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Tensor splitting preconditioners for multilinear systems

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Abstract. In this paper, we propose some new preconditioners for solving multilinear system $\mathscr{A}\mathbf{x}^{m-1} = \mathbf{b}$. These preconditioners are based on tensor splitting. We also present some theorems for analyzing and convergence of the preconditioned Jacobi-, Gauss-Seidel-, and SOR-type iterative methods. Numerical examples are presented to verify the efficiency of the proposed preconditioned methods.

Keywords: Multilinear system, $\mathscr{M}$-tensor, tensor splitting, preconditioned methods.
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1 Introduction

Recently, solving multilinear system

$$\mathscr{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1}$$

where $\mathscr{A} = (a_{i_1 i_2 \ldots i_m}),\ a_{i_1 i_2 \ldots i_m} \in \mathbb{C},\ 1 \le i_j \le n_j,\ j = 1, \ldots, m$ is an m order n-dimensional tensor, $\mathbf{x}$ and $\mathbf{b}$ are vectors in $\mathbb{C}^n$ has become a hot topic due to its applications in fields such as data analysis, engineering and scientific computing [6, 8, 15]. The n-dimensional vector $\mathscr{A}\mathbf{x}^{m-1}$ is defined by [29]

$$(\mathscr{A}\mathbf{x}^{m-1})_i = \sum_{i_2=1}^{n} \ldots \sum_{i_m=1}^{n} a_{i i_2 \ldots i_m} x_{i_2} \ldots x_{i_m}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where x_i denotes the ith component of $\mathbf{x}$. Many theoretical analysis and algorithms for solving multilinear systems (1) have also studied in [1, 5, 9–21, 23, 25, 30, 31].

We know that preconditioning techniques play a fundamental role in solving multilinear systems, in particular, when the coefficient tensor is an $\mathscr{M}$-tensor. Liu et al. in [24], presented the preconditioned SOR method for solving multilinear systems whose coefficient tensor is an $\mathscr{M}$-tensor. Also, the corresponding comparison for spectral radii of the tensor iterative methods was given. Cui et al. in [7], proposed a preconditioned iterative method based on tensor splitting for solving the multilinear system

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(1). For this purpose, they suggested the preconditioner $\mathbf{I}+\mathbf{S}_{\max}$.

In this paper, we propose new preconditioners for solving multilinear system (1) that are more efficient than the existing methods. Also, we give some theorems for analyzing and convergence of the new preconditioned methods.

The rest of this paper is organized as follows. Section 2 is preliminary, in which we introduce some related definitions and lemmas. In the 3rd section, some new preconditioners are proposed, and the corresponding theoretical analysis is given. In Section 4, numerical examples are given to show the efficiency of the proposed preconditioned iterative methods. Section 5 is the concluding remarks.

2 Preliminaries

In this section, we introduce some definitions, notations and lemmas. Let $\mathbf{0}$, $\mathbf{O}$ and $\mathscr{O}$ denote null vector, null matrix and null tensor, respectively. Suppose that $\mathscr{A}$ and $\mathscr{B}$ are tensors with the same size. The order $\mathscr{A} \geq \mathscr{B} (> \mathscr{B})$ means that each element of $\mathscr{A}$ is no less than (larger than) the corresponding one of $\mathscr{B}$. A tensor $\mathscr{A} \in \mathbb{C}^{n_1\times\ldots\times n_m}$ consists of $\prod_{i=1}^m n_i$ elements in the complex field $\mathbb{C}$. If $n_1 = \ldots = n_m = n$, $\mathscr{A}$ is called an m order n-dimensional tensor. By $\mathbb{C}^{n_1\times\ldots\times n_m}$, we denote all m order tensors consisting of $\prod_{i=1}^m n_i$ entries and by $\mathbb{C}^{[m,n]}$ we denote the set of all m order n-dimensional tensors. When $m = 1$, $\mathbb{C}^{[1,n]}$ is simplified as $\mathbb{C}^n$, which is the set of all n-dimensional complex vectors. Similarly, the above notions can be used for the real number field $\mathbb{R}$. Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$, if each entry of $\mathscr{A}$ is nonnegative, then $\mathscr{A}$ is called a nonnegative tensor. The set of all m order n-dimensional nonnegative tensors is denoted by $\mathbb{R}_+^{[m,n]}$. The m order n-dimensional identity tensor is denoted by $\mathscr{I}_m = (\delta_{i_1 i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ where

$$\delta_{i_1 i_2\ldots i_m} = \begin{cases} 1, & if\ i_1 = i_2 = \ldots = i_m \\ 0, & otherwise. \end{cases}$$

The identity matrix of size $n\times n$, is denoted by $\mathbf{I}$.

Definition 1. *[32] $\mathscr{A} \in \mathbb{R}^{[m,n]}$ is called a $\mathscr{Z}$-tensor if its off-diagonal entries are non-positive. $\mathscr{A}$ is an $\mathscr{M}$-tensor if there exists a tensor $\mathscr{B} \in \mathbb{R}_+^{[m,n]}$ and a positive real number $\eta \geq \rho(\mathscr{B})$ such that $\mathscr{A} = \eta\mathscr{I} - \mathscr{B}$. If $\eta > \rho(\mathscr{B})$, then $\mathscr{A}$ is called a strong $\mathscr{M}$-tensor.*

Definition 2. *[23] Let $\mathbf{A} \in \mathbb{R}^{[2,n]}$ and $\mathscr{B} \in \mathbb{R}^{[m,n]}$. $\mathscr{C} = \mathbf{A}\mathscr{B} \in \mathbb{C}^{[m,n]}$ is defined by*

$$c_{ji_2\ldots i_m} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2\ldots i_m}, \tag{3}$$

which can be written as $\mathscr{C}_{(1)} = (\mathbf{A}\mathscr{B})_{(1)} = \mathbf{A}\mathscr{B}_{(1)}$, where $\mathscr{C}_{(1)}$ and $\mathscr{B}_{(1)}$ are the matrices obtained from $\mathscr{C}$ and $\mathscr{B}$ flattened along the first index, respectively.

Definition 3. *[28] Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. The majorization matrix of $\mathscr{A}$, denoted by $\mathbf{M}(\mathscr{A})$, is defined as a square matrix of size $n\times n$ where its entries are*

$$(\mathbf{M}(\mathscr{A}))_{ij} = a_{ij\ldots j},\ \ i,j = 1,2,\ldots,n.$$

Definition 4. *[21] If $\mathbf{M}(\mathscr{A})$ is a nonsingular matrix and $\mathscr{A} = \mathbf{M}(\mathscr{A})\mathscr{I}_m$, then $(\mathbf{M}(\mathscr{A}))^{-1}$ is the order 2 left-inverse of $\mathscr{A}$, i.e. $(\mathbf{M}(\mathscr{A}))^{-1}\mathscr{A} = \mathscr{I}_m$, and then we call $\mathscr{A}$ a left-invertible tensor or left-nonsingular tensor.*

Definition 5. *[29] Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{\mathbf{0}\})$ is called an eigenvalue-eigenvector(or simply eigenpair) of $\mathscr{A}$ if they satisfy the equation*

$$\mathscr{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{4}$$

where $\mathbf{x}^{[m-1]} = (x_1^{m-1}, \dots, x_n^{m-1})^\mathsf{T}$. $\rho(\mathscr{A}) = \max\{|\lambda||\lambda \in \sigma(\mathscr{A})\}$ is called the spectral radius of $\mathscr{A}$, where $\sigma(\mathscr{A})$ is the set of all eigenvalues of $\mathscr{A}$.

Lemma 1. *[23] If $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, then $\mathbf{M}(\mathscr{A})$ is a nonsingular $\mathbf{M}$-matrix.*

Definition 6. *[23] Suppose that $\mathscr{A}, \mathscr{E}, \mathscr{F} \in \mathbb{R}^{[m,n]}$. $\mathscr{A} = \mathscr{E} - \mathscr{F}$ is said to be a splitting of $\mathscr{A}$ if $\mathscr{E}$ is a left-nonsingular; a regular splitting of $\mathscr{A}$ if $\mathscr{E}$ is left-nonsingular with $(\mathbf{M}(\mathscr{E}))^{-1} \geq \mathbf{O}$ and $\mathscr{F} \geq \mathscr{O}$; a weak regular splitting of $\mathscr{A}$ if $\mathscr{E}$ is left-nonsingular with $(\mathbf{M}(\mathscr{E}))^{-1} \geq \mathbf{O}$ and $(\mathbf{M}(\mathscr{E}))^{-1}\mathscr{F} \geq \mathscr{O}$; a convergent splitting if $\rho((\mathbf{M}(\mathscr{E}))^{-1}\mathscr{F}) < 1$.*

Lemma 2. *[33] If $\mathscr{A}$ is a $\mathscr{Z}$-tensor, then the following conditions are equivalent*

1. *$\mathscr{A}$ is a strong $\mathscr{M}$-tensor;*
2. *$\mathscr{A}$ has a convergent (weak) regular splitting;*
3. *All (weak) regular splittings of $\mathscr{A}$ are convergent;*
4. *There exists a vector $\mathbf{x} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^{m-1} > \mathbf{0}$.*

Lemma 3. *[15] If $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, then for every positive vector $\mathbf{b}$, the multilinear system (1) has a unique positive solution.*

Lemma 4. *[22] Suppose that $\mathscr{A} \in \mathbb{R}^{[m,n]}$. Let $\mathscr{A} = \mathscr{E}_1 - \mathscr{F}_1$ and $\mathscr{A} = \mathscr{E}_2 - \mathscr{F}_2$ be a weak regular splitting and a regular splitting, respectively, and $\mathscr{F}_2 \leq \mathscr{F}_1, \mathscr{F}_2 \neq \mathscr{O}$. One of the following statements holds.*

1. *$\rho((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2) \leq \rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1) < 1$;*
2. *$\rho((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2) \geq \rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1) \geq 1$.*

If $\mathscr{F}_2 < \mathscr{F}_1, \mathscr{F}_2 \neq \mathscr{O}$ and $\rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1) > 1$, then the first inequality in part 2 is strict.

Lemma 5. *[33] Let $\mathscr{A}$ be a strong $\mathscr{M}$-tensor, and $\mathscr{A} = \mathscr{E}_1 - \mathscr{F}_1 = \mathscr{E}_2 - \mathscr{F}_2$ be two weak regular splittings with $(\mathbf{M}(\mathscr{E}_1))^{-1} \leq (\mathbf{M}(\mathscr{E}_2))^{-1}$. If the Perron vector $\mathbf{x}$ of $(\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2$ satisfies $\mathscr{A}\mathbf{x}^{m-1} \geq \mathbf{0}$ then $\rho((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2) \leq \rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)$.*

A general tensor splitting iterative method for solving (1) is

$$\mathbf{x}_{j+1} = [(\mathbf{M}(\mathscr{E}))^{-1}\mathscr{F}\mathbf{x}_j^{m-1} + (\mathbf{M}(\mathscr{E}))^{-1}\mathbf{b}]^{[\frac{1}{m-1}]}, \; j = 0, 1, \dots. \tag{5}$$

$(\mathbf{M}(\mathscr{E}))^{-1}\mathscr{F}$ is called the iterative tensor of the splitting method (5). Taking $\mathscr{A} = \mathscr{D} - \mathscr{L} - \mathscr{F}$, Liu et al. in [23], considered $\mathscr{E} = \mathscr{D}$, $\mathscr{E} = \mathscr{D} - \mathscr{L}$ and $\mathscr{E} = \frac{1}{\tau}(\mathscr{D} - \tau\mathscr{L})$,for the Jacobian, the Gauss-Seidel and the SOR iterative methods, respectively, where $\mathscr{D} = \mathbf{D}\mathscr{I}_m$ and $\mathscr{L} = \mathbf{L}\mathscr{I}_m$, where $\mathbf{D}$ and $\mathbf{L}$ are the positive diagonal matrix and the strictly lower triangle nonnegative matrix, respectively. Without loss of generality, we always assume that $a_{ii\ldots i}=1,\ i=1,2,\ldots,n$. Consider the splitting of $\mathscr{A}=\mathscr{I}-\mathscr{L}-\mathscr{F}$, where $\mathscr{L}=\mathbf{L}\mathscr{I}_m$ and $\mathbf{L}$ is the strictly lower triangle part of $\mathbf{M}(\mathscr{A})$.

Using iterative methods for solving (1) may have a poor convergence or even fail to converge. To overcome this problem, it is efficient to apply these methods which combine preconditioning techniques. These iterative methods usually involve some matrices that transform the iterative tensor $(\mathbf{M}(\mathscr{E}))^{-1}\mathscr{F}$ into a favorable tensor. The transformation matrices are called preconditioners. Li et al., in [21], considered the preconditioner $\mathbf{P}_{\boldsymbol{\alpha}}=\mathbf{I}+\mathbf{S}_{\boldsymbol{\alpha}}$ for solving preconditioned multilinear system

$$\mathbf{P}_{\boldsymbol{\alpha}}\mathscr{A}\mathbf{x}^{m-1}=\mathbf{P}_{\boldsymbol{\alpha}}\mathbf{b},$$

with

$$\mathbf{S}_{\boldsymbol{\alpha}}=\begin{bmatrix} 0 & -\alpha_1 a_{12\ldots 2} & 0 & \ldots & 0 \\ 0 & 0 & -\alpha_2 a_{23\ldots 3} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -\alpha_{n-1}a_{n-1,n\ldots n} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In [24], Liu et al. considered the preconditioned SOR method for solving multilinear systems with preconditioner $\mathbf{P}_{\boldsymbol{\beta}}=\mathbf{I}+\mathbf{C}_{\boldsymbol{\beta}}$ where

$$\mathbf{C}_{\boldsymbol{\beta}}=\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ -\beta_1 a_{21\ldots 1} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\beta_{n-2}a_{(n-1)1\ldots 1} & 0 & 0 & \ldots & 0 \\ -\beta_{n-1}a_{n1\ldots 1} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Herein, we consider new preconditioners $\mathbf{P}_{\boldsymbol{\alpha\beta}}(s,k)=\mathbf{D}+\mathbf{S}^s_{\boldsymbol{\alpha}}+\mathbf{K}^k_{\boldsymbol{\beta}}$, where $1\leq s,k\leq n-1$, $\mathbf{D}$ is the diagonal part of majorization of $\mathscr{A}$ and $\mathbf{S}^s_{\boldsymbol{\alpha}}$, $\mathbf{K}^k_{\boldsymbol{\beta}}$ are square matrices with elements equal to zero except the sth upper and kth lower diagonals, respectively, i.e.

$$\mathbf{S}^s_{\boldsymbol{\alpha}}=\begin{bmatrix} 0 & \ldots & 0 & -\alpha_1 a_{1(1+s)\ldots(1+s)} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & -\alpha_2 a_{2(2+s)\ldots(2+s)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & -\alpha_{n-s}a_{n-s,n\ldots n} \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix},$$

$$\mathbf{K}^k_{\boldsymbol{\beta}}=\begin{bmatrix} 0 & 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \ldots & 0 \\ -\beta_{k+1}a_{(k+1)1\ldots 1} & 0 & \ldots & 0 & \ldots & 0 \\ 0 & -\beta_{k+2}a_{(k+2)2\ldots 2} & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & -\beta_n a_{n(n-k)\ldots(n-k)} & 0 & \ldots & 0 \end{bmatrix}.$$

Applying $\mathbf{P}_{\alpha\beta}(s,k)$ on the left side of Equation (1), we get the new preconditioned multilinear system

$$\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} = \mathbf{b}_{\alpha\beta}(s,k), \tag{6}$$

where $\mathscr{A}_{\alpha\beta}(s,k) = \mathbf{P}_{\alpha\beta}(s,k)\mathscr{A}$ and $\mathbf{b}_{\alpha\beta}(s,k) = \mathbf{P}_{\alpha\beta}(s,k)\mathbf{b}$.

Recently, F.P. Ali beik et al. [1] proposed a class of preconditioners in the form $\tilde{P} = I + \tilde{S}$ where

$$\tilde{S} = (\tilde{S}_{ij}) = \begin{cases} -\alpha_{ij}a_{ij\dots j} & i \neq j, \\ 0 & i = j, \end{cases}$$

and the parameter $\alpha_{ij} \in [0,1]$ is given for $i,j = 1,2,...,n$. Note that, for suitable choices of parameters, the matrix $\tilde{S}$ reduces to $\mathbf{S}_{\alpha}^{s} + \mathbf{K}_{\beta}^{k}$. Therefore, $\mathbf{P}_{\alpha\beta}(s,k)$ is a special case of $\tilde{P}$.

However, by using the new preconditioner $\mathbf{P}_{\alpha\beta}(s,k)$, we propose and establish the comparison results between the spectra radii of several Jacobi-, Gauss-Seidel- and SOR-type splittings of the preconditioned multilinear system

$$\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} = \mathbf{b}_{\alpha\beta}(s,k),$$

where $\mathscr{A}_{\alpha\beta}(s,k) = \mathbf{P}_{\alpha\beta}(s,k)\mathscr{A}$ and $\mathbf{b}_{\alpha\beta}(s,k) = \mathbf{P}_{\alpha\beta}(s,k)\mathbf{b}$. Hence, the differences between this paper and [2] lie in the assumptions used to establish the comparison results between the spectra radii of splittings given in Section 3.

Remark 1. *We denote $\mathscr{A}_{\alpha}(s) = \mathbf{P}_{\alpha}(s)\mathscr{A}$ and $\mathscr{A}_{\beta}(k) = \mathbf{P}_{\beta}(k)\mathscr{A}$, where $\mathbf{P}_{\alpha}(s) = \mathbf{D} + \mathbf{S}_{\alpha}^{s}$ and $\mathbf{P}_{\beta}(k) = \mathbf{D} + \mathbf{K}_{\beta}^{k}$, respectively.*

Proposition 1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a $\mathscr{Z}$-tensor. If $\mathscr{A}$ is a strong $\mathscr{M}$-tensor, then for any $\alpha_i, \beta_j \in [0,1],\ i = 1,\dots,n-s,\ j = k+1,\dots,n$, $\mathscr{A}_{\alpha\beta}(s,k)$ is a strong $\mathscr{M}$-tensor.*

Proof. Without loss of generality, we assume that $s = k = 1$. Let $\mathscr{A}_{\alpha\beta}(s,k) = \mathbf{P}_{\alpha\beta}(s,k)\mathscr{A} = (\hat{a}_{i_1 i_2\dots i_m})$. Then for $1 \leq i_2,...,i_m \leq n$, we have

$$\hat{a}_{ji_2\dots i_m} = \begin{cases} a_{1i_2\dots i_m} - \alpha_1 a_{12\dots 2}a_{2i_2\dots i_m}, & j = 1 \\ a_{ji_2\dots i_m} - \beta_j a_{j(j-1)\dots(j-1)}a_{(j-1)i_2\dots i_m} - \alpha_j a_{j(j+1)\dots(j+1)}a_{(j+1)i_2\dots i_m}, & 2 \leq j \leq n-1 \\ a_{ni_2\dots i_m} - \beta_n a_{n(n-1)\dots(n-1)}a_{(n-1)i_2\dots i_m}, & j = n. \end{cases}$$

For $(j,i_2,...,i_m) \neq (j,j,\dots,j)$ and $\alpha_i, \beta_j \in [0,1],\ i = 1,\dots,n-s,\ j = k+1,\dots,n$, we have $\hat{a}_{ji_2\dots i_m} \leq 0$, i.e. $\mathscr{A}_{\alpha\beta}(s,k)$ is a $\mathscr{Z}$-tensor. According to Lemma 2, there exists a vector $\mathbf{x} > \mathbf{0}$ such that $\mathscr{A}\mathbf{x}^{m-1} > \mathbf{0}$. We also have $\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} = (\mathbf{D} + \mathbf{S}_{\alpha}^{s} + \mathbf{K}_{\beta}^{k})\mathscr{A}\mathbf{x}^{m-1} = \mathbf{D}\mathscr{A}\mathbf{x}^{m-1} + \mathbf{S}_{\alpha}^{s}\mathscr{A}\mathbf{x}^{m-1} + \mathbf{K}_{\beta}^{k}\mathscr{A}\mathbf{x}^{m-1} > \mathbf{0}$. Thus by Lemma 2, $\mathscr{A}_{\alpha\beta}(s,k)$ is a strong $\mathscr{M}$-tensor. □

Since $\mathbf{b}_{\alpha\beta}(s,k) \geq \mathbf{b} > \mathbf{0}$ for any $\alpha_i, \beta_j \in [0,1],\ i = 1,\dots,n-s,\ j = k+1,\dots,n$, by Lemma 3 and Proposition 1, the following proposition is easily proved.

Proposition 2. *The preconditioned multilinear system (6) has the same unique positive solution with multilinear system (1).*

Remark 2. *Let in the multilinear system (1), $\mathscr{A}$ be an $\mathscr{M}$-tensor. We can write $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$. Also, from now on, if no other special illustration, we suppose that $\mathscr{A}$ is an $\mathscr{M}$-tensor.*

3 The preconditioned Jacobi-, Gauss-Seidel- and SOR-type iteration schemes

3.1 The preconditioned Jacobi-type iterative schemes

We consider the following five Jacobi-type splittings

$$\begin{aligned}
\mathscr{A}_{\alpha\beta}(s,k) &= \mathbf{P}_{\alpha\beta}(s,k)\mathscr{I}_m - \mathbf{P}_{\alpha\beta}(s,k)(\mathscr{L}+\mathscr{F}) = \mathscr{E}_1 - \mathscr{F}_1,\\
\mathscr{A}_{\alpha\beta}(s,k) &= \mathscr{I}_m - (\mathbf{P}_{\alpha\beta}(s,k)(\mathscr{L}+\mathscr{F}) - (\mathbf{S}^s_{\alpha}+\mathbf{K}^k_{\beta})\mathscr{I}_m) = \mathscr{E}_2 - \mathscr{F}_2,\\
\mathscr{A}_{\alpha}(s) &= \mathscr{I}_m - (\mathbf{P}_{\alpha}(s)(\mathscr{L}+\mathscr{F}) - \mathbf{S}^s_{\alpha}\mathscr{I}_m) = \mathscr{E}_3 - \mathscr{F}_3,\\
\mathscr{A}_{\beta}(k) &= \mathscr{I}_m - (\mathbf{P}_{\beta}(\mathscr{L}+\mathscr{F}) - \mathbf{K}^k_{\beta}\mathscr{I}_m) = \mathscr{E}_4 - \mathscr{F}_4.
\end{aligned}$$

Remark 3. *The splitting $\mathscr{A}_{\alpha}(s) = \mathscr{E}_3 - \mathscr{F}_3$, where $s = 1$, is the same as the splitting in [21].*

Remark 4. *When $s = k = 1$, we denote $\mathbf{K}^1_{\beta}$ by $\mathbf{K}_{\beta}$ and $\mathbf{S}^1_{\alpha}$ by $\mathbf{S}_{\alpha}$. Also denote $\mathbf{K}_{\beta}$ by $\mathbf{K}$ and $\mathbf{S}_{\alpha}$ by $\mathbf{S}$ for all $\alpha_i = \beta_j = 1,\ i = 1,2,\dots,n-1,\ j = 2,3,\dots,n$. Let $\mathscr{L} = \mathbf{K}\mathscr{I}_m + \mathscr{L}'$ and $\mathscr{F} = \mathbf{S}\mathscr{I}_m + \mathscr{F}'$. Thus we have the following Jacobi-type splitting*

$$\begin{aligned}
\mathscr{A}_{\alpha\beta}(1,1) &= (\mathbf{I} - \mathbf{S}_{\alpha}\mathbf{K} - \mathbf{K}_{\beta}\mathbf{S})\mathscr{I}_m - [\mathscr{L}+\mathscr{F} - (\mathbf{S}_{\alpha}+\mathbf{K}_{\beta})\mathscr{I}_m + \mathbf{S}_{\alpha}(\mathscr{L}'+\mathscr{F}) + \mathbf{K}_{\beta}(\mathscr{L}+\mathscr{F}')]\\
&= \mathscr{E}_5 - \mathscr{F}_5.
\end{aligned}$$

Theorem 1. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. Then for every $\beta_j \in [0,1],\ j = k+1,\dots,n$ and $\alpha_i \in [0,1],\ i = 1,\dots,n-s$, $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_1 - \mathscr{F}_1 = \mathscr{E}_2 - \mathscr{F}_2$, $\mathscr{A}_{\alpha}(s) = \mathscr{E}_3 - \mathscr{F}_3$ and $\mathscr{A}_{\beta}(k) = \mathscr{E}_4 - \mathscr{F}_4$ are convergent. Moreover if*

$$\begin{cases}
0 < \alpha_1 a_{12\dots2}a_{21\dots1} < 1,\\
0 < \alpha_i a_{i(i+1)\dots(i+1)}a_{(i+1)i\dots i} + \beta_i a_{i(i-1)\dots(i-1)}a_{(i-1)i\dots i} < 1,\ i = 2,\dots,n-1,\\
0 < \beta_n a_{n(n-1)\dots(n-1)}a_{(n-1)n\dots n} < 1,
\end{cases} \tag{7}$$

then the tensor splitting $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_5 - \mathscr{F}_5$ is convergent.

Proof. Suppose $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_1 - \mathscr{F}_1$. Since $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$ is a strong $\mathscr{M}$-tensor, then $\rho(\mathscr{L}+\mathscr{F}) < 1$. Thus $\rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1) = \rho(\mathscr{L}+\mathscr{F}) < 1$. Hence $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_1 - \mathscr{F}_1$ is a convergent splitting. Let $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_2 - \mathscr{F}_2$. We have $(\mathbf{M}(\mathscr{E}_1))^{-1} = \mathbf{I} \geq \mathbf{O}$ and since $\alpha_i, \beta_j \in [0,1]$, it is easy to see that $\mathscr{F}_2 \geq \mathscr{O}$. Thus $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_2 - \mathscr{F}_2$ is a regular splitting. By Proposition 1, $\mathscr{A}_{\alpha\beta}(s,k)$ is a strong $\mathscr{M}$-tensor and using Lemma 2, $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_2 - \mathscr{F}_2$ is a convergent regular splitting. For $\mathscr{A}_{\alpha}(s) = \mathscr{E}_3 - \mathscr{F}_3$ and $\mathscr{A}_{\beta}(k) = \mathscr{E}_4 - \mathscr{F}_4$, the proof is similar to the proof of the case $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_2 - \mathscr{F}_2$. Suppose that $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_5 - \mathscr{F}_5$ and equation (7) holds. Thus $(\mathbf{M}(\mathscr{E}_5))^{-1}$ exists and

$$(\mathbf{M}(\mathscr{E}_5))^{-1}_{ii} = \begin{cases}
(1 - \alpha_1 a_{12\dots2}a_{21\dots1})^{-1}, & i = 1,\\
(1 - \alpha_i a_{i(i+1)\dots(i+1)}a_{(i+1)i\dots i} - \beta_i a_{i(i-1)\dots(i-1)}a_{(i-1)i\dots i})^{-1}, & i = 2,\dots,n-1,\\
(1 - \beta_n a_{n(n-1)\dots(n-1)}a_{(n-1)n\dots n})^{-1}, & i = n,
\end{cases} \tag{8}$$

which implies that $(\mathbf{M}(\mathscr{E}_5))^{-1} \geq \mathbf{O}$. It is not difficult to see that $\mathscr{F}_5 = \mathscr{E}_5 - \mathscr{A}_{\alpha\beta}(s,k) \geq \mathscr{O}$. Using Proposition 1, $\mathscr{A}_{\alpha\beta}(s,k)$ is a strong $\mathscr{M}$-tensor and from Lemma 2, $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_5 - \mathscr{F}_5$ is a convergent regular splitting. □

Theorem 2. *Let $\mathscr{A}$ be a strong $\mathscr{M}$-tensor and equation (7) holds. Then the following relations hold.*

1. $\exists\, \mathbf{x}_1 \in \mathbb{R}^n_+$, $((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}\mathbf{x}_1^{m-1} \leq ((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{m-1}$.
2. $\exists\, \mathbf{x}_2 \in \mathbb{R}^n_+$, $\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}_2^{m-1} \geq \mathbf{0}$.
3. $\rho(((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta})$.

Proof. Case 1. Since $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$ is a strong $\mathscr{M}$-tensor, $\rho((\mathbf{M}(\mathscr{E}_1))^{-1}(\mathscr{F}_1)) = \rho(\mathscr{L}+\mathscr{F}) < 1$. Thus, for the nonnegative Jacobi iteration tensor $(\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1 = \mathscr{L}+\mathscr{F}$ and by the Perron-Frobneius theorm, there exists a nonnegative vector $\mathbf{x}_1$ such that $(\mathbf{M}(\mathscr{E}_1))^{-1}(\mathscr{F}_1)\mathbf{x}_1^{m-1} = \rho(\mathscr{F}_1)\mathbf{x}_1^{[m-1]}$. Thus, we have

$$\begin{aligned}
((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}\mathbf{x}_1^{m-1} &= (\mathbf{P}_{\alpha\beta}(s,k)(\mathscr{L}+\mathscr{F}) - (\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)\mathscr{I}_m)\mathbf{x}_1^{m-1}\\
&= (\mathbf{I}+\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)(\mathscr{L}+\mathscr{F})\mathbf{x}_1^{m-1} - (\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)\mathscr{I}_m\mathbf{x}_1^{m-1}\\
&= (\mathscr{L}+\mathscr{F})\mathbf{x}_1^{m-1} + (\mathbf{K}^k_\beta + \mathbf{S}^s_\alpha)(\mathscr{L}+\mathscr{F})\mathbf{x}_1^{m-1} - (\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)\mathscr{I}_m\mathbf{x}_1^{m-1}\\
&= ((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{m-1} - (\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)(\mathscr{I}_m - (\mathscr{L}+\mathscr{F}))\mathbf{x}_1^{m-1}\\
&= ((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{m-1} - (\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)(1-\rho((\mathbf{M}(\mathscr{E}_1))^{-1})\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{[m-1]}.
\end{aligned}$$

This results in

$$((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}\mathbf{x}_1^{m-1} - ((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{m-1} = -(\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta)(1-\rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1))_{\alpha\beta}\mathbf{x}_1^{[m-1]} \leq \mathbf{0},$$

due to $\mathbf{S}^s_\alpha + \mathbf{K}^k_\beta \geq \mathbf{O}$ and $0 < \rho((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1) < 1$.

Case 2. By Theorem 1, we know that $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{E}_5 - \mathscr{F}_5$ is convergent, i.e. $0 < \rho((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5) < 1$ and thus, for the nonnegative Jacobi iteration tensor $(\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5$ and by the Perron-Frobenius theorem, there exists a nonnegative vector $\mathbf{x}_2$ such that $(\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5\mathbf{x}_2^{m-1} = \rho((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)\mathbf{x}_2^{[m-1]}$. Therefore we have

$$\begin{aligned}
\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}_2^{m-1} &= \mathscr{E}_5\mathbf{x}_2^{m-1} - \mathscr{F}_5\mathbf{x}_2^{m-1}\\
&= \mathscr{E}_5\mathbf{x}_2^{m-1} - \mathbf{M}(\mathscr{E}_5)(\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5\mathbf{x}_2^{m-1}\\
&= (\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})\mathbf{x}_2^{[m-1]} - \rho((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)(\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})\mathscr{I}_m\mathbf{x}_2^{m-1}\\
&= (\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})\mathbf{x}_2^{[m-1]} - \rho((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)(\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})\mathbf{x}_2^{[m-1]}\\
&= (1-\rho((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5))(\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})\mathbf{x}_2^{[m-1]} \geq \mathbf{0}.
\end{aligned}$$

Case 3. Since $(\mathbf{M}(\mathscr{E}_2)_{\alpha\beta})^{-1} = \mathbf{I}$ and $(\mathbf{M}(\mathscr{E}_5)_{\alpha\beta})^{-1} = (\mathbf{I}-\mathbf{S}_\alpha\mathbf{K}-\mathbf{K}_\beta\mathbf{S})^{-1}$, thus

$$(\mathbf{M}(\mathscr{E}_5)_{\alpha\beta})^{-1} \geq (\mathbf{M}(\mathscr{E}_2)_{\alpha\beta})^{-1}.$$

Let $(\rho(((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\alpha\beta}),\mathbf{x})$ be a Perron eigenpair of $((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\alpha\beta}$, then by part 2, we have $\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} \geq \mathbf{0}$ and by Lemma 5, we have $\rho(((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta})$. Now suppose that $\mathbf{x}$ is a nonnegative Perron vector of $((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}$, then by part 1, we have

$$((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}\mathbf{x}_1^{m-1} \leq ((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta}\mathbf{x}_1^{m-1} = \rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta})\mathbf{x}_1^{[m-1]}.$$

Since $((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta} \geq \mathscr{O}$, then we have $\rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\alpha\beta})$, and the proof is completed. □

Remark 5. *It is easy to see that for every Perron vector* $\mathbf{x}$ *of nonnegative Jacobi iteration tensor of convergence splitting method, we have,* $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k)\mathbf{x}^{m-1} \geq \mathbf{0}$.

Proposition 3. *[33] Let* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *be a strong* $\mathscr{M}$*-tensor. If equation (7) holds for any* $\beta_{1,j},\ \beta_{2,j} \in [0,1],\ j = k+1,\ldots,n,\ \alpha_{1,i}, \alpha_{2,i} \in [0,1],\ i = 1,\ldots,n-s,\ \boldsymbol{\alpha}' = (\alpha_{1,i}),\ \boldsymbol{\alpha}'' = (\alpha_{2,i}),\ \beta' = (\beta_{1,j}),\ \beta'' = (\beta_{2,j})$ *and* $\boldsymbol{\alpha}' \geq \boldsymbol{\alpha}'',\ \beta' \geq \beta''$*, then we have*

1. $\rho(((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\boldsymbol{\alpha}'\beta'}) \leq \rho(((\mathbf{M}(\mathscr{E}_1))^{-1}\mathscr{F}_1)_{\boldsymbol{\alpha}''\beta''})$;
2. $\rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\boldsymbol{\alpha}'\beta'}) \leq \rho(((\mathbf{M}(\mathscr{E}_2))^{-1}\mathscr{F}_2)_{\boldsymbol{\alpha}''\beta''})$;
3. $\rho(((\mathbf{M}(\mathscr{E}_3))^{-1}\mathscr{F}_3)_{\boldsymbol{\alpha}'}) \leq \rho(((\mathbf{M}(\mathscr{E}_3))^{-1}\mathscr{F}_3)_{\boldsymbol{\alpha}''})$;
4. $\rho(((\mathbf{M}(\mathscr{E}_4))^{-1}\mathscr{F}_4)_{\beta'}) \leq \rho(((\mathbf{M}(\mathscr{E}_4))^{-1}\mathscr{F}_4)_{\beta''})$;
5. $\rho(((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\boldsymbol{\alpha}'\beta'}) \leq \rho(((\mathbf{M}(\mathscr{E}_5))^{-1}\mathscr{F}_5)_{\boldsymbol{\alpha}''\beta''})$.

3.2 Gauss-Seidel-type iterative schemes

We consider the following four Gauss-Seidel-type splittings:

$$\begin{aligned}
\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) &= \mathbf{P}_{\boldsymbol{\alpha\beta}}(s,k)(\mathscr{I}_m - \mathscr{L}) - \mathbf{P}_{\boldsymbol{\alpha\beta}}(s,k)\mathscr{F} = \mathscr{M}_1 - \mathscr{N}_1,\\
\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) &= (\mathscr{I}_m - \mathscr{L} + \mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{I}_m - \mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{L} - \mathscr{D}_{\boldsymbol{\alpha}} - \mathscr{L}_{\boldsymbol{\alpha}} - \mathscr{D}_{\boldsymbol{\beta}} - \mathscr{L}_{\boldsymbol{\beta}}) - (\mathscr{F} - \mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{I}_m + \mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{F} + \mathscr{F}_{\boldsymbol{\alpha}} + \mathscr{F}_{\boldsymbol{\beta}})\\
&= \mathscr{M}_2 - \mathscr{N}_2,\\
\mathscr{A}_{\boldsymbol{\alpha}}(s) &= (\mathscr{I}_m - \mathscr{L} - \mathscr{D}_{\boldsymbol{\alpha}} - \mathscr{L}_{\boldsymbol{\alpha}}) - (\mathscr{F} - \mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{I}_m + \mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{F} + \mathscr{F}_{\boldsymbol{\alpha}}) = \mathscr{M}_3 - \mathscr{N}_3,\\
\mathscr{A}_{\boldsymbol{\beta}}(k) &= ((\mathbf{I} + \mathbf{K}^k_{\boldsymbol{\beta}})(\mathscr{I}_m - \mathscr{L}) - \mathscr{D}_{\boldsymbol{\beta}} - \mathscr{L}_{\boldsymbol{\beta}}) - (\mathscr{F} + \mathscr{F}_{\boldsymbol{\beta}}) = \mathscr{M}_4 - \mathscr{N}_4,
\end{aligned}$$

where $\mathscr{D}_{\boldsymbol{\alpha}} = \mathbf{D}_{\boldsymbol{\alpha}}\mathscr{I}_m,\ \mathscr{L}_{\boldsymbol{\alpha}} = \mathbf{L}_{\boldsymbol{\alpha}}\mathscr{I}_m,\ \mathscr{D}_{\boldsymbol{\beta}} = \mathbf{D}_{\boldsymbol{\beta}}\mathscr{I}_m,\ \mathscr{L}_{\boldsymbol{\beta}} = \mathbf{L}_{\boldsymbol{\beta}}\mathscr{I}_m$, and $\mathbf{D}_{\boldsymbol{\alpha}},\ \mathbf{D}_{\boldsymbol{\beta}}, \mathbf{L}_{\boldsymbol{\alpha}},\ \mathbf{L}_{\boldsymbol{\beta}}$ are the diagonal parts and the strictly lower triangle parts of $\mathbf{M}(\mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{L})$ and $\mathbf{M}(\mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{F})$, respectively, i.e.

$$\mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{L} = \mathscr{D}_{\boldsymbol{\alpha}} + \mathscr{L}_{\boldsymbol{\alpha}} + \mathscr{F}_{\boldsymbol{\alpha}},\ \mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{F} = \mathscr{D}_{\boldsymbol{\beta}} + \mathscr{L}_{\boldsymbol{\beta}} + \mathscr{F}_{\boldsymbol{\beta}}.$$

Remark 6. *Splitting* $\mathscr{A}_{\boldsymbol{\alpha}}(1) = \mathscr{M}_3 - \mathscr{N}_3$*, is the same as the splitting in [21].*

Remark 7. *If* $k = s = 1$*, similar to Remark 4, we have*

$$\mathscr{A}_{\boldsymbol{\alpha\beta}}(1,1) \quad = \quad ((\mathbf{I} + \mathbf{K}_{\boldsymbol{\beta}})(\mathscr{I}_m - \mathscr{L}) - \mathbf{S}_{\boldsymbol{\alpha}}\mathscr{L} - \mathbf{K}_{\boldsymbol{\beta}}\mathbf{S}\mathscr{I}_m) - ((\mathbf{I} + \mathbf{S}_{\boldsymbol{\alpha}})\mathscr{F} - \mathbf{S}_{\boldsymbol{\alpha}}\mathscr{I}_m + \mathbf{K}_{\boldsymbol{\beta}}\mathscr{F}') = \mathscr{M}_5 - \mathscr{N}_5.$$

Theorem 3. *Let* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *be a strong* $\mathscr{M}$*-tensor. Then for any* $\beta_j \in [0,1],\ j = k+1,\ldots,n$ *and* $\alpha_i \in [0,1],\ i = 1,\ldots,n-s,\ \mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) = \mathscr{M}_1 - \mathscr{N}_1$ *is convergent. Also,*
when $k < s$ *if*

$$\begin{cases}
0 < \alpha_i a_{i(n-i)\ldots(n-i)} a_{(n-i)i\ldots i} < 1,\ i = 1,2,\ldots,k,\\
0 < \alpha_i a_{i(n-i)\ldots(n-i)} a_{(n-i)i\ldots i} + \beta_i a_{i(i-k)\ldots(i-k)} a_{(i-k)i\ldots i} < 1, i = k+1,\ldots,s,\\
0 < \beta_i a_{i(i-k)\ldots(i-k)} a_{(i-k)i\ldots i} < 1, i = s+1,\ldots,n,
\end{cases} \tag{9}$$

for $k > s$ if

$$\begin{cases} 0 < \alpha_i a_{i(n-i)\dots(n-i)} a_{(n-i)i\dots i} < 1, i = 1,2,\dots,s, \\ 0 < \alpha_i a_{i(n-i)\dots(n-i)} a_{(n-i)i\dots i} + \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i} < 1, i = s+1,\dots,k, \\ 0 < \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i} < 1, i = k+1,\dots,n, \end{cases} \tag{10}$$

and for $k = s$ if

$$\begin{cases} 0 < \alpha_i a_{i(i+k)\dots(i+k)} a_{(i+k)i\dots i} < 1, i = 1,2,\dots,k, \\ 0 < \alpha_i a_{i(i+k)\dots(i+k)} a_{(i+k)i\dots i} + \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i} < 1, i = k+1,\dots,n-k, \\ 0 < \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i} < 1, i = n-k+1,\dots,n. \end{cases} \tag{11}$$

hold, then the tensor splitting $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) = \mathscr{M}_2 - \mathscr{N}_2$ is convergent. Besides, if

$$0 < \alpha_i a_{i(n-i)\dots(n-i)} a_{(n-i)i\dots i} < 1,\ i = 1,2,\dots,s,$$

holds, then $\mathscr{A}_{\boldsymbol{\alpha}}(s) = \mathscr{M}_3 - \mathscr{N}_3$ is convergent. Finally, if

$$0 < \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i} < 1,\ i = k+1,\dots,n,$$

holds, then $\mathscr{A}_{\boldsymbol{\beta}}(k) = \mathscr{M}_4 - \mathscr{N}_4$ is convergent.

Proof. Let $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) = \mathscr{M}_1 - \mathscr{N}_1$. Due to Proposition 1, $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k)$ is a strong $\mathscr{M}$-tensor, and $\mathscr{N}_1 \geq \mathscr{O}$. Since

$$(\mathbf{M}(\mathscr{M}_1))^{-1}\mathscr{N}_1 = (\mathbf{I}-\mathbf{L})^{-1}(\mathbf{P}_{\boldsymbol{\alpha\beta}}(s,k))^{-1}\mathbf{P}_{\boldsymbol{\alpha\beta}}(s,k)\mathscr{N}_1 \geq \mathscr{O},$$

then, $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) = \mathscr{M}_1 - \mathscr{N}_1$ is a weak regular splitting and, using Lemma 2, it is convergent. Suppose that $\mathscr{A}_{\boldsymbol{\alpha\beta}}(s,k) = \mathscr{M}_2 - \mathscr{N}_2$ and $k = s$. Since $\mathscr{M}_2 = \mathscr{I}_m - \mathscr{L} + \mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{I}_m - \mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{L} - \mathscr{D}_{\boldsymbol{\alpha}} - \mathscr{L}_{\boldsymbol{\alpha}} - \mathscr{D}_{\boldsymbol{\beta}} - \mathscr{L}_{\boldsymbol{\beta}}$, then we have

$$\mathbf{M}(\mathscr{M}_2) = \mathbf{I} - \mathbf{D}_{\boldsymbol{\alpha}} - \mathbf{D}_{\boldsymbol{\beta}} - \mathbf{L} + \mathbf{K}^k_{\boldsymbol{\beta}} - \mathbf{K}^k_{\boldsymbol{\beta}}\mathbf{L} - \mathbf{L}_{\boldsymbol{\alpha}} - \mathbf{L}_{\boldsymbol{\beta}}.$$

Notice that $\mathbf{D}_{\boldsymbol{\alpha}}$ and $\mathbf{D}_{\boldsymbol{\beta}}$ are diagonal part of $\mathbf{M}(\mathbf{S}^s_{\boldsymbol{\alpha}}\mathscr{L})$ and $\mathbf{M}(\mathbf{K}^k_{\boldsymbol{\beta}}\mathscr{F})$, respectively. It is not difficult to see that

$$(\mathbf{I} - \mathbf{D}_{\boldsymbol{\alpha}} - \mathbf{D}_{\boldsymbol{\beta}})_{ii} = \begin{cases} 1 - \alpha_i a_{i(i+k)\dots(i+k)} a_{(i+k)i\dots i}, & i = 1,2,\dots,k, \\ 1 - \alpha_i a_{i(i+k)\dots(i+k)} a_{(i+k)i\dots i} - \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i}, & i = k+1,\dots,n-k, \\ 1 - \beta_i a_{i(i-k)\dots(i-k)} a_{(i-k)i\dots i}, & i = n-k+1,\dots,n. \end{cases} \tag{12}$$

Since equation (11) holds, $(\mathbf{I} - \mathbf{D}_{\boldsymbol{\alpha}} - \mathbf{D}_{\boldsymbol{\beta}})^{-1}$ exists and $(\mathbf{I} - (\mathbf{D}_{\boldsymbol{\alpha}} + \mathbf{D}_{\boldsymbol{\beta}}))^{-1} = \mathbf{I} + (\mathbf{D}_{\boldsymbol{\alpha}} + \mathbf{D}_{\boldsymbol{\beta}}) + \dots + (\mathbf{D}_{\boldsymbol{\alpha}} + \mathbf{D}_{\boldsymbol{\beta}})^{n-1} + \dots \geq \mathbf{I}$. By taking $\mathbf{H} := \mathbf{L} + \mathbf{L}_{\boldsymbol{\alpha}} + \mathbf{L}_{\boldsymbol{\beta}} - \mathbf{K}^k_{\boldsymbol{\beta}} + \mathbf{K}^k_{\boldsymbol{\beta}}\mathbf{L}$, it is easy to show that $\mathbf{H}$ is a lower triangular matrix. To prove $\mathbf{H} \geq \mathbf{O}$, it is sufcient to show that $(\mathbf{L} - \mathbf{K}^k_{\boldsymbol{\beta}})_{i+1,i} \geq 0$ for any $i = 1,\dots,n-1$. Actually, this is shown by

$$(\mathbf{L} - \mathbf{K}^k_{\boldsymbol{\beta}})_{i+1,i} = -a_{i+1,i\dots i} - (-\beta_{i+1} a_{i+1,i\dots i}) = a_{i+1,i\dots i}(\beta_{i+1} - 1) \geq 0.$$

By the Neumanns series [26], we have

$$\begin{aligned}(\mathbf{M}(\mathscr{M}_2))^{-1} &= [(\mathbf{I}-\mathbf{D}_\alpha-\mathbf{D}_\beta)-\mathbf{H}]^{-1}\\ &= [\mathbf{I}-(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}]^{-1}(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\\ &= \{\mathbf{I}+(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}+[(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta)^{-1}\mathbf{H}]^2+\dots\\ &\quad +[(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}]^{n-1}+\dots\}(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\\ &\geq \mathbf{O}.\end{aligned}$$

Since $\mathscr{N}_2 \geq \mathscr{O}$ (like what was said in the proof $\mathbf{H} \geq \mathbf{O}$), $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{M}_2 - \mathscr{N}_2$ is a weak regular splitting and, using Lemma 2, it is convergent. A similar proof can be used for the cases $k < s$ and $k > s$. $\mathscr{A}_\alpha(s) = \mathscr{M}_3 - \mathscr{N}_3$ and $\mathscr{A}_\beta(k) = \mathscr{M}_4 - \mathscr{N}_4$ can be proved similarly. □

Proposition 4. *Let $\mathscr{A}$ be a strong $\mathscr{M}$-tensor and equations (9)-(11) hold. The following relations hold:*

1. $\exists\, \mathbf{x} \in \mathbb{R}^n_+, \mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} \geq \mathbf{0}$;
2. $\rho((\mathbf{M}(\mathscr{M}_2)^{-1}\mathscr{N}_2)_{\alpha\beta}) \leq \rho((\mathbf{M}(\mathscr{M}_3)^{-1}\mathscr{N}_3)_{\alpha\beta}) \leq \rho((\mathbf{M}(\mathscr{M}_1)^{-1}\mathscr{N}_1)_{\alpha\beta}) < 1$;
3. $\rho((\mathbf{M}(\mathscr{M}_2)^{-1}\mathscr{N}_2)_{\alpha\beta}) \leq \rho((\mathbf{M}(\mathscr{M}_4)^{-1}\mathscr{N}_4)_{\alpha\beta}) \leq \rho((\mathbf{M}(\mathscr{M}_1)^{-1}\mathscr{N}_1)_{\alpha\beta}) < 1$.

Proof. Case 1. $\mathscr{A}_{\alpha\beta}(s,k)$ is a strong $\mathscr{M}$-tensor by Proposition 1. Using Lemma 2, there exists $\mathbf{x} \in \mathbb{R}^n_+$ such that $\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} \geq \mathbf{0}$.
Case 2. From Theorem 3, $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{M}_2 - \mathscr{N}_2$ and $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{M}_3 - \mathscr{N}_3$ are two weak regular splitting. By taking $\mathbf{H}' := \mathbf{L} + \mathbf{L}_\alpha \geq \mathbf{O}$ and using Neumanns series, we have

$$\begin{aligned}(\mathbf{M}(\mathscr{M}_2))^{-1} &= [(\mathbf{I}-\mathbf{D}_\alpha-\mathbf{D}_\beta)-\mathbf{H}]^{-1}\\ &= [\mathbf{I}-(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}]^{-1}(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\\ &= \{\mathbf{I}+(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}+[(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta)^{-1}\mathbf{H}]^2\\ &\quad +\dots+[(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\mathbf{H}]^{n-1}+\dots\}(\mathbf{I}-(\mathbf{D}_\alpha+\mathbf{D}_\beta))^{-1}\\ &\geq \{\mathbf{I}+(\mathbf{I}-\mathbf{D}_\alpha)^{-1}\mathbf{H}'+[(\mathbf{I}-\mathbf{D}_\alpha)^{-1}\mathbf{H}']^2\\ &\quad +\dots+[(\mathbf{I}-\mathbf{D}_\alpha)^{-1}\mathbf{H}']^{n-1}+\dots\}(\mathbf{I}-\mathbf{D}_\alpha)^{-1}\\ &= [(\mathbf{I}-\mathbf{D}_\alpha)-\mathbf{H}']^{-1}\\ &= (\mathbf{M}(\mathscr{M}_3))^{-1}.\end{aligned}$$

By Theorem 3, we know that $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{M}_2 - \mathscr{N}_2$ is convergent, i.e. $0 < \rho((\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2) < 1$ and thus, for the nonnegative Gauss-Seidel iteration tensor $(\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2$, there exists a nonnegative vector $\mathbf{x}$ such that $(\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2\mathbf{x}^{m-1} = \rho((\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2)\mathbf{x}^{[m-1]}$. Using Theorem 2, we have $\mathscr{A}_{\alpha\beta}(s,k)\mathbf{x}^{m-1} \geq 0$. By Lemma 5, we have $\rho(((\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{M}_3))^{-1}\mathscr{N}_3)_{\alpha\beta})$. Similar discussion give us $\rho(((\mathbf{M}(\mathscr{M}_3))^{-1}\mathscr{N}_3)_{\alpha\beta}) \leq \rho(((\mathbf{M}(\mathscr{M}_1))^{-1}\mathscr{N}_1)_{\alpha\beta})$. According to Theorem 3 $\mathscr{A}_{\alpha\beta}(s,k) = \mathscr{M}_1 - \mathscr{N}_1$ is convergent and therefore $\rho(((\mathbf{M}(\mathscr{M}_1))^{-1}\mathscr{N}_1)_{\alpha\beta}) < 1$.
Case 3. The proof of this case is similar to the case 2. □

Proposition 5. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. If equations (9)-(11) hold for any $\beta_{1,j},\ \beta_{2,j} \in [0,1],\ j = k+1,\dots,n,\ \alpha_{1,i}, \alpha_{2,i} \in [0,1],\ i = 1,\dots,n-s,\ \alpha' = (\alpha_{1,i}),\ \alpha'' = (\alpha_{2,i})$, then*

1. $\rho(((\mathbf{M}(\mathscr{M}))^{-1}\mathscr{N}_1)_{\alpha'\beta'}) \leq \rho(((\mathbf{M}(\mathscr{M}_1))^{-1}\mathscr{N}_1)_{\alpha''\beta''})$;
2. $\rho(((\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2)_{\alpha'\beta'}) \leq \rho(((\mathbf{M}(\mathscr{M}_2))^{-1}\mathscr{N}_2)_{\alpha''\beta''})$;
3. $\rho(((\mathbf{M}(\mathscr{M}_3))^{-1}\mathscr{N}_3)_{\alpha'}) \leq \rho(((\mathbf{M}(\mathscr{M}_3))^{-1}\mathscr{N}_3)_{\alpha''})$;
4. $\rho(((\mathbf{M}(\mathscr{M}_4))^{-1}\mathscr{N}_4)_{\beta'}) \leq \rho(((\mathbf{M}(\mathscr{M}_4))^{-1}\mathscr{N}_4)_{\beta''})$.

3.3 The preconditioned SOR-type method

In [23], the SOR-type method was given by taking $\mathscr{E} = \frac{1}{\omega}(\mathscr{I}_m - \omega\mathscr{L})$ for solving equations (1) as follows

$$\mathbf{x}_{j+1} = ((\mathbf{M}(\mathscr{I}_m - \omega\mathscr{L}))^{-1}((1-\omega)\mathscr{I}_m + \omega\mathscr{F})\mathbf{x}_j^{m-1} + \omega(\mathbf{M}(\mathscr{I}_m - \omega\mathscr{L}))^{-1}\mathbf{b})^{[\frac{1}{m-1}]}.$$

In this paper, we consider the following preconditioned SOR-type method

$$\mathbf{x}_{j+1} = (\mathscr{H}_{\alpha\beta}(\omega)\mathbf{x}_j^{m-1} + \mathbf{h}_{\alpha\beta}(\omega))^{[\frac{1}{m-1}]},$$

where

$$\mathscr{H}_{\alpha\beta}(\omega) = \mathbf{M}(\mathscr{E}_{\alpha\beta}(\omega))^{-1}\mathscr{F}_{\alpha\beta}(\omega),\ \mathbf{h}_{\alpha\beta}(\omega) = \mathbf{M}(\mathscr{E}_{\alpha\beta}(\omega))^{-1}\mathbf{b}_{\alpha\beta}(s,k),$$
$$\mathscr{E}_{\alpha\beta}(\omega) = \frac{1}{\omega}(\mathscr{D}_{\alpha\beta} - \omega\mathscr{L}_{\alpha\beta}),\ \mathscr{F}_{\alpha\beta}(\omega) = \frac{1}{\omega}((1-\omega)\mathscr{D}_{\alpha\beta} + \omega\mathscr{F}_{\alpha\beta}),$$

and

$$\mathscr{D}_{\alpha\beta} = \mathscr{I}_m - \mathscr{D}_\alpha - \mathscr{D}_\beta,$$
$$\mathscr{L}_{\alpha\beta} = \mathscr{L} - \mathbf{K}_\beta^k\mathscr{I}_m + \mathbf{K}_\beta^k\mathscr{L} + \mathscr{L}_\alpha + \mathscr{L}_\beta,$$
$$\mathscr{F}_{\alpha\beta} = \mathscr{F} - \mathbf{S}_\alpha^s\mathscr{I}_m + \mathbf{S}_\alpha^s\mathscr{F} + \mathscr{F}_\alpha + \mathscr{F}_\beta.$$

Remark 8. *When $s = 1$ and $k = 0$, the new preconditioned SOR method is similar to the preconditioned SOR method which is proposed in [21].*

In the following, we present some propositions and omit their proof due to the similarity of their proof with those of in [24].

Proposition 6. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. If $\mathscr{A} = \mathscr{I}_m - \mathscr{L} - \mathscr{F}$ and $0 < \omega_1 < \omega_2 \leq 1$, then $\rho(\mathscr{H}_{\alpha\beta}(\omega_2)) \leq \rho(\mathscr{H}_{\alpha\beta}(\omega_1)) < 1$.*

Proposition 7. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor. For any $\omega \in (0,1]$, $\rho(\Theta_{\alpha\beta}) \leq \rho(\mathscr{H}_{\alpha\beta}(\omega))$, where $\Theta_{\alpha\beta}$ is the iteration tensor of the preconditioned Gauss-Seidel-type methods.*

Proposition 8. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor and $\alpha_i, \beta_j \in [0,1]$, $i = 1,2,\ldots,n-1$. If $0 < \omega \leq 1$, $a_{i(i+1)\ldots(i+1)}a_{(i+1)i\ldots i} > 0$, $i = 1,2,\ldots,n-1$ and $0 < a_{i1\ldots1}a_{1i\ldots i} < 1$, $i = 2,3,\ldots,n$, then we have $\rho(\mathscr{H}_{\alpha\beta}(\omega)) \leq \rho(\mathscr{H}(\omega)) < 1$.*

Proposition 9. *Let $\mathscr{A} \in \mathbb{R}^{[m,n]}$ be a strong $\mathscr{M}$-tensor with $a_{i(i+1)\ldots(i+1)}a_{(i+1)i\ldots i} > 0$, $i = 1,2,\ldots,n-1$ and $0 < a_{i1\ldots1}a_{1i\ldots i} < 1$, $i = 2,3,\ldots,n$. If $\beta_{1,j}$, $\beta_{2,j} \in [0,1]$, $j = k+1,\ldots,n$, $\alpha_{1,i}, \alpha_{2,i} \in [0,1]$, $i = 1,\ldots,n-s$, $\alpha' = (\alpha_{1,i})$, $\alpha'' = (\alpha_{2,i})$, $\beta' = (\beta_{1,j})$, $\beta'' = (\beta_{2,j})$ and $\alpha' \geq \alpha''$, $\beta' \geq \beta''$, then we have $\rho(\mathscr{H}_{\alpha'\beta'}(\omega)) \leq \rho(\mathscr{H}_{\alpha''\beta''}(\omega)) < 1$.*

Proposition 10. *Let* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *be a strong* $\mathscr{M}$*-tensor. If* $0 < \omega_1 < \omega_2 \leq 1$*, then*

$$\rho((\mathbf{M}(\mathscr{I}_m - \omega_2\mathscr{L}))^{-1}(\omega_2\mathscr{F} + (1-\omega_2)\mathscr{I}_m)) \leq \rho((\mathbf{M}(\mathscr{I}_m - \omega_1\mathscr{L}))^{-1}(\omega_1\mathscr{F} + (1-\omega_1)\mathscr{I}_m)) < 1.$$

Proposition 11. *Let* $\mathscr{A} \in \mathbb{R}^{[m,n]}$ *be a strong* $\mathscr{M}$*-tensor. If* $0 < \omega_1 < \omega_2 \leq 1$ *and* $\alpha_i, \beta_j \in [0,1]$, $i = 1,\ldots,n-s,\ j = k+1,\ldots,n$*, then* $\rho(\mathscr{H}_{\alpha\beta}(\omega_2)) \leq \rho(\mathscr{H}_{\alpha\beta}(\omega_1)) < 1$.

4 Numerical Examples

In this section, we give some numerical examples to show the performance of the proposed preconditioned methods. All experiments are carried out in double precision in MATLAB on a machine with CPU 2.70 GHz and 8 GB of RAM. All used codes came from the MATLAB tensor toolbox developed by Bader and Kolda [3,4]. We denote by PJ, PGS and PSOR the preconditioned Jacobi, preconditioned Gauss-Seidel and preconditioned SOR tensor splittings proposed in [33], [22] and [24], respectively. From the five splittings proposed to Jacobi and Gauss-Seidel, we choose the 2nd and 5th splitting and denote $\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$, $\mathbf{P}_{\alpha\beta}\mathscr{E}_5\mathscr{F}_5$ and $\mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$, $\mathbf{P}_{\alpha\beta}\mathscr{M}_5\mathscr{N}_5$ the preconditioned Jacobi and preconditioned Gauss-Seidel, respectively.

In addition, $\mathbf{P}_{\alpha\beta}SOR$ denotes the preconditioned SOR-type splitting method. In all tables, Iter and Time denote the number of iterations and elapsed CPU Time in seconds, respectively. The stoping criterion is $\|\mathbf{r}_j\| < 10^{-12}$, where $\mathbf{r}_j = \mathbf{b} - \mathscr{A}\mathbf{x}_j^{m-1}$, and the initial guess is $\mathbf{x}_0 = \mathbf{0}$, the right hand side vector $\mathbf{b}$ is $\mathbf{1} = (1,\ldots,1)^\top$. We also take that the maximum number of iterations up to 2000. Also, we suppose that $\boldsymbol{\beta} = \beta\mathbf{1}$ and $\boldsymbol{\alpha} = \alpha\mathbf{1}$, where the scalars α and β are given.

Example 1. Consider a strong $\mathscr{M}$-tensor $\mathscr{A} \in \mathbb{R}^{3\times3\times3}$as follows

$$\begin{aligned}
\mathscr{A}(:,:,1) &= \begin{pmatrix} 1.00 & -0.01 & -0.02 \\ -0.02 & -0.03 & -0.04 \\ -0.04 & -0.05 & -0.06 \end{pmatrix}, \\
\mathscr{A}(:,:,2) &= \begin{pmatrix} -0.06 & -0.07 & -0.08 \\ -0.08 & 1.00 & -0.09 \\ -0.01 & -0.02 & -0.03 \end{pmatrix}, \\
\mathscr{A}(:,:,3) &= \begin{pmatrix} -0.03 & -0.04 & -0.05 \\ -0.05 & -0.06 & -0.07 \\ -0.07 & -0.08 & 1.00 \end{pmatrix}.
\end{aligned}$$

We compared PJ, PGS and PSOR with $\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2, \mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$ and $\mathbf{P}_{\alpha\beta}SOR$, respectively. We chose $\omega = 1.2$ for the SOR method. We took $\alpha = \beta$ in the interval $[0,10]$ with the step size 0.5 and $s,k = 2$. The numerical results are reported in Table 1.

In addition, we selected ω in the interval $[0.5, 1.8]$ with the step size 0.1 and obtained the solution by using the proposed preconditioned SOR method for $\alpha = 0, \beta = 5$ and $s = 1, k = 2$. We have reported the results in Table 2.

From Table 1, we find that all the preconditioned methods perform better in CPU times and iteration numbers than the ones with unpreconditioned ($\alpha = \beta = 0$). Also, the proposed preconditioned schemes of Jacobi, Gauss-Seidel and SOR methods are all better than the corresponding ones that are considered

Table 1: Iteration number (Iter) and CPU Time (Time) for Example 1.

	PJ		PGS		PSOR		$\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$		$\mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$		$\mathbf{P}_{\alpha\beta}SOR$	
α	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.0	51	0.0066	50	0.0065	39	0.0100	51	0.0048	50	0.0055	39	0.0050
0.5	51	0.0046	49	0.0041	39	0.0044	50	0.0018	49	0.0030	38	0.0018
1.0	50	0.0044	47	0.0032	39	0.0033	49	0.0028	48	0.0019	37	0.0014
1.5	50	0.0039	46	0.0040	39	0.0024	48	0.0017	47	0.0018	36	0.0017
2.0	50	0.0040	45	0.0020	39	0.0021	46	0.0017	46	0.0017	35	0.0012
2.5	49	0.0040	44	0.0020	39	0.0021	45	0.0017	45	0.0017	35	0.0016
3.0	49	0.0030	43	0.0024	39	0.0020	44	0.0016	44	0.0016	34	0.0012
3.5	48	0.0028	43	0.0019	39	0.0019	43	0.0015	43	0.0013	33	0.0011
4.0	48	0.0032	42	0.0023	39	0.0027	42	0.0022	42	0.0013	32	0.0012
4.5	47	0.0021	43	0.0021	39	0.0019	41	0.0015	41	0.0012	31	0.0013
5.0	47	0.0023	43	0.0023	39	0.0028	40	0.0016	40	0.0012	30	0.0010
5.5	46	0.0023	44	0.0024	39	0.0021	39	0.0021	39	0.0014	29	0.0023
6.0	46	0.0026	44	0.0022	39	0.0019	38	0.0022	38	0.0014	28	0.0016
6.5	45	0.0024	45	0.0022	39	0.0020	37	0.0016	36	0.0019	27	0.0012
7.0	44	0.0022	46	0.0024	39	0.0020	35	0.0020	35	0.0014	26	0.0022
7.5	44	0.0027	48	0.0024	39	0.0019	34	0.0017	33	0.0013	24	0.0012
8.0	43	0.0023	49	0.0025	39	0.0021	**29**	**0.0014**	31	0.0013	22	0.0009
8.5	43	0.0029	50	0.0027	39	0.0021	31	0.0016	28	0.0012	**21**	**0.0008**
9.0	42	0.0026	52	0.0028	39	0.0022	33	0.0018	**27**	**0.0010**	23	0.0013
9.5	42	0.0024	53	0.0028	39	0.0022	34	0.0017	31	0.0014	25	0.0014
10.0	41	0.0024	55	0.0028	39	0.0022	34	0.0017	32	0.0014	28	0.0017

in this paper when the parameters α and β are taken suitably. The best answers in terms of CPU times and iteration numbers are bold numbers in Table 1. In Table 2 and for different choices of ω, we have marked some of the best results in terms of CPU times and iteration number which are obtained for the cases $\beta = 0$ and $s = 2$.

Example 2. Let $\mathscr{B} \in \mathbb{R}^{[3,n]}$ be a nonnegative tensor with $\mathbf{M}(\mathscr{B})$=hilb(n,n), where hilb is the function of MATLAB, for $i = 2,3,\ldots,n, b_{ii-1i} = b_{iii-1} = b_{ii+1i} = b_{iii+1} = \frac{1}{3}$ and other entries are zeros. Let $\mathscr{A} = n^2\mathscr{I} - 0.01\mathscr{B}$. We took $\alpha = \beta = 1,\ s = k = n-1$ and applied $\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2, \mathbf{P}_{\alpha\beta}\mathscr{E}_5\mathscr{F}_5, \mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$, $\mathbf{P}_{\alpha\beta}\mathscr{M}_5\mathscr{N}_5$ and $\mathbf{P}_{\alpha\beta}SOR$ for solving multiliear (1). Also we obtained experimentally the optimal parameter ω in the interval $[0,2]$.

The numerical results are reported in Table 3 which illustrate that the proposed preconditioned methods perform better in CPU times than the ones with the others. From Table 3, we find that when n increases, the CPU times for obtaining the appropriate solution increase. Also, if the parameters α,β, and ω are chosen suitably, the proposed preconditioned schemes of Jacobi, Gauss-Seidel, and SOR methods are all better than the corresponding ones that are considered in this paper. The best outcomes in terms of CPU times and iteration numbers for every n are bold numbers in Table 3, which show that the proposed second scheme of the preconditioned Jacobi method is the best. Note that in this table,

Table 2: Iteration numbers (Iter) and CPU times (Time) for the preconditioned SOR-type method.

	$\mathbf{P}_{\alpha=5}(1)$		$\mathbf{P}_{\alpha=5}(2)$		$\mathbf{P}_{5,5}(1,1)$		$\mathbf{P}_{5,5}(1,2)$		$\mathbf{P}_{5,5}(2,1)$		$\mathbf{P}_{5,5}(2,2)$	
ω	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0.5	103	0.0206	95	0.0106	105	0.0119	**94**	**0.0133**	97	0.0128	96	0.0148
0.6	83	0.0089	77	0.0025	85	0.0036	**76**	**0.0026**	79	0.0027	77	0.0027
0.7	69	0.0018	63	0.0018	70	0.0020	63	0.0018	**60**	**0.0019**	64	0.0024
0.8	58	0.0017	**53**	**0.0015**	60	0.0022	53	0.0019	56	0.0017	54	0.0016
0.9	50	0.0014	**45**	**0.0014**	51	0.0018	46	0.0014	48	0.0014	46	0.0016
1.0	43	0.0014	**39**	**0.0012**	44	0.0013	40	0.0012	42	0.0014	40	0.0012
1.1	37	0.0012	**34**	**0.0007**	39	0.0007	35	0.0010	37	0.0011	35	0.0010
1.2	33	0.0009	**29**	**0.0007**	34	0.0010	30	0.0009	32	0.0019	30	0.0008
1.3	29	0.0008	**25**	**0.0007**	30	0.0008	26	0.0007	29	0.0008	26	0.0008
1.4	31	0.0009	**28**	**0.0008**	30	0.0009	29	0.0008	33	0.0011	**28**	**0.0010**
1.5	40	0.0010	**35**	**0.0009**	39	0.0013	37	0.0010	44	0.0011	**35**	**0.0010**
1.6	53	0.0014	**45**	**0.0012**	51	0.0013	47	0.0012	59	0.0017	46	0.0019
1.7	71	0.0028	**60**	**0.0022**	70	0.0020	64	0.0017	81	0.0013	61	0.0010
1.8	105	0.0015	**84**	**0.0012**	101	0.0014	92	0.0013	135	0.0020	85	0.0019

Table 3: Iteration number (Iter) and CPU Time (Time) for Example 2.

	PJ		PGS		PSOR		$\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$		$\mathbf{P}_{\alpha\beta}\mathscr{E}_5\mathscr{F}_5$		$\mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$		$\mathbf{P}_{\alpha\beta}\mathscr{M}_5\mathscr{N}_5$		$\mathbf{P}_{\alpha\beta}SOR$	
n	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
30	4	0.0360	5	0.0169	3	0.0228	**3**	**0.0151**	3	0.0242	3	0.0196	3	0.0230	3	0.0240
40	4	0.0597	5	0.0273	3	0.0299	**3**	**0.0174**	3	0.0327	3	0.0233	3	0.0276	3	0.0243
50	4	0.0739	5	0.0293	3	0.0339	**3**	**0.0201**	3	0.0362	3	0.0257	3	0.0306	3	0.0288
60	4	0.0608	5	0.0443	3	0.0483	**3**	**0.0292**	3	0.0499	3	0.0417	3	0.0550	3	0.0425
70	4	0.0730	5	0.0558	3	0.0645	**3**	**0.0416**	3	0.0787	3	0.0575	3	0.0604	3	0.0657
80	4	0.0985	5	0.0829	3	0.0918	**3**	**0.0497**	3	0.1008	3	0.0733	3	0.0829	3	0.0929
90	4	0.1173	5	0.2064	3	0.1300	**3**	**0.0687**	3	0.1329	3	0.0859	3	0.0977	3	0.1179
100	4	0.1480	5	0.1406	3	0.1735	**3**	**0.1173**	3	0.2311	3	0.1264	3	0.1373	3	0.1449
110	4	0.2133	5	0.3544	3	0.1968	**3**	**0.1314**	3	0.2432	3	0.1525	3	0.1618	3	0.1790
120	4	0.2250	5	0.2073	3	0.2406	**3**	**0.1464**	3	0.2773	3	0.1978	3	0.1967	3	0.2544

regarding the structure of the $\mathscr{M}$-tensor $\mathscr{A} = \eta\mathscr{I} - 0.01\mathscr{B}, \eta = n^2 >> \rho(0.01\mathscr{B}) \approx 0.0328$ and that the spectral radius of the iteration tensor increases slowly by increasing the tensor size, there are no noticeable change on the iterations number.

Example 3. Let $\mathscr{B} \in \mathbb{R}^{[3,10]}$ be a nonnegative tensor and $b_{i_1 i_2 i_3} = |\tan(i_1 + i_2 + i_3)|$. It is not difficult to see that $\rho(\mathscr{B}) \approx 1450$, thus $\mathscr{A} = 2000\mathscr{I} - \mathscr{B}$ is a strong $\mathscr{M}$-tensor [27].

For mentioned methods, we obtained experimentally the optimal parameter ω in the interval $[1,2]$, the values of α, β are chosen from 0 to 30 and $s, k = 1$. The numerical results are reported in Table 4. In this table, † indicates no convergence up to 2000 iterations. We see that the proposed preconditioned methods perform better in CPU times than the ones with the others.
Moreover, for unpreconditioning schemes($\alpha = \beta = 0$), the proposed preconditioned schemes of Jacobi,

Table 4: Iteration number (Iter) and CPU Time (Time) for Example 3.

		PJ		PGS		PSOR		$\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$		$\mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$		$\mathbf{P}_{\alpha\beta}SOR$	
α	β	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time	Iter	Time
0	0	91	0.0308	87	0.0181	69	0.0302	91	0.0168	87	0.0186	69	0.0191
0.5	0.5	†	0.1253	†	0.0831	†	0.0820	90	0.0143	86	0.0184	68	0.0158
1	1	†	0.1253	†	0.0831	†	0.0820	89	0.0147	85	0.0195	67	0.0227
2	2	†	0.0960	†	0.0596	†	0.0853	87	0.0149	83	0.0241	65	0.0157
3	2	†	0.1025	†	0.0807	†	0.0949	85	0.0169	81	0.0167	64	0.0152
4	2	†	0.1051	†	0.0740	†	0.0788	83	0.0145	79	0.0204	62	0.0148
5	5	†	0.1023	†	0.0745	†	0.0849	81	0.0167	77	0.0201	60	0.0160
7	5	†	0.0953	†	0.0999	†	0.0831	77	0.0144	73	0.0185	57	0.0160
9	5	†	0.0835	†	0.0822	†	0.0989	73	0.0142	69	0.0164	54	0.0150
10	8	†	0.1347	†	0.1228	†	0.1262	71	0.0155	67	0.0193	52	0.0146
12	10	†	0.0975	†	0.0848	†	0.0760	67	0.0159	64	0.0159	49	0.0145
15	12	†	0.0934	†	0.0875	†	0.0987	61	0.0134	58	0.0167	44	0.0153
18	10	†	0.0912	†	0.0985	†	0.1044	55	0.0166	52	0.0186	39	0.0150
20	15	†	0.0987	†	0.0924	†	0.0901	49	0.0138	48	0.0157	38	0.0137
20	20	†	0.0912	†	0.0914	†	0.0926	49	0.0144	48	0.0159	38	0.0182
25	20	†	0.0989	†	0.0932	†	0.0911	**40**	**0.0138**	**40**	**0.0151**	38	0.0139
25	25	†	0.0999	†	0.0924	†	0.0937	42	0.0139	**40**	**0.0153**	**37**	**0.0137**
30	20	†	0.0974	†	0.0978	†	0.0945	47	0.0140	48	0.0166	45	0.0140
30	25	†	0.0910	†	0.0934	†	0.0922	48	0.0139	48	0.0170	46	0.0140
30	30	†	0.0900	†	0.0944	†	0.0891	48	0.0141	49	0.0158	46	0.0140

Gauss-Seidel and SOR methods obtained the same outcomes as the corresponding ones that are considered in this paper. When the parameters α and β are considered nonzero, we see that the PJ, PGS, and PSOR methods are not convergent while the proposed methods are convergent which improve the iteration numbers and CPU times compared to the corresponding unpreconditioned methods. The best results in terms of CPU times and the iteration numbers are marked in this Table.

In the following, we give an example of some test problems to evaluate the comparison results between the spectra radii of the splittings of the proposed iterative methods.

Example 4. Consider the following test problems:
Case I : Let $\mathscr{B} \in \mathbb{R}^{[3,200]}$ be a nonnegative tensor and $b_{i_1i_2i_3} = |\tan(i_1+i_2+i_3)|$. Using the power method ([27]), we obtained the spectral radius of $\mathscr{B}$, $\rho(\mathscr{B}) \approx 1.8452e+05$, and thus $\mathscr{A} = (1.8800e+05)\mathscr{I} - \mathscr{B}$ is a strong $\mathscr{M}$-tensor.
Case II : Let $\mathscr{B} \in \mathbb{R}^{[3,50]}$ be a nonnegative tensor and $\mathscr{M}(\mathscr{B}) = sprand(S)$ where *sprand* is a MATLAB function which generates uniformly distributed random entries with the same sparsity structure as S and other entries are $1e-1$. We choose S as a tridiagonal matrix. Using the power method ([27]), we obtained the spectral radius of $\mathscr{B}$, $\rho(\mathscr{B}) \approx 2.4660e+02$, and thus $\mathscr{A} = (1+\rho(\mathscr{B}))\mathscr{I} - \mathscr{B}$ is a strong $\mathscr{M}$-tensor.

Case III : Let $\mathscr{A} \in \mathbb{R}^{[3,200]}$ with

$$\begin{cases} a_{1,1,1} = a_{200,200,200} = 1, & \\ a_{i,i,i} = 2, & i = 2,\ldots,199, \\ a_{i,i-1,i} = a_{i,i,i-1} = \frac{-1}{3}, & i = 2,\ldots,199, \\ a_{i,i+1,i} = a_{i,i,i+1} = \frac{-1}{3}, & i = 2,\ldots,199. \end{cases}$$

Case IV : Let $\mathscr{B} \in \mathbb{R}^{[3,3]}$ be a nonnegative tensor and $b_{i_1 i_2 i_3} = |\sin(i_1 + i_2 + i_3)|$. Using the power method ([27]), we obtained the spectral radius of $\mathscr{B}$, $\rho(\mathscr{B}) \approx 5.8147$, and thus $\mathscr{A} = 5.8800\mathscr{I} - \mathscr{B}$ is a strong $\mathscr{M}$-tensor.

We applied $\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2, \mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2, \mathbf{P}_{\alpha\beta}SOR$ for solving (1), with selected parameters $\alpha = \beta = \omega = 1$. For two cases I, III, we choose $b = \mathscr{A}\, ones(n,1)^{m-1}$ in (1) where MATLAB function $ones(n,1)$ is an n-by-1 vector of ones and for the cases II, IV, $b = \mathscr{A} e_2{}^{m-1}$ in (1) where e_2 is the second column of the identity matrix. The numerical results are given in Table 5. In this table, Iter, Time and $\rho(M(\mathscr{E})^{-1}\mathscr{F})$ denote iteration number, performance CPU time and the spectral radius of iterative tensor of the corresponding tensor splitting iterative method. The results show that $\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$ in terms of CPU time is relatively superior than the other methods.

Table 5: Numerical results for Example 4.

		$\mathbf{P}_{\alpha\beta}\mathscr{E}_2\mathscr{F}_2$	$\mathbf{P}_{\alpha\beta}\mathscr{M}_2\mathscr{N}_2$	$\mathbf{P}_{\alpha\beta}SOR$
Case I:	Iter	2196	2191	2191
	Time	20.4094	21.3205	21.1293
	$\rho(M(\mathscr{E})^{-1}\mathscr{F})$	0.9815	0.9814	0.981
Case II:	Iter	4993	4993	4993
	Time	0.5218	0.5410	0.5011
	$\rho(M(\mathscr{E})^{-1}\mathscr{F})$	0.9959	0.9959	0.9959
Case III:	Iter	54	54	54
	Time	0.5061	1.7410	1.7787
	$\rho(M(\mathscr{E})^{-1}\mathscr{F})$	0.6025	0.6025	0.6025
Case IV:	Iter	1193	1104	1104
	Time	0.0236	0.0463	0.0366
	$\rho(M(\mathscr{E})^{-1}\mathscr{F})$	0.9857	0.9846	0.9846

5 Conclusion

In this paper, we proposed some new preconditioners based on tensor splitting for solving multilinear system $\mathscr{A}\mathbf{x}^{m-1} = \mathbf{b}$. We also presented some theorems for analyzing and convergence of the preconditioned Jacobi-, Gauss-Seidel-, and SOR-type iterative methods. Numerical results illusterated the efficiency and superiority of the proposed methods.

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