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1-absorbing prime property in lattices

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Abstract. Let $\pounds$ be a bounded distributive lattice. Following the concept of 1-absorbing prime ideals, we define 1-absorbing prime filters of $\pounds$. A proper filter F of $\pounds$ is called 1-absorbing prime filter of $\pounds$ if whenever non-zero elements $a, b, c \in \pounds$ and $a \vee b \vee c \in F$, then either $a \vee b \in F$ or $c \in F$. We will make an intensive investigate the basic properties and possible structures of these filters.

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1 Introduction

All lattices considered in this paper are assumed to have a least element denoted by 0 and a greatest element denoted by 1, in other words they are bounded. As algebraic structures, lattices are definitely a natural choice of generalizations of rings, and it is appropriate to ask which properties of rings can be extended to lattices. The main aim of this article is that of extending some results obtained for ring theory to the theory of lattices. Nevertheless, growing interest in developing the algebraic theory of lattices can be found in several papers and books (see for example [2, 4–10]).

Since prime ideals have an important role in the theory of commutative rings, there are several ways to generalize the concept of prime ideals. In 2021, Yassine et al. in [13] defined 1-absorbing prime ideals which is a generalization of prime ideals . A proper ideal I of R is said to be a 1-absorbing prime ideal if whenever $abc \in I$ for some non-units $a, b, c \in R$, then $ab \in I$ or $c \in I$ (also see [1, 3] and [5, 11]). Let $\pounds$ be a bounded distributive lattice. Our objective in this paper is to extend the notion of 1-absorbing property in commutative rings to 1-absorbing property in the lattices, and to investigate the relations between 1-absorbing prime filters, 2-absorbing filters and prime filters. Among many results in this paper, the first, preliminaries section contains elementary observations needed later on.

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In Section 3, we give basic properties of 1-absorbing prime filters. At first, we give the definition of 1-absorbing prime filters (Definition 1) and we give an example (Example 1) of a 1-absorbing prime filter of $\mathcal{L}$ which is not that a prime filter. It is shown (Theorem 1) that if $\mathcal{L}$ admits a 1-absorbing prime filter that is not a prime filter, then $\mathcal{L}$ is a local lattice. We give an example (Example 2) of a 2-absorbing filter of $\mathcal{L}$ that is not a 1-absorbing prime filter. It is shown (Theorem 3) that If every 2-absorbing filter of a lattice $\mathcal{L}$ is a 1-absorbing prime filter, then prime filters of $\mathcal{L}$ are comparable; in particular, $\mathcal{L}$ is a local lattice. It is proved (Theorem 4) that F is a 1-absorbing prime filter of $\mathcal{L}$ if and only if for any proper filters F_1, F_2, F_3 of $\mathcal{L}$ such that $F_1 \vee F_2 \vee F_3 \subseteq F$ implies that either $F_1 \vee F_2 \subseteq F$ or $F_3 \subseteq F$. It is shown (Theorem 5) that if $\mathcal{L}$ is a non-local lattice, then every nontrivial filter of $\mathcal{L}$ is a 1-absorbing prime filter if and only if $\mathcal{L} = S_1 \oplus S_2$, where S_1 and S_2 are simple filters of $\mathcal{L}$. In the rest of this section, we provide an example of lattices for which their 1-absorbing prime filters and prime filters are the same. It is proved (Theorem 6) that if $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$ is a direct product of lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, then P is a 1-absorbing prime filter of $\mathcal{L}$ if and only if P is a prime filter of $\mathcal{L}$.

Section 4 is dedicated to the study the stability of 1-absorbing prime filters in various lattice-theoretic constructions. Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. There are many different definition of a quotient lattice appearing in the literature. Here, quotient 1-absorbing prime filters are studied and some possible properties of these filters are investigated. It is proved (Theorem 8) that if F and G are proper filters of a complemented lattice $\mathcal{L}$ with $G \subseteq F$, then F is a 1-absorbing prime filter of $\mathcal{L}$ if and only if F/G is a 1-absorbing prime filter of $\mathcal{L}/G$. It is shown (Theorem 9) that if F is a proper filter of $\mathcal{L}$ and $\mathcal{L}$ is a $\vee$-lattice, then $\mathcal{L}/F$ is a $\vee$-lattice. It is proved (Theorem 12) that if F is a proper filter of $\mathcal{L}$, then the following hold: (1) $F = \bigcap_{H \in \min_{ab}(F)} H$, (2) If F is a 1-absorbing prime filter, then $|\min_{ab}(F)| = 1$ and (3) If F has a $\vee$-factorization, then $\min_{ab}(F)$ is finite. It is shown (Theorem 14) that if F is a 1-absorbing prime filter of a divided lattice $\mathcal{L}$ and $(F :_{\mathcal{L}} a)$ is a minimal prime filter over F for some $a \in \mathcal{L} \setminus F$, then F is a prime filter.

Section 5 is devoted to prove that the 1-absorbing prime avoidance theorem. More precisely, by using the technique of efficient covering of filters, in Theorem 15, the 1-absorbing prime avoidance theorem for 1-absorbing prime filters of $\mathcal{L}$ is proved (also, some applications of this theorem are given (Theorem 16 and Theorem 17)).

2 Preliminaries

A *poset* $(\mathcal{L}, \leq)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in \mathcal{L}$ (and call $\wedge$ the *meet* and $\vee$ the *join*). A lattice $\mathcal{L}$ is *complete* when each of its subsets X has a least upper bound and a greatest lower bound in $\mathcal{L}$. Setting $X = \mathcal{L}$, we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that $\mathcal{L}$ is a lattice with 0 and 1). A lattice $\mathcal{L}$ is called a *distributive* lattice if $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all a, b, c in $\mathcal{L}$ (equivalently, $\mathcal{L}$ is distributive if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ for all a, b, c in $\mathcal{L}$). A non-empty subset F of a lattice $\mathcal{L}$ is called a *filter*, if for $a \in F$, $b \in \mathcal{L}$, $a \leq b$ implies $b \in F$, and $x \wedge y \in F$ for all $x, y \in F$ (so if $\mathcal{L}$ is a lattice with $0, 1$, then $1 \in F$, $0 \in F$ if and only if $F = \mathcal{L}$ and $\{1\}$ is a filter of $\mathcal{L}$). A proper filter F of $\mathcal{L}$ is called *prime* if $x \vee y \in F$, then $x \in F$ or $y \in F$. A proper filter F of $\mathcal{L}$ is said to be *maximal* if G is a filter in $\mathcal{L}$ with $F \subsetneqq G$, then

$G = \pounds$. The intersection of all filters containing a given subset A of $\pounds$ is the *filter generated by* it, is denoted by $T(A)$. A filter F is called *finitely generated* if there is a finite subset A of F such that $F = T(A)$. A proper filter F of a lattice $\pounds$ is called a 2-*absorbing* filter if whenever $a, b, c \in \pounds$ and $a \vee b \vee c \in F$, then $a \vee b \in F$ or $a \vee c \in F$ or $b \vee c \in F$.

A lattice $\pounds$ is called *local* if it has exactly one maximal filter that contains all proper filters. If $x \in \pounds$, then a *complement* of x in $\pounds$ is an element $y \in \pounds$ such that $x \vee y = 1$ and $x \wedge y = 0$. The lattice $\pounds$ is *complemented* if every element of $\pounds$ has a complement in $\pounds$. If F is a proper filter of $\pounds$, then by a $\vee$-*factorization* of F we mean an expression of F as an intersection $\bigcap_{i=1}^{n} F_i$ of 1-absorbing prime filters. We call $\pounds$ a $\vee$-*lattice* if every proper filter has a $\vee$-factorization. We denote by $\mathrm{abSpec}(\pounds)$ the set of all 1-absorbing prime filters of $\pounds$. If F is a filter in $\pounds$, the set of all *minimal* 1-absorbing prime filters over F will be denoted by $\min_{ab}(F)$ and $V_{ab}(F) = \{H \in \mathrm{abSpec}(\pounds) : F \subseteq H\}$. If $\pounds$ and $\pounds'$ are lattices, then a *lattice homomorphism* $f : \pounds \to \pounds'$ is a map from $\pounds$ to $\pounds'$ satisfying $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$ for $x, y \in \pounds$. We say that a subset $S \subseteq \pounds$ is *join subset* if $0 \in S$ and $s_1 \vee s_2 \in S$ for all $s_1, s_2 \in S$ (clearly, if P is a prime filter of $\pounds$, then $\pounds \setminus P$ is a join subset of $\pounds$). For undefined notations or terminologies in lattice theory, we refer the reader to [2,4]. First we need the following lemma proved in [2,5,9].

Lemma 1. *Let F and G be filters of a lattice $\pounds$ and $x \in \pounds$.*

(1) A non-empty subset K of $\pounds$ is a filter of $\pounds$ if and only if $x \vee z \in K$ and $x \wedge y \in K$ for all $x, y \in K$, $z \in \pounds$. Moreover, since $x = x \vee (x \wedge y)$, $y = y \vee (x \wedge y)$ and K is a filter, $x \wedge y \in K$ gives $x, y \in K$ for all $x, y \in \pounds$.

(2) Let A be an arbitrary non-empty subset of $\pounds$. Then

$$T(A) = \{x \in L : \ a_1 \wedge a_2 \wedge \cdots \wedge a_n \leq x \text{ for some } a_i \in A \ (1 \leq i \leq n)\}.$$

Moreover, if $a \in \pounds$, then $T(\{a\}) = T(a) = \{x \vee a : x \in \pounds\}$.

(3) $F \vee G = \{a \vee b : a \in F, b \in G\}$ and $x \vee F = \{x \vee y : y \in F\}$ are filters of $\pounds$ with $F \vee G = F \cap G$.

(4) If $\pounds$ is distributive, then $F \wedge G = \{a \wedge b : a \in F, b \in G\}$ is a filter of $\pounds$ with $F, G \subseteq F \wedge G$.

(5) If $\pounds$ is distributive, F, G are filters of $\pounds$, and $y \in \pounds$, then $(G :_{\pounds} F) = \{x \in \pounds : x \vee F \subseteq G\}$ and $(F :_{\pounds} T(\{y\})) = (F :_L y) = \{a \in L : a \vee y \in F\}$ are filters of L.

(6) If $f : \mathcal{L} \to \mathcal{L}'$ is a lattice homomorphism with $f(1) = 1$, then $\mathrm{Ker}(f) = \{x \in \mathcal{L} : f(x) = 1\}$ is a filter of $\mathcal{L}$.

A proper filter F of $\pounds$ is said to be a *direct meet* of $\pounds$ if $\pounds = F \wedge G$ and $F \cap G = \{1\}$ for some filter G of $\pounds$. In this case we write $L = F \oplus G$. A *simple* filter is a filter that has no filters besides the $\{1\}$ and itself.

3 Characterization of 1-absorbing prime filters

In this section, we collect some basic properties concerning 1-absorbing prime filters. We remind the reader with the following definition.

Definition 1. *A proper filter F of a lattice $\pounds$ is called* 1-absorbing prime *if for all non-zero elements $a, b, c \in \pounds$ such that $a \vee b \vee c \in F$, then either $a \vee b \in F$ or $c \in F$.*

Proposition 1. *If F is a* 1*-absorbing prime filter of $\pounds$, then $(F :_{\pounds} a)$ is a prime filter of $\pounds$ for every $a \in \pounds \setminus F$.*

Proof. Let $x \vee y \in (F :_{\pounds} a)$ for some non-zero elements $x, y \in \pounds$ with $x \notin (F :_{\pounds} a)$ (so $x \vee a \notin F$). Then by assumption, $x \vee y \vee a \in F$ and $x \vee a \notin F$ gives $y \in F \subseteq (F :_{\pounds} a)$. Hence $(F :_{\pounds} a)$ is a prime filter. □

It can be easily seen that every prime filter of $\pounds$ is 1-absorbing prime. But generally these two classes are different. The following result provides an example to construct a 1-absorbing prime filter that is not a prime filter (which is mentioned in [6, Example 2.4]).

Example 1. Let $\mathbb{N}$ be the set of natural numbers and $\pounds = H \cup \{\mathbb{N}\}$, where $H = \{X \subseteq \mathbb{N} : X \text{ is finite}\}$. Then $\pounds$ is a distributive lattice (note that if $X, Y \in \pounds$, then $X \vee Y = X \cup Y$ and $X \wedge Y = X \cap Y$). Every (resp. prime) filter of $\pounds$ has the form $[X) = \{G \in \pounds : X \subseteq G\}$, where $X \in \pounds$ (resp. $[\{a\}) = \{G \in \pounds : a \in G\}$, where $a \in \mathbb{N}$). Set $X = \{1, 2\}$. Then $[X)$ is a 1-absorbing prime filter of $\pounds$ which is not that a prime filter. Thus a 1-absorbing prime filter need not be a prime filter.

In the following theorem, we prove that if $\pounds$ is not a local lattice, then every 1-absorbing prime filter is prime. Compare the next theorem with Theorem 2.4 in [13].

Theorem 1. *If F is a* 1*-absorbing prime filter of a lattice $\pounds$ that is not prime for some filter F of $\pounds$, then $\pounds$ is a local lattice.*

Proof. By hypothesis, there are non-zero elements $a, b \in \pounds$ such that $a \vee b \in F$ and $a, b \notin F$. Assume to the contrary, let M_1 and M_2 be two distinct maximal filters of $\pounds$. Then $M_1 \subsetneqq M_1 \wedge M_2 \subseteq \pounds$ gives $M_1 \wedge M_2 = \pounds$; so $0 = m_1 \wedge m_2$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Therefore $a \vee b \vee m_1 \in F$ and $b \notin F$ implies that $a \vee m_1 \in F$, as F is a 1-absorbing prime filter. Similarly, $a \vee m_2 \in F$. It follows that $(a \vee m_1) \wedge (a \vee m_2) = a \vee (m_1 \wedge m_2) = a \in F$ which is impossible. This completes the proof. □

Corollary 1. *If a lattice $\pounds$ is not local, then a proper filter F is a* 1*-absorbing prime filter if and only if F is a prime filter of $\pounds$.*

Proof. This is a direct consequence of Theorem 1. □

Theorem 2. *If P is a* 1*-absorbing prime filter of a local lattice $\pounds$ with unique maximal filter M, then either P is prime or $P \wedge M \subseteq P \subseteq M$.*

Proof. Let P be a 1-absorbing prime filter such that P is not a prime filter. Since P is proper, we infer that $P \subsetneqq M$ by [7, Lemma 2.1]. There are $a, b \in M \setminus P$ such that $a \vee b \in P$, as P is not prime. To see that $P \wedge M \subseteq P$, it is enough to show that $x \wedge y \in P$ for all $x \in P$ and $y \in M$. Let $x \in P$ and $y \in M$. Set $z = x \wedge y$ (so $a \vee b \vee z \in P$). Since a, b, z are non-zero, $b \notin P$ and P is a 1-absorbing prime filter, we have $z \vee a \in P$. Again, since z, a are non-zero, $a \notin \mathbf{p}$ and P is 1-absorbing prime, we have $z \in P$, as required. □

It is obvious that any 1-absorbing prime filter of $\pounds$ is a 2-absorbing filter. The following example shows that 1-absorbing prime filters and 2-absorbing filters are not coincide generally.

Example 2. Let $\pounds = \{0, a, b, c, 1\}$ be a lattice with the relations $0 \leq a \leq c \leq 1$, $0 \leq b \leq c \leq 1$, $a \vee b = c$ and $a \wedge b = 0$. An inspection will show that the nontrivial filters of $\pounds$ (i.e. different from $\{1\}$ and $\pounds$) are $F_1 = \{1, a, c\}$, $F_2 = \{1, b, c\}$ and $F_3 = \{1, c\}$.

(1) $\pounds$ is not a local lattice and F_3 is not a prime filter of $\pounds$ since F_1, F_2 are distinct maximal filters, $a \vee b = c \in F_3$ and $a, b \notin F_3$. Then it follows from Corollary 1 that F_3 is not a 1-absorbing prime filter of $\pounds$.

(2) Using [5, Theorem 2.8 (iii)], $F_1 \cap F_2 = F_3$ is a 2-absorbing filter, but it is not a 1-absorbing prime filter by (1). Thus a 2-absorbing filter need not be a 1-absorbing prime filter.

Theorem 3. *If every* 2*-absorbing filter of a lattice* $\pounds$ *is a* 1*-absorbing prime, then prime filters of* $\pounds$ *are comparable; in particular,* $\pounds$ *is a local lattice.*

Proof. Let P_1 and P_2 be two prime filters of $\pounds$. Using [5, Theorem 2.8 (iii)], $P_1 \cap P_2$ is a 2-absorbing filter; so by assumption, it is a 1-absorbing prime filter which gives $(P_1 \cap P_2 :_\pounds x) = (P_1 :_\pounds x) \cap (P_2 :_\pounds x)$ is a prime filter for every $x \in \pounds \setminus (P_1 \cap P_2)$ by Proposition 1. Let $x \in \pounds \setminus (P_1 \cup P_2)$. Then by [6, Lemma 2.1], either $(P_1 :_\pounds x) \subseteq (P_1 :_\pounds x) \cap (P_2 :_\pounds x)$ or $(P_2 :_\pounds x) \subseteq (P_1 :_\pounds x) \cap (P_2 :_\pounds x)$ which implies that either $(P_1 :_\pounds x) \subseteq (P_2 :_\pounds x)$ or $(P_2 :_\pounds x) \subseteq (P_1 :_\pounds x)$. Therefore, without loss of generality, we can assume that $(P_1 :_\pounds x) \subseteq (P_2 :_\pounds x)$. Let $p_1 \in P_1$. Then $p_1 \in P_1 \subseteq (P_1 :_\pounds x) \subseteq (P_2 :_\pounds x)$ gives $p_1 \in P_2$, as P_2 is a prime filter; so $P_1 \subseteq P_2$. This shows that the prime filters of $\pounds$ are comparable. Accordingly, $\pounds$ is a local lattice. □

Lemma 2. *Let* F *be a* 1*-absorbing prime filter of* $\pounds$*. If* $(a \vee b) \vee G \subseteq F$ *for some non-zero elements* $a, b \in \pounds$ *and a proper filter* G *of* $\pounds$*, then* $a \vee b \in F$ *or* $G \subseteq F$*.*

Proof. Suppose on the contrary that $(a \vee b) \vee G \subseteq F$, but $a \vee b \notin F$ and $G \nsubseteq F$. Then there is an element $g \in G$ such that $g \notin F$. By assumption, $a \vee b \vee g \in F$ gives $a \vee b \in F$ or $g \in F$ which is impossible. □

Theorem 4. *Let* F *be a proper filter of* $\pounds$*. The following statements are equivalent:*

(1) F *is a* 1*-absorbing prime filter of* $\pounds$*;*

(2) For any proper filters F_1, F_2, F_3 *of* $\pounds$ *such that* $F_1 \vee F_2 \vee F_3 \subseteq F$ *implies that either* $F_1 \vee F_2 \subseteq F$ *or* $F_3 \subseteq F$*.*

Proof. (1) ⇒ (2) Suppose that $F_1 \vee F_2 \vee F_3 \subseteq F$ for some filters F_1, F_2, F_3 of $\pounds$ and $F_1 \vee F_2 \nsubseteq F$. Then there are non-zero elements $f_1 \in F_1$ and $f_2 \in F_2$ such that $f_1 \vee f_2 \notin F$. Since $(f_1 \vee f_2) \vee F_3 \subseteq F$ and $f_1 \vee f_2 \notin F$, it follows from lemma 2 that $F_3 \subseteq F$.

(2) ⇒ (1) Suppose that $x \vee y \vee z \in F$ for some non-zero elements x, y, z of $\pounds$ and $x \vee y \notin F$. Set $F_1 = T(\{x\})$, $F_2 = T(\{y\})$ and $F_3 = T(\{z\})$. Then by (2), $F_1 \vee F_2 \vee F_3 \subseteq F$ and $F_1 \vee F_2 \nsubseteq F$ gives $z \in F_3 \subseteq F$, as needed. □

We need the following lemma proved in [8, Lemma 2.2].

Lemma 3. *If* A *and* B *are nontrivial filters of* $\pounds$ *such that* $\pounds = A \wedge B$*, then* F *is a filter of* $\pounds$ *if and only if* $F = C \wedge D$ *for some subfilter* C *of* A *and some subfilter* D *of* B*.*

The following theorem is a lattice counterpart of Theorem 2 in [3] describing the structure of 1-absorbing prime filters.

Theorem 5. *Let $\pounds$ be a non-local lattice. The following statements are equivalent:*

(1) Every nontrivial filter of $\pounds$ is a 1-absorbing prime filter;

(2) $\pounds = S_1 \oplus S_2$, where S_1 and S_2 are simple filters of $\pounds$.

Proof. (1) $\Rightarrow$ (2) By assumption, $\pounds$ has at least two maximal filters. Let S_1 and S_2 be two distinct maximal filters of $\pounds$. Then $S_1 \subsetneqq S_1 \wedge S_2 \subseteq \pounds$ gives $\pounds = S_1 \wedge S_2$. We claim that $S_1 \cap S_2 = \{1\}$. On the contrary, assume that $S_1 \cap S_2 \neq \{1\}$. Then by using Corollary 1 and our hypothesis, we conclude that $S_1 \cap S_1$ is a prime filter of $\pounds$. Then by [6, Lemma 2.1], $S_1 \cap S_2 \subseteq S_1 \cap S_2$ gives $S_1 = S_2$ which is impossible. So $S_1 \cap S_2 = \{1\}$ and then $\pounds = S_1 \oplus S_2$. It remains to show that S_1 and S_2 are simple filters. If S_1 is not simple, then there exists a nontrivial filter S of $\pounds$ such that $S \subsetneqq S_1$ with $S \nsubseteq S_2$. It follows that $S \wedge S_2 = \pounds$. If $x \in S_1$, then $x = x \wedge 1 \in S_1 \wedge S_2 = S \wedge S_2$ which implies that $x = s \wedge b$ for some $s \in S$ and $b \in S_2$. Now S_1 is a filter gives $b \in S_1 \cap S_2 = \{1\}$ and so $x = s \in S$; hence $S_1 = S$, a contradiction. Similarly, S_2 is a simple filter, i.e. (2) holds.

(2) $\Rightarrow$ (1) If $\pounds = S_1 \oplus S_2$, where S_1 and S_2 are simple filters of $\pounds$, it is clear that every nontrivial filter of $\pounds$ is of the form $\{1\} \wedge S_2 = S_2$ and $S_1 \wedge \{1\} = S_1$ by Lemma 3. Let $S_1 \subseteq S \subseteq \pounds$ for some filter S of $\pounds$. If $S \cap S_2 \neq \{1\}$, then $S_2 \subseteq S$; so $S = \pounds$. Assume that $S \cap S_2 = \{1\}$ and let $x \in S$. Then $x = s_1 \wedge s_2$ for some $s_1 \in S_1$ and $s_2 \in S_2$. Then S is a filter gives $s_2 \in S \cap S_2 = \{1\}$ and so $x = s_1 \in S_1$; hence $S_1 = S$. Thus S_1 is a maximal filter (so prime by [7, Lemma 2.1]). Similarly, S_2 is a maximal filter. Hence, every nontrivial filter of $\pounds$ is prime and so is 1-absorbing prime, as needed. □

Suppose that R_1 and R_2 are commutative rings with identity. It is well known that the ideals of $R_1 \times R_2$ have the form $I_1 \times I_2$ where I_1 is an ideal of R_1 and I_2 is an ideal of R_2. It easily follows that the prime (resp. maximal) ideals of $R_1 \times R_2$ have the form $P \times R_2$ or $R_1 \times Q$ where P is a prime (resp. maximal) ideal of R_1 or Q is a prime (resp. maximal) ideal of R_2. The following results (i.e. Lemma 4, Lemma 5, Lemma 6, Proposition 2 and Proposition 3) are lattice counterpart of these results. In fact, we provide an example of lattices for which their 1-absorbing prime filters and prime filters are the same.

Assume that $(\mathcal{L})_1, \leq_1), (\mathcal{L})_2, \leq_2)$ are lattices and let $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$. We set up a partial order $\leq_c$ on $\mathcal{L}$ as follows: for each $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{L}$, we write $x \leq_c y$ if and only if $x_i \leq_i y_i$ for each $i \in \{1, 2\}$. The following notation below will be used in this paper: It is straightforward to check that $(\mathcal{L}, \leq_c)$ is a lattice with $x \vee_c y = (x_1 \vee y_1, x_2 \vee y_2)$ and $x \wedge_c y = (x_1 \wedge y_1, x_2 \wedge y_2)$. In this case, we say that $\mathcal{L}$ is a *decomposable lattice*.

Lemma 4. *If $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice, then every filter of $\pounds$ is of the form $P_1 \times P_2$, where P_1 is a filter of $\pounds_1$ and P_2 is a filter of $\pounds_2$.*

Proof. Let P be any filter of $\pounds$, $P_1 = \{x_1 \in \pounds_1 : (x_1, y_2) \in P$ for some $y_2 \in \pounds_2\}$ and $P_2 = \{y_2 \in \pounds_2 : (x_1, y_2) \in P$ for some $x_1 \in \pounds_1\}$. Let $x_1, x_1' \in P_1$ and $t_1 \in \pounds_1$. Then $(x_1, y_1), (x_1', y_1') \in P$ for some $y_1, y_1' \in \pounds_2$. Therefore, $(x_1 \wedge x_1', y_1 \wedge y_1') = (x_1, y_1) \wedge_c (x_1', y_1') \in P$ and $(x_1 \vee t_1, y_1) = (x_1, y_1) \vee_c (t_1, 0) \in P$ gives $x_1 \wedge x_1', x_1 \vee t_1 \in P_1$. Thus P_1 is a filter of $\pounds_1$. Similarly, P_2 is a filter of $\pounds_2$. We claim that $P = P_1 \times P_2$. Since the inclusion $P \subseteq P_1 \times P_2$ is clear, we will prove the reverse inclusion. Let $(a, b) \in P_1 \times P_2$. There exist $b' \in \pounds_2$ and $a' \in \pounds_1$ such that $(a, b'), (a', b) \in P$. Since P is a filter, $(0, 1) \vee_c (a, b') = (a, 1) \in P$ and $(1, 0) \vee_c (a', b) = (1, b) \in P$ which gives $(a, 1) \wedge (1, b) = (a, b) \in P$, and so we have equality. □

Lemma 5. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and let P, P_1 and P_2 be as in Lemma 4. If P is a prime filter of $\pounds$, then P_1 is a prime filter of $\pounds_1$ and P_2 is a prime filter of $\pounds_2$.*

Proof. Let $a \vee b \in P_1$ for some $a, b \in \pounds_1$. Then $(a \vee b, c) = (a, c) \vee_c (b, 0) \in P$ for some $c \in \pounds_2$. Then P is prime gives either $(a, c) \in P$ or $(b, 0) \in P$; hence either $a \in P_1$ or $b \in P_1$. Thus P_1 is a prime filter of $\pounds_1$. Similarly, P_2 is a prime filter of $\pounds_2$. □

Lemma 6. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and let P, P_1 and P_2 be as in Lemma 4. If P is a maximal filter of $\pounds$, then P_1 is a maximal filter of $\pounds_1$ and P_2 is a maximal filter of $\pounds_2$.*

Proof. Let $P_1 \subsetneqq K \subseteq \pounds_1$ for some filter K of $\pounds_1$. Then there exist $k \in K \setminus P_1$ and $y \in \pounds_2$ such that $(k, y) \notin P$. Then $P \wedge T(\{(k, y)\}) = \pounds$ gives $(0, 0) = ((k, y) \vee (a, b)) \wedge (c, d) = ((k \vee a) \wedge c, ((y \vee b) \wedge d)$ for some $(a, b) \in \pounds$ and $(c, d) \in P$ (so $c \in P_1 \subseteq K$). It follows that $0 = (k \vee a) \wedge c \in K$, as K is a filter; so $K = \pounds_1$. Thus P_1 is a maximal filter of $\pounds_1$. Similarly, P_2 is a maximal filter of $\pounds_2$. □

Proposition 2. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and let P be a proper filter of $\pounds$. Then P is a prime filter of $\pounds$ if and only if $P = P_1 \times \pounds_2$ for some prime filter P_1 of $\pounds_1$ or $P = \pounds_1 \times P_2$ for some prime filter P_2 of $\pounds_2$.*

Proof. Suppose that P is a prime filter of $\pounds$; we show that either $P = P_1 \times \pounds_2$ or $P = \pounds_1 \times P_2$, where P_1 and P_2 are as in Lemma 4. On the contrary, assume that $P \subsetneqq P_1 \times \pounds_2$ and $P \subsetneqq \pounds_1 \times P_2$. There are elements $p_1 \in P_1$, $y_2 \in \pounds_2$, $x_1 \in \pounds_1$ and $p_2 \in P_2$ such that $(p_1, y_2) \in (P_1 \times \pounds_2) \setminus P$ and $(x_1, p_2) \in (\pounds_1 \times P_2) \setminus P$. Since $x_1 \vee p_1 \in P_1$, $(x \vee p_1, z) \in P$ for some $z \in \pounds_2$, we have $(x_1 \vee p_1, z) \vee_c (0, y_2 \vee p_2) =$

$$(x_1 \vee p_1, z \vee y_2 \vee p_2) = (p_1, y_2) \vee_c (x_1, z \vee p_2) \in P,$$

as P is a filter. Then P is prime gives $(x_1, p_2) \vee_c (0, z) \in P$; hence $(0, z) \in P$. This shows that $0 \in P_1$, a contradiction. Therefore either $P = P_1 \times \pounds_2$ or $P = \pounds_1 \times P_2$. Conversely, suppose that either $P = G \times \pounds_2$ or $P = \pounds_1 \times H$, where G is prime in $\pounds_1$ and H is prime in $\pounds_2$. Let

$$(x, y) \vee_c (x', y') = (x \vee x', y \vee y') \in P$$

for some $(x, y), (x', y') \in \pounds$. We can assume that $P = G \times \pounds_2$. Then $x \vee x' \in G$ gives either $x \in G$ or $x' \in G$ which implies that either $(x, y) \in P$ or $(x', y') \in P$; so P is prime. □

Proposition 3. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and let P be a proper filter of $\pounds$. Then P is a maximal filter of $\pounds$ if and only if $P = P_1 \times \pounds_2$ for some maximal filter P_1 of $\pounds_1$ or $P = \pounds_1 \times P_2$ for some maximal filter P_2 of $\pounds_2$.*

Proof. If P is a maximal filter of $\pounds$, then by an argument like that in Proposition 2, we have either $P = P_1 \times \pounds_2$ or $P = \pounds_1 \times P_2$, where P_1 and P_2 are as in Lemma 4. Conversely, suppose that either $P = G \times \pounds_2$ or $P = \pounds_1 \times H$, where G is a maximal filter of $\pounds_1$ and H is a maximal filter of $\pounds_2$. On the contrary, suppose that $P \subsetneqq H \subsetneqq L$ for some filter H of $\pounds$. We may assume that $P = G \times \pounds_2$. So by Lemma 4, H is of the form $F_1 \times \pounds_2$ such that $G \subsetneqq F_1 \subsetneqq \pounds_1$. Consider $x \in F_1 \setminus G$. Then $T(\{x\}) \wedge G = \pounds_1$ gives $0 = g \wedge (x \vee a) = (g \wedge a) \vee (g \wedge x)$ which gives $g \wedge x = 0 \in F_1$, as F_1 is a filter, a contradiction. Thus P is a maximal filter of $\pounds$. □

By our previous result, it can be easily seen that every direct product of lattices is not a local lattice.

Corollary 2. *If $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice, then a proper filter P of $\pounds$ is a 1-absorbing prime filter of $\pounds$ if and only if P is a prime filter of $\pounds$.*

Proof. This is a direct consequence of Corollary 1. □

In view of Corollary 2 and Proposition 2, we have the following result.

Theorem 6. *Suppose that $\pounds = \pounds_1 \times \pounds_2$ is a decomposable lattice and let P be a proper filter of $\pounds$. The following statements are equivalent:*

(1) P is a 1-absorbing prime filter of $\pounds$;

(2) P is a prime filter of $\pounds$;

(3) $P = P_1 \times \pounds_2$ for some prime filter P_1 of $\pounds_1$ or $P = \pounds_1 \times P_2$ for some prime filter P_2 of $\pounds_2$.

4 Further results

We begin this section by investigation the stability of prime 1-absorbing filters in various lattice-theoretic constructions.

Theorem 7. *Let $\pounds_1$ and $\pounds_2$ be lattices and $f : \pounds_1 \to \pounds_2$ be a lattice homomorphism such that $f(1) = 1$ and $f(a)$ is non-zero in $\pounds_2$ for every non-zero element a of $\pounds_1$. The following statements hold:*

(1) If F is a 1-absorbing prime filter of $\pounds_2$, then $f^{-1}(F))$ is a 1-absorbing prime filter of $\pounds_1$.

(2) Let $\pounds_1$ be a complemented lattice. If f is epimorphism and H is a 1-absorbing prime filter of $\pounds_1$ with $\mathrm{Ker}(f) \subseteq H$, then $f(H)$ is a 1-absorbing prime filter of $\pounds_2$.

Proof. (1) Let $x, y \in f^{-1}(F)$ and $t \in \pounds_1$. Then $f(x \wedge y) = f(x) \wedge f(y)$, $f(x \vee t) = f(x) \vee f(t) \in F$ gives $x \wedge y, x \vee t \in f^{-1}(F)$, as F is a filter. Thus $f^{-1}(F)$ is a filter of $\pounds_1$. Suppose that $a \vee b \vee c \in f^{-1}(F)$ for some non-zero elements $a, b, c \in \pounds_1$. Then $f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) \in F$ which implies that $f(a) \vee f(b) = f(a \vee b) \in F$ or $f(c) \in F$. It follows that $a \vee b \in f^{-1}(F)$ or $c \in f^{-1}(F)$. Thus $f^{-1}(F)$ is a 1-absorbing prime filter of $\pounds_1$.

(2) Clearly, $f(H)$ is a filter of $\pounds_2$. Suppose that $x \vee y \vee z \in f(H)$ for some non-zero elements $x, y, z \in \pounds_2$. Then there are non-zero elements $a, b, c \in \pounds_1$ such that $x = f(a)$, $y = f(b)$ and $z = f(c)$. Therefore $f(a \vee b \vee c) = f(a) \vee f(b) \vee f(c) = x \vee y \vee z \in f(H)$; so $f(a \vee b \vee c) = f(d)$ for some $d \in H$. By the hypothesis, there exists $e \in \pounds_1$ such that $e \vee d = 1$ and $d \wedge e = 0$. Set $u = a \vee b \vee c$ (so $u \vee d \in H$). Then $f(u \vee e) = f(u) \vee f(e) = f(d) \vee f(e) = f(1) = 1$; hence $u \vee e \in \mathrm{Ker}(f) \subseteq H$. Now H is a filter gives $(u \vee d) \wedge (u \vee e) = u \in H$. Therefore $a \vee b \in H$ or $c \in H$, and so $x \vee y \in f(H)$ or $z \in f(H)$. Hence $f(H)$ is a 1-absorbing prime filter of $\pounds_2$. □

Quotient lattices are determined by equivalence relations rather than by ideals as in the ring case. If F is a filter of a lattice $(\pounds, \leq)$, we define a relation on $\pounds$, given by $x \sim y$ if and only if there exist $a, b \in F$ satisfying $x \wedge a = y \wedge b$. Then $\sim$ is an equivalence relation on $\pounds$, and we denote the equivalence class of a by $a \wedge F$ and these collection of all equivalence classes by

$\pounds/F$. We set up a partial order $\leq_Q$ on $\pounds/F$ as follows: for each $a \wedge F, b \wedge F \in \pounds/F$, we write $a \wedge F \leq_Q b \wedge F$ if and only if $a \leq b$. The following notation below will be kept in this paper: It is straightforward to check that $(\pounds/F, \leq_Q)$ is a lattice with $(a \wedge F) \vee_Q (b \wedge F) = (a \vee b) \wedge F$ and $(a \wedge F) \wedge_Q (b \wedge F) = (a \wedge b) \wedge F$ for all elements $a \wedge F, b \wedge F \in \pounds/F$. Note that $f \wedge F = F$ if and only if $f \in F$ (see [10, Remark 4.2 and Lemma 4.3]).

Compare the next theorem with Corollary 2.17 in [13].

Theorem 8. *Let F and G be proper filters of a complemented lattice $\pounds$ with $G \subseteq F$. Then F is a 1-absorbing prime filter of $\pounds$ if and only if F/G is a 1-absorbing prime filter of $\pounds/G$.*

Proof. Let $f : \pounds \to \pounds/G$ such that $f(x) = x \wedge G$. So we have

$$f(x \vee y) = (x \vee y) \wedge G = (x \wedge G) \vee_Q (y \wedge G) = f(x) \vee_Q f(y).$$

similarly, $f(x \wedge y) = f(x) \wedge_Q f(y)$. Then f is a lattice homomorphism from $\pounds$ onto $\pounds/G$, $f(1) = 1 \wedge G = 1_{\pounds/G}$, $\mathrm{Ker}(f) = \{x \in \pounds : x \wedge G = 1 \wedge G\} = G \subseteq F$ by [9, Lemma 4.2 (2)] and $f(F) = \{f(z) : z \in F\} = \{z \wedge G : z \in F\} = F/G$ by [10, Lemma 4.2 (4)]. Suppose that F is a 1-absorbing prime filter. Since $\mathrm{Ker}(f) \subseteq F$ and f is onto, we conclude that $f(F) = F/G$ is a prime 1-absorbing filter of $\pounds/G$ by Theorem 7 (2). Suppose that F/G is a 1-absorbing prime filter of $\pounds/G$. Then $f^{-1}(F/G) = F$ by [10, Lemma 4.2 (4)]; so F is a 1-absorbing prime filter of $\pounds$ by Theorem 7 (1). $\square$

Theorem 9. *Let F be a proper filter of $\pounds$. If $\pounds$ is a $\vee$-lattice, then $\pounds/F$ is a $\vee$-lattice.*

Proof. Let H be a proper filter of $\pounds/F$. Then $H = K/F$ for some proper filter K of $\pounds$ by [10, Lemma 4.2 (5)]. Let $K = \bigcap_{i=1}^{n} F_i$ be a $\vee$-factorization. Then $H = (\bigcap_{i=1}^{n} F_i)/F = \bigcap_{i=1}^{n} F_i/F$ by [10, Lemma 4.2 (7)]. It is enough to show that F_i/F is a 1-absorbing prime filter of $\pounds/F$ for each $i \in \{1, 2, \ldots, n\}$. Let

$$(a \wedge F) \vee_Q (b \wedge F) \vee_Q (c \wedge F) = a \vee b \vee c \wedge F \in F_i/F$$

for some non-zero elements $a \wedge F, b \wedge F, c \wedge F \in \pounds/F$. Clearly, a, b, c are non-zero elements of $\pounds$ and $a \vee b \vee c \in F_i$ by [10, Lemma 4.2 (2)]. Since F_i is a 1-absorbing prime filter, we get that $a \vee b \in F_i$ or $c \in F_i$ which implies that $(a \wedge F) \vee_Q (b \wedge F) \in F_i/F$ or $c \wedge F \in F_i/F$. Therefore $\pounds/F$ is a $\vee$-lattice. $\square$

Theorem 10. *For each 1-absorbing prime filter P and a join subset S, a 1-absorbing prime filter Q that satisfies $P \subseteq Q$ is constructible.*

Proof. Set $Q = \{x \in \pounds : x \vee s \in P$ for some $s \in S\}$. If $p \in P$, then $p \vee s \in P$ $(s \in S)$ gives $P \subseteq Q$. Let $x_1, x_2 \in Q$ and $t \in \pounds$. Then $x_1 \vee s_1, x_2 \vee s_2 \in P$ for some $s_1, s_2 \in S$ (so $s_1 \vee s_2 \in S$) gives $(x_1 \wedge x_2) \vee (s_1 \vee s_2), (x_1 \vee t) \vee s_1 \in P$; hence $x_1 \wedge x_2, x_1 \vee t \in Q$. Thus Q is a filter of $\pounds$. Let $a \vee b \vee c \in Q$ for some non-zero elements $a, b, c \in \pounds$ such that $a \vee b \notin Q$. Then $a \vee b \vee c \vee s \in P$ for some $s \in S$ and $a \vee b \notin P$. Now P is a 1-absorbing prime filter gives $c \vee s \in Q$ which implies that $c \in Q$. Hence Q is a 1-absorbing prime filter of $\pounds$ such that $P \subseteq Q$. $\square$

Let $\mathbb{F}(\pounds)$ be the set of all filters of $\pounds$.

Lemma 7. *If F is a filter and S a join subset S with $S \cap F = \emptyset$, then the set $\Omega = \{G \in \mathbb{F}(\pounds) : F \subseteq G$ and $S \cap G = \emptyset\}$ of filters under the relation of inclusion has at least one maximal element, and any such maximal element of Ω is a 1-absorbing prime filter.*

Proof. Clearly, $F \in \Omega$, and so $\Omega \neq \emptyset$. Moreover, $(\Omega, \subseteq)$ is a partial order. It is easy to see that Ω is closed under taking unions of chains and so Ω has at least one maximal element by Zorn's Lemma, say M. Since $S \cap M = \emptyset$ and $0 \in S$, we see that $0 \notin M$ and $M \subsetneqq \pounds$. It remains to show that M is a 1-absorbing prime filter. Now let $a \vee b, c \notin M$; we must show that $a \vee b \vee c \notin M$ for some non-zero elements $a, b, c \in \pounds$. Since $a \vee b \notin M$, we have $F \subseteq M \subsetneqq M \wedge T(\{a \vee b\})$. By maximality of M in Ω, we must have $S \cap (M \wedge T(\{a \vee b\})) \neq \emptyset$, and so there exist $s \in S$, $t \in \pounds$ and $e \in M$ such that $s = e \wedge (a \vee b \vee t)$. Similarly, there exist $s' \in S$, $t' \in \pounds$ and $e' \in M$ such that $s' = e' \wedge (c \vee t')$. Put $A = a \vee b \vee t$ and $c \vee t' = B$. Then $s \vee s' = ((e \wedge A) \vee e') \wedge ((e \wedge A) \vee B) = ((e \wedge A) \vee e') \wedge (e \vee B) \wedge (a \vee b \vee c \vee t \vee t')$. Since $s \vee s' \in S$ and $((e \wedge A) \vee e') \wedge (e \vee B) \in M$, we must have $a \vee b \vee c \notin M$ since $S \cap M = \emptyset$, as needed. □

Theorem 11. *If F is a filter of $\pounds$, then $F = \bigcap_{H \in V_{ab}(F)} H$.*

Proof. Since the inclusion $F \subseteq \bigcap_{H \in V_{ab}(F)} H$ is clear, we will prove the reverse inclusion. Let $x \in \bigcap_{H \in V_{ab}(F)} H$. We suppose that $x \notin F$, and look for a contradiction. Set $S = \{0, x\}$. Then S is a join subset of $\pounds$ with $S \cap F = \emptyset$. Hence, by Lemma 7, there exists a 1-absorbing prime filter P of $\pounds$ such that $F \subseteq P$ and $P \cap S = \emptyset$. It follows that $P \in V_{ab}(F)$, so that $x \in S \cap P$, a contradiction, i.e. we have equality. □

Proposition 4. *If F, G are filters with G a 1-absorbing prime and $F \subseteq G$, then there exists a minimal 1-absorbing prime H over F such that $H \subseteq G$.*

Proof. Set $\Omega = \{K \in \text{abSpec}(\pounds) : F \subseteq K \subseteq G\}$. Then $G \in \Omega$, and so $\Omega \neq \emptyset$. Put $S = \pounds \setminus (G :_{\pounds} x)$ for every non-zero element $x \in \pounds \setminus G$. Then S is a join closed subset of $\pounds$ by Proposition 1. By an argument like that in Lemma 7, the set Ω of 1-absorbing prime filters of $\pounds$ has a minimal member with respect to inclusion (by partially ordering Ω by reverse inclusion and using Zorn's Lemma) which is a 1-absorbing prime filter, i.e. the result holds. □

Theorem 12. *Let F be a proper filter of $\pounds$. The following hold:*

(1) $F = \bigcap_{H \in \min_{ab}(F)} H$;

(2) If F is a 1-absorbing prime filter, then $|\min_{ab}(F)| = 1$;

(3) If F has a $\vee$-factorization, then $\min_{ab}(F)$ is finite.

Proof. (1) Since $\min_{ab}(F) \subseteq V_{ab}(F)$, it is clear that $\bigcap_{H \in V_{ab}(F)} H$ is a subset of $\bigcap_{H \in \min_{ab}(F)} H$ by Theorem 11. However, the reverse inclusion is immediate from Proposition 4 (since every prime 1-absorbing filter in $V_{ab}(F)$ contains a minimal 1-absorbing prime filter over F).

(2) This follows from (1).

(3) If $F = \bigcap_{i=1}^{k} F_i$ is a $\vee$-factorization, then $\min_{ab}(F) \subseteq \bigcup_{i=1}^{k} \min_{ab}(F_i)$, and so $|\min_{ab}(F)| \leq k$ by (2), i.e. (3) holds. □

Compare the next Theorem with Theorem 2.1, p. 2 in [12].

Theorem 13. *Let $F \subseteq G$ be filters of £ with G is a 1-absorbing prime filter and $a \in £ \setminus G$. The following statements are equivalent:*

(1) $(G :_£ a)$ is a minimal 1-absorbing prime filter over F;

(2) $S = £ \setminus (G :_£ a)$ is a join subset that is maximal with $S \cap F = \emptyset$;

(3) For each $x \in (G :_£ a)$, there is a $y \notin (G :_£ a)$ such that $x \vee y \in F$.

Proof. (1) $\Rightarrow$ (2) Since $F \subseteq (G :_£ a)$, $(£ \setminus (G :_£ a)) \cap F = \emptyset$. Then the set Ω of all join closed sets, say U, with $U \cap F = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Ω. Now Ω is easily seen to be inductive under inclusion, so by Zorn's Lemma Ω has a maximal element S (so $F \cap S = \emptyset$). By Lemma 7, there is a 1-absorbing prime filter H of £ containing F that is maximal with respect to being disjoint from S which implies that $(£ \setminus (G :_£ a)) \cap H = \emptyset$. It follows that $H = (G :_£ a)$ by (1); hence $S = £ \setminus (G :_£ a)$.

(2) $\Rightarrow$ (3) Let $1 \neq x \in (G :_£ a)$ and set $S = \{x \vee y : y \in £ \setminus (G :_£ a)\}$. Then S is a join subset that properly contains $£ \setminus (G :_£ a)$; so $F \cap S \neq \emptyset$ by maximality of $£ \setminus (G :_£ a)$. Thus there exists $y \in £ \setminus (G :_£ a)$ such that $x \vee y \in F$.

(3) $\Rightarrow$ (1) Let K be a 1-absorbing prime filter such that $F \subsetneqq K \subseteq (G :_£ a)$. If $K \neq (G :_£ a)$, then there is an element $x \in (G :_£ a)$ with $x \notin K$; so $x \vee y \in F \subsetneqq K$ for some $y \notin (G :_£ a)$ which is a contradiction. Therefore $(G :_£ a) = K$. □

Theorem 14. *Suppose that F is a 1-absorbing prime filter of a divided lattice £ and let $P = (F :_£ a)$ be a minimal prime filter over F for some $a \in £ \setminus F$. Then F is a prime filter. In particular, $F = (F :_£ a)$.*

Proof. Let $x \vee y \in F$ for some $x, y \in £$ such that $y \notin F$. We claim that $y \notin P$. On the contrary, assume that $y \in P$. Then by Theorem 13, there exists $z \notin P$ such that $y \vee y \vee z = z \vee y \in F$. But $y \notin F$ and F is a 1-absorbing prime filter of £, so $z \in F$, a contradiction. Thus $y \notin P$. Now $x \vee y \in F \subseteq P$ gives $x \in P$ and since £ is divided, we conclude that $P \subseteq T(\{y\})$; so $x = y \vee u$ for some $u \in £$. Then $x \vee y = y \vee y \vee u \in F$. But $y \notin F$ and F is a 1-absorbing prime filter of £; hence $u \in F$. Therefore $x \in F$, and thus F is a prime filter of £. The "in particular" statement is clear. □

5 1-Absorbing prime avoidance theorem

In this section, we state the 1-absorbing prime avoidance theorem for 1-absorbing prime filters of a lattice £.

Let $F, F_1, F_2, \dots, F_n$ be filters of £. We call a covering $F \subseteq \bigcup_{i=1}^n F_i$ efficient if no F_i is superfluous. Analogously, we say that $F = \bigcup_{i=1}^n F_i$ is an efficient union if none of the F_i may be excluded. Any cover or union consisting of filters of £ can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms. We need the following lemma proved in [5, Lemma 3.2] and [5, Remark 2.3 (i)], respectively.

Lemma 8. *For the lattice £ the following hold:*

(1) Let F and F_i $(i = 1, 2, \dots, n)$ be filters such that $F \subseteq \bigcup_{i=1}^n F_i$ is an efficient covering of filters of £, where $n \geq 3$. Then The intersection of any $n-1$ of the filters $F \cap F_i$ coincides with $\bigcap_{i=1}^n (F \cap F_i)$.

(2) If F, F_1, F_2 are filters of £ with $F \subseteq F_1 \cup F_2$, then either $F \subseteq F_1$ or $F \subseteq F_2$.

Proposition 5. *Let F and Fi $(i = 1, 2, \dots, n)$ be filters such that $F \subseteq \bigcup_{i=1}^{n} F_i$ is an efficient covering of filters of $\pounds$, where $n \geq 2$. If $F_i \nsubseteq (F_j, :_\pounds x)$ for all $x \in \pounds \setminus F_j$ whenever $i \neq j$, then no F_i is a 1-absorbing prime filter of $\pounds$, for every $1 \leq i \leq n$.*

Proof. On the contrary, assume that F_j is a 1-absorbing prime filter for some $j \in \{1, 2, \dots, n\}$. Since $F \subseteq \bigcup_{i=1}^{n} F_i$ is an efficient union, we have $F = \bigcup_{i=1}^{n}(F \cap F_i)$ is an efficient union. Therefore there is an element $a_j \in F \setminus F_j$ for every $1 \leq j \leq n$. Since $F = \bigcup_{i=1}^{n}(F \cap F_i)$ is an efficient union, we conclude that $\bigcap_{j \neq i}(F \cap F_i) \subseteq F \cap F_j$ by Lemma 8 (1). By assumption, there is an element $b_i \in F_i$ such that $b_i \notin (F_j :_\pounds a_j)$ for every $i \neq j$. Set $b = \bigvee_{j \neq i} b_i$. Then F and F_i are filters gives $b \vee a_j \in \bigcap_{j \neq i}(F \cap F_i)$. We claim that $b \vee a_j \notin F \cap F_j$. Assume to the contrary, that $b \vee a_j \in F_j \cap F$. Since F_j is 1-absorbing prime and $(F_j :_\pounds a_j)$ is a prime filter of $\pounds$ by Proposition 1, we have $b_i \in (F_j :_\pounds a_j)$ for some $i \neq j$ which is a contradiction. It follows that $b \vee a_j \notin F \cap F_j$ and this contradicts the fact that $\bigcap_{j \neq i}(F \cap F_i) \subseteq F \cap F_j$, as required. □

Theorem 15. *(1-Absorbing prime avoidance theorem). Let F, $F_1, F_2, \dots, F_n$ $(n \geq 2)$ be filters of $\pounds$ such that at most two of $F_1, F_2, \dots, F_n$ are not 1-absorbing prime. If $F \subseteq \bigcup_{i=1}^{n} F_i$ and $F_i \nsubseteq (F_j :_\pounds x)$ for all $x \in \pounds \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some $i \in \{1, 2, \dots, n\}$.*

Proof. By Lemma 8 (2), we may assume that $n \geq 3$. Let $F \subseteq F_i$ for all i with $1 \leq i \leq n$. Then $F \subseteq \bigcup_{i=1}^{n} F_i$ is an efficient covering of filters of $\pounds$. So by Proposition 5, no F_i is 1-absorbing prime that contradicts the assumption. Therefore $F \subseteq F_i$ for some i with $1 \leq i \leq n$. □

Compare the next theorem with Corollary 3.3 in [13].

Theorem 16. *Let $F = T(\{x_1, x_2, \dots, x_s\})$, where $\{x_1, x_2, \dots, x_s\} \subseteq \pounds$. Let $F_1, F_2, \cdots, F_n$ be 1-absorbing prime filters of $\pounds$, $F \nsubseteq F_i$ for every $1 \leq i \leq n$ and $F_i \nsubseteq (F_j :_\pounds x)$ for every $x \in \pounds \setminus F_j$ and $i \neq j$. Then There are elements $b_2, b_3, \cdots, b_s \in \pounds$ such that $x = x_1 \wedge (\bigwedge_{i=2}^{s}(b_i \vee x_i)) \notin \bigcup_{i=1}^{n} F_i$.*

Proof. We use induction on n. If $n = 1$, then the result is clear. So assume that $n \geq 2$ and that the result has been proved for smaller values than n. then there are elements $c_2, \dots, c_s \in \pounds$ such that $c = x_1 \wedge (\bigwedge_{i=2}^{s}(c_i \vee x_i)) \notin \bigcup_{i=1}^{n-1} F_i$ (so $c \notin F_i$ for each $i \in \{1, \cdots, n-1\}$). If $c \notin F_n$, then $c \notin \bigcup_{i=1}^{n} F_i$ and so we are done. Thus we may assume that $c \in F_n$. This shows that $x_1 \in F_n$ by Lemma 1 (1). Since $F \nsubseteq F_n$, there is a $2 \leq i \leq n$ such that $x_i \notin F_n$, say x_2. By assumption, there exists $d_i \in F_i$ such that $d_i \notin (F_n :_\pounds x_2)$ for every $i \neq n$. Set $d = \bigvee_{i=1}^{n-1} d_i$. Then $d \in F_i$ for every $i \neq n$ but $d \notin (F_n :_\pounds x_2)$ by Proposition 3.2. Thus $d \in F_i \setminus (F_n :_\pounds x_2)$ for every $i \neq n$. Let $e = x_1 \wedge ((c_2 \wedge d) \vee x_2) \wedge (\bigwedge_{i=3}^{s}(c_i \vee x_i)) =$

$$x_1 \wedge (x_2 \vee c_2) \wedge (x_2 \vee d) \wedge (\bigwedge_{i=3}^{s}(c_i \vee x_i)) = c \wedge (x_2 \vee d).$$

If $F \subseteq \bigcup_{i=1}^{n} F_i$, then $F \subseteq F_i$ for some $1 \leq i \leq n$ by Theorem 15 which is impossible. So suppose that $F \nsubseteq \bigcup_{i=1}^{n} F_i$. Then by an argument like that as above, we assume $x_2 \notin F_n$. If $e \in F_n$, then $x_2 \vee d \in F_n$ by Lemma 1, a contradiction. If $e \in F_i$ for some i $(1 \leq i \leq n-1)$, then $c \in F_i$ which is impossible. Thus $e \notin \bigcup_{i=1}^{n} F_i$, as needed. □

Compare the next theorem with Corollary 3.4 in [13].

Theorem 17. *Let $F_1, F_2, \ldots, F_n$ be 1-absorbing prime filters of $\pounds$, F be a filter of $\pounds$ and $F_i \nsubseteq (F_j :_\pounds x)$ for every $x \in \pounds \setminus F_j$ and $i \neq j$. If $a \in \pounds$ and $T(\{a\}) \wedge F \nsubseteq \bigcup_{i=1}^n F_i$, then there exists $x \in F$ such that $a \wedge x \notin \bigcup_{i=1}^n F_i$.*

Proof. Let $a \in \bigcap_{i=1}^k F_i$ such that $a \notin \bigcup_{k+1}^n F_i$. If $k = 0$, then $a \wedge 1 \notin \bigcup_{i=1}^n F_i$. So suppose that $k > 1$. If $F \subseteq \bigcup_{i=1}^k F_i$, then $F \subseteq F_i$ for some $i \in \{1, 2, \cdots, k\}$ by Theorem 15 which gives $T(\{a\}) \wedge F \subseteq F_i \subseteq \bigcup_{i=1}^n F_i$, a contradiction. So $F \nsubseteq \bigcup_{i=1}^k F_i$. Thus there is an element $b \in F$ such that $b \notin \bigcup_{i=1}^k F_i$ (so $b \notin F_i$ for every $i \in \{1, 2, \cdots, k\}$). We claim that $\bigcap_{i=k+1}^n F_i \nsubseteq \bigcup_{j=1}^k (F_j :_\pounds x)$ for every $x \in \pounds \setminus F_j$. Let $\bigcap_{i=k+1}^n F_i \subseteq \bigcup_{j=1}^k (F_j :_\pounds b)$ for $b \in F \setminus F_j$. By Theorem 15, we obtain $\bigcap_{i=k+1}^n F_i \subseteq (F_j :_\pounds b)$ for some $1 \leq j \leq k$ and $b \in F \setminus F_j$. Since $(F_j :_\pounds b)$ is a prime filter by Proposition1, we conclude by [6, Lemma 2.1] that $F_i \subseteq (F_j :_\pounds b)$, where $1 \leq j \leq k$ and $k+1 \leq i \leq n$ which contradicts the assumption. Therefore there is an element $c \in \bigcap_{i=k+1}^n F_i$ such that $c \notin \bigcup_{i=1}^k (F_j :_\pounds b)$. Set $z = b \vee c$. Since F and $\bigcap_{i=k+1}^n F_i$ are filters, we get $z \in F \cap (\bigcap_{i=k+1}^n F_i)$. If $z = b \vee c \in \bigcup_{i=1}^k F_i$, then $z \in F_i$ for some $1 \leq i \leq k$ and hence $c \in (F_i :_\pounds b)$ for some $1 \leq i \leq k$ which is a contradiction. Thus $z \notin \bigcup_{i=1}^n F_i$ (so $z \notin F_i$ for every $1 \leq i \leq k$). If $a \wedge z \in \bigcup_{i=1}^n F_i$, then $a \wedge z \in F_i$ for some $i \in \{1, 2, \cdots, n\}$. If $i \leq k$, then F_i is a filter gives $z \in F_i$ by Lemma 1 (1), a contradiction. If $k+1 \leq i \leq n$, then $a \in F_i$ which is a contradiction. Thus $a \wedge z \notin \bigcup_{i=1}^n F_i$. □

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