of a. But this is impossible. Thus a is uniquely strongly clean.

□

Proposition 21. *A ring R is strongly win-clean and $2 \in J(R)$ if and only if R is strongly π-regular ring and $U(R) = -1 + Nil(R)$.*

Proof. ($\Longrightarrow$) Suppose R is strongly win-clean and $a \in R$. Then its win-clean decomposition is $a = n + w$ where $n \in Nil(R)$ and $w \in wi(R)$. So its strongly π-regular decomposition is $a = (n + w - 1 + w^2) + (1 - w^2)$ by Proposition 15.
Next we show that $n + w + w^2$ is nilpotent. Now $(w + w^2)^2 = 2(w^2 + w^3) \in J(R)$ since $2 \in J(R)$. As $J(R)$ is nil, $[(w + w^2)^2]^k = 0$ for some positive integer k. So $(w + w^2)^{2k} = 0$. This implies that $w + w^2$ is nilpotent in R. As R is strongly win-clean, $n_1 = n + (w + w^2)$ is nilpotent. Hence, $(n + w + w^2) - 1 = n_1 - 1 = u$ for some unit u in R.
($\Longleftarrow$) Suppose R is strongly π-regular and $U(R) = -1 + Nil(R)$. Let $r \in R$. Then its strongly π-regular decomposition is $r = u + e$, $ue = eu$, $re = er$ and er is nilpotent. By assumption, $u = n - 1$ for some $n \in Nil(R)$ so that $r = n - 1 + e = n + (e - 1)$. Also, $(e - 1)^2 = 1 - e = (e - 1)^4$ and hence $e - 1 \in wi(R)$. Moreover, $(e - 1)n = en - n = ne - n = n(e - 1)$ since $eu = ue$ and $u = n - 1$. Thus r is strongly win-clean element. Hence, R is strongly win-clean ring.
Take $1 \in R$. Then $1 = u + e$ implies $1 - u = e$ and hence $1 - u = e = e^2 = (1 - u)^2$. Thus $u^2 = u$ implies $u = 1$ and hence $e = 0$. So $1 = u + e = n - 1$, i.e., $n = 2$. Hence, $2 \in J(R)$. □

Proposition 22. *If R is strongly win-clean and $2 \in J(R)$, then R satisfies nil-involution property.*

Proof. Let $r \in R$. Then $r = n + w$, $nw = wn$, $n \in Nil(R)$ and $w \in wi(R)$. Thus $r = (n + w - 1 + 2w^2) + (1 - 2w^2) = [(n + w + 2w^2) - 1] + (1 - 2w^2)$. Since $2 \in J(R)$, $2w^2 \in J(R)$. So $(2w^2)^k = 0$ for some positive integer k, i.e., $2w^2$ is nilpotent. Now, $(w + 2w^2)^k = w^k + \binom{k}{1}w^{k-1}(2w^2) + \cdots + \binom{k}{k-1}w(2w^2)^{k-1} + (2w^2)^k \in J(R)$. Thus $w^k \in J(R)$. Since $J(R)$ is nil, we have $(w^k)^t$ for some $t \in \mathbb{N}$. So w is nilpotent and hence $w + 2w^2$ is nilpotent. As R is strongly win-clean, $n + w + 2w^2$ is nilpotent. Also $(1 - 2w^2)^2 = 1$. Hence, r satisfies nil-involution property. □

Lemma 2 ([6]). *Let R be a ring, and I be a nilpotent ideal of R. An element $x \in R$ is strongly π-regular if and only if $\bar{x}$ is strongly π-regular in $\bar{R} = R/I$.*

Proposition 23. *Let R be a ring and I be a nilpotent ideal of R. Let $\bar{R} = R/I$. If $\bar{a}$ is an element of R such that strongly win-clean in $\bar{R}$, then a is strongly win-clean in R.*

Proof. Since $\bar{a}$ is strongly win-clean, we may write $\bar{a} = \bar{n} + \bar{w}$ for some nilpotent $\bar{n}$ and weak idempotent $\bar{w}$ which commute. By Proposition 2, $\bar{a} = \overline{n + w - 1 + w^2} + \overline{1 - w^2}$ is a strongly π-regular decomposition of $\bar{a}$. By Lemma 15, there exists an idempotent f, lifting $1 - w^2$, and a unit u such that $a = f + u$ is a strongly π-regular decomposition in R by Lemma 2. By Proposition 18, we need only show that $1 - u^2 - 3uf - 3f$ is nilpotent. Since $\bar{f} = \overline{1 - w^2}$ and $\bar{u} = \overline{n + w - 1 + w^2}$, we can calculate that $\overline{1 - u^2 - 3uf - 3f} = \overline{1 - (n + w - 1 + w^2) - 3(n + w - 1 + w^2)(1 - w^2) - 3(1 - w^2)} = \overline{n(w^2 - n - 1 - 2w) + (w^3 - w)}$ $= \overline{n_1}$. Since n_1 is nilpotent modulo I, $1 - u^2 - 3uf - 3f$ is nilpotent. □

Corollary 12. *Suppose that R is a ring with a nilpotent ideal I. Then R is strongly win-clean if and only if R/I is strongly win-clean.*

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