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A nonautonomous delayed viscoelastic wave equation with a linear damping: well-posedness and exponential stability

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Abstract. In this paper, we consider a nonautonomous viscoelastic wave equation with linear damping and delayed terms. Under some appropriate assumptions, we prove the global existence using the semigroup theory. Furthermore, for a small enough coefficient of delay, we obtained a stability result via a suitable Lyapunov function where the kernel function decays exponentially.

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1 Introduction

We take into consideration the following linear viscoelastic wave equation, which has a constant internal feedback delay and linear damping:

$$\begin{cases} u_{tt}(x,t) - a(t)\Delta u(x,t) + \displaystyle\int_0^\infty g(s)b(t)\Delta u(x,t-s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \quad t \geq 0, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, \quad t \in (0,\tau), \end{cases} \tag{1}$$

where Ω be a regular domain of $\mathbb{R}^n$, a, b are given functions of class $C^1(\mathbb{R}_+, \mathbb{R}_+^*)$ and Δ design the Laplacian operator. The decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ and the positive constant τ represent, respectively,

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the kernel of the viscoelastic term and time delay, μ_1 is a positive constant and μ_2 is a real number, such that

$$|\mu_2| \leq \mu_1. \tag{2}$$

The initial datum (u_0, u_1, f_0) belongs to a suitable space.

In recent years, delayed equations have been addressed by several of authors in the literature, and it was proved that the delay may destabilize the system (see [1, 4, 30, 33–36, 41]). Also, viscoelastic equations got a great part of research [3, 8, 9, 14, 26, 27, 39, 44]. For instance, the viscoelastic wave equation of the form

$$u_{tt} - \Delta u + f(x,t,u) + \int_0^t g(t-s)\Delta u(x,s)\,ds + a(x)u_t = 0,$$

where $a : \Omega \to \mathbb{R}$ is a non-negative and bounded function, $f : \bar{\Omega} \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a function of class C^1 and $g : \mathbb{R}_+ \to \mathbb{R}_+$, has been considered by Cavalcanti et al. in [8]. Under some restrictions on a and g, they showed that the solution decays exponentially. Later, under weaker condition, the previous result was improved by Berrimi and Messaoudi in [6]. For the case $\mu_1 = \mu_2 = 0$, in [21], Guesmia proved two general decay estimates of solution (polynomial and logarithmic) under a general assumption on the kernel function g, see [25,29,45,46] for other related works. To stabilize the system even in the presence of delay, there are different decay results for equations equipped with both viscoelastic damping term and time delay feedback, see [7, 19, 28, 38] and the references therein. In this context, it was proved in [17,22,30,31] that additional conditions or control terms are enough to ensure the stabilization of the solution in the presence of delay.

In [23], Kirane et al. studied the following equation

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \quad x \in \Omega,\ t > 0.$$

They obtained the exponential stability of solutions under a suitable condition between the weight of the delay term in the feedback and the weight of the term without delay. The same equation has been considered in [15] by Dai and Yang where the authors obtained the exponential decay result for energy without any restrictions on μ_1 and μ_2. This work was later extended by involving the constant delay in the nonlinear non-external feedback in [5] by Benaissa et al., where they studied

$$u_{tt}(x,t) - \Delta_x u(x,t) + \int_0^t g(t-s)\Delta_x(x,s)\,ds + \mu_1 a_1(u_t(x,t)) + \mu_2 a_2(u_t(x,t-\tau)) = 0,$$

where a_1 is a non-decreasing function of class $C(\mathbb{R})$ and $a_2 : \mathbb{R} \to \mathbb{R}$ be a function of class C^1 such that it is odd and non-decreasing function. The writers obtained the global existence result using the energy method combined with the Faedo-Galerkin argument. Furthermore, they studied the asymptotic behavior of solutions using a perturbed energy method. After that, Remil and Hakem [43] treated the case when μ_1 and μ_2 were real functions. Precisely, they investigated the viscoelastic wave equation below

$$u_{tt} - k_0\Delta u + \alpha \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1(t)u_t(x,t) + \mu_2(t)u_t(x,t-\tau) = 0,$$

where k_0 and α are positive real numbers. They used a multiplier method to establish a decay estimate for the energy, which is depends on the behavior of α and g. Many authors proved that the solution is asymptotic stable for $\mu_1 = 0$ [10, 20, 42]. For time-dependent delay, Baowei in [18] established the general decay of energy of the problem by using the energy perturbation method; for related work, we refer to [12, 32].

A large part of the literature addresses the autonomous abstract evolution equation. In [11], for the abstract problem with past history and constant delay, Chellaoua and Boukhatem [11] considered the following abstract viscoelastic equation

$$u_{tt}(t) + Au(t) - \int_0^\infty g(s)Bu(t-s)\,ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = 0,$$

where $A : D(A) \to H$ and $B : D(B) \to H$ are self-adjoint linear positive operators. They proved the well-posedness result by using semi-group theory. They established explicit and general decay results of the energy solution for a larger class of kernel functions where the exponential and polynomial are particular cases. The previous authors established the same results for the above problem with source term and time-varying delay, see [12, 13].

For time-dependent operators A and B, there has been an increasing interest in studying evolution equations with nonautonomous feedback. The reader is referred to [16, 24]. It has been noted that the existence of a solution to this type of equation is related to the existence of an evolution family, which is not fully direct because the domain of the operators may depend on the time variable. Very recently, on the other hand, there are few recent works that have been dedicated to the study of abstract equations with nonautonomous feedback, that is, the operators are time-dependent. It has been noted that existing results in the case of nonautonomous are only partially direct for these reasons. First, the domain of operators may depend on the time variable. Second, the existence of a solution is related to the existence of an evolution family. However, only some evolution families solve such a problem. Here, we mention the work of Al-Khulaifi et al. in [2], who studied a class of nonautonomous second-order evolution equations without delay and obtained the well-posedness and stability of the solution.

Our main goal of this work is to establish the well-posedness result and exponential decay of energy of the problem (1) in the nonautonomous case, where we are considering the varying-time operators $a(t)\Delta$ and $b(t)\Delta$ with a constant delay. According to our knowledge, there are no decay results for problems involving time-dependent operators with delay and infinite memory. Moreover, our problem generalizes the advance results unescorted by delay to those with delay and the nonautonomous case of evolution equation.

The plan of this paper is as follows: in Section 2, we state some assumptions on the considered datum. Then, we prove the global existence by using the semi-group arguments. Section 3 presents some technical lemmas needed to get the main results. Section 4 is devoted to establishing the decay results of the solution based on the energy method by choosing a suitable Lyapunov functional.

2 Well-posedness

In this section, we will present the well-posedness result of system using semi-group approach. Throughout this paper, we use standard functional space $L^2(\Omega)$ endowed with the inner product $\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$ and the induced norm $\|u\| = \sqrt{\langle u, v\rangle}$, and we denote c_0 the Poincary's constant. Now, we make the following assumptions:

($\mathbf{H}_1$) The non-increasing C^1 kernel function $g:\mathbb{R}_+\to\mathbb{R}_+$ satisfies

$$g_0=\int_0^\infty g(s)\,ds<\min\left\{\frac{a(t)}{b(t)},\frac{a'(t)}{b'(t)}\right\}, \tag{3}$$

and there exist positives constants θ_1 and θ_2 such that

$$-\frac{1}{\theta_1}g'(s)\le g(s)\le -\frac{1}{\theta_2}g'(s). \tag{4}$$

($\mathbf{H}_2$) There exists a positive constant θ_3 such that

$$b'(t)\le -\theta_3 b(t), \tag{5}$$

and

$$\|a'(t)-g_0b'(t)\|_{L^\infty(\mathbb{R}_+)} \quad \text{is small enough.} \tag{6}$$

Following a method developed in [14] (see also [39]) and the idea of Nicaise and Pignotti in [30] (see also [36, 37]) by producing the new auxiliary variables η and z, system (1) can be reformulated as the following abstract linear first order evolution

$$\begin{cases} U_t(t)=\mathscr{A}(t)U(t),\\ U(0)=U_0,\end{cases} \tag{7}$$

where $U=(u,u_t,\eta,z)^T$, $U_0=(u_0,u_1,\eta_0,f_0(-\tau\rho))^T$ are elements on the space $\mathscr{H}(t)$, which given by

$$\mathscr{H}(t)=V\times L^2(\Omega)\times L_g(t)\times L^2(0,1),\qquad V=H^2(\Omega)\cap H_0^1(\Omega),$$

and

$$\begin{cases} \eta(t,s)=u(t)-u(t-s), & t,s\ge 0,\\ z(\rho,t)=u_t(t-\rho\tau), & \rho\in(0,1),\quad t\ge 0,\end{cases} \tag{8}$$

with

$$\begin{cases} \eta_0(s)=\eta(0,s)=u_0(0)-u_0(s), & s\ge 0,\\ z_0(\rho)=z(\rho,0)=f_0(-\tau\rho), & \rho\in(0,1).\end{cases} \tag{9}$$

The spaces $L_g(t)$ and $L^2(0,1)$, respectively, are defined by

$$L_g(t)=\left\{\phi:\mathbb{R}_+\to V,\ \int_0^\infty g(s)b(t)\|\nabla\phi\|^2\,ds<\infty\right\},$$

and

$$L^2(0,1)=\left\{\phi:]0,1[\to L^2(\Omega):\int_0^1\|\phi\|^2\,d\rho<\infty\right\},$$

endowed with the inner products

$$\langle\phi_1,\phi_2\rangle_{L_g(t)}=\int_0^\infty g(s)b(t)\,\langle\nabla\phi_1(s),\nabla\phi_2(s)\rangle\,ds,$$

and

$$\langle\phi_1,\phi_2\rangle_{L^2(0,1)}=\int_0^1\langle\phi_1,\phi_2\rangle\,d\rho. \tag{10}$$

We define the operator $\mathscr{A}(t)$ by

$$\mathscr{A}(t)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}=\begin{pmatrix}\phi_2\\ (a(t)-g_0b(t))\Delta\phi_1+\int_0^\infty g(s)b(t)\Delta\phi_3(s)\,ds-\mu_1\phi_2-\mu_2\phi_4(1)\\ \phi_2-\dfrac{\partial\phi_3}{\partial s}\\ -\dfrac{1}{\tau}\dfrac{\partial\phi_4}{\partial\rho}\end{pmatrix}, \tag{11}$$

with domain

$$D(\mathscr{A}(t))=\left\{\begin{array}{c}(\phi_1,\phi_2,\phi_3,\phi_4)\in V\times L^2(\Omega)\times L_g(t)\times L^2(0,1),\\ (a(t)-g_0b(t))\,\phi_1+\int_0^\infty g(s)b(t)\phi_3(s)\,ds \text{ in } V,\\ \dfrac{\partial\phi_3}{\partial s}\in L_g(t),\quad \phi_3(0)=0,\phi_4(0)=\phi_2\end{array}\right\}. \tag{12}$$

By the definitions of η and z, we have

$$\begin{cases}\eta_t(t,s)+\eta_s(t,s)=u_t(t), & t,s\geq 0,\\ \eta(t,0)=0, & t\geq 0,\end{cases} \tag{13}$$

and

$$\begin{cases}\tau z_t(\rho,t)+z_\rho(\rho,t)=0, & \rho\in(0,1),\quad t\geq 0,\\ z(0,t)=u_t(t), & t\geq 0.\end{cases} \tag{14}$$

Owing to (13) and (14), we conclude that systems (1) and (7) are equivalent.

Hence, from (3), $\mathscr{H}(t)$ is a Hilbert space endowed with the inner product

$$\begin{aligned}\left\langle(\phi_1,\phi_2,\phi_3,\phi_4)^T,(\tilde\phi_1,\tilde\phi_2,\tilde\phi_3,\tilde\phi_4)^T\right\rangle_{\mathscr{H}(t)}= & \ (a(t)-g_0b(t))\langle\nabla\phi_1,\nabla\tilde\phi_1\rangle+\langle\phi_2,\tilde\phi_2\rangle_{L^2(\Omega)}\\ & +\langle\phi_3,\tilde\phi_3\rangle_{L_g(t)}+\tau\xi\langle\phi_4,\tilde\phi_4\rangle_{L^2(0,1)},\end{aligned}$$

while ξ be a constant satisfies $\xi\geq 0$ and

$$|\mu_2|<\xi<2\mu_1-|\mu_2|. \tag{15}$$

From (19), we observe that ξ exists.

The global existence results of (7) is given by the following.

Theorem 1. *Assuming that*

1) $\forall t>0$, $D(\mathscr{A}(t))=D(\mathscr{A}(0))$.

2) $\forall t\in[0,T]$, *$\mathscr{A}(t)$ is the infinitesimal generator of a C_0−semi-group on $\mathscr{H}(t)$ and $\mathscr{A}=\{\mathscr{A}(t),t\in[0,T]\}$ is a stable family.*

3) *$\partial_t\mathscr{A}$ belongs to $L^\infty([0,T],B(D(\mathscr{A}(0)),\mathscr{H}(t))$, which is the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(D(\mathscr{A}(0)),\mathscr{H}(t))$ of bounded operators from $D(\mathscr{A}(0))$ into $\mathscr{H}(t)$.*

Under the assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$, *for any* $U_0 \in \mathscr{H}(t)$, *system* (7) *has a unique solution* $U \in C(\mathbb{R}_+, \mathscr{H}(t))$. *Moreover, if* $U_0 \in D(\mathscr{A}(0))$, *then* $U \in C^1(\mathbb{R}_+, \mathscr{H}(t)) \cap C(\mathbb{R}_+, D(\mathscr{A}(0)))$.

Proof. We will prove that the linear operator $\mathscr{A}(t)$ generates a linear C_0-semi-group on $\mathscr{H}(t)$. Firstly, $\mathscr{A}(t)$ is dissipative. Let $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ be element of $D(\mathscr{A}(t))$, then

$$\begin{aligned}\langle \mathscr{A}(t)\Phi, \Phi\rangle =&(a(t)-g_0b(t))\langle \nabla\phi_2, \nabla\phi_1\rangle + (a(t)-g_0b(t))\langle \Delta\phi_1, \phi_2\rangle + \left\langle \int_0^\infty g(s)b(t)\Delta\phi_3(s)ds, \phi_2\right\rangle \\ &-\mu_1\langle\phi_2,\phi_2\rangle - \mu_2\langle\phi_4(1),\phi_2\rangle + \langle\phi_2,\phi_3\rangle_{L_g(t)} - \left\langle \frac{\partial\phi_3}{\partial s}, \phi_3\right\rangle_{L_g(t)} - \xi\left\langle \frac{\partial\phi_4}{\partial\rho}, \phi_4\right\rangle_{L^2(0,1)}.\end{aligned} \tag{16}$$

We have by the Green's formula

$$\langle \Delta\phi_1, \phi_2\rangle = -\langle \nabla\phi_1, \nabla\phi_2\rangle. \tag{17}$$

Definition (12) together with the Green's formula lead to

$$\langle\phi_2,\phi_3\rangle_{L_g(t)} = \left\langle \phi_2, \int_0^\infty g(s)b(t)\Delta\phi_3(s)ds\right\rangle = -\int_0^\infty g(s)b(t)\langle\nabla\phi_2,\nabla\phi_3\rangle\,ds. \tag{18}$$

The Cauchy-Schwartz's and Young's inequalities imply that

$$-\mu_2\langle\phi_4(1),\phi_2\rangle \le \frac{|\mu_2|}{2}\left(\|\phi_4(1)\|^2 + \|\phi_2\|^2\right). \tag{19}$$

Let integrate by parts and using the condition $\phi_3(0) = 0$, we obtain

$$-\langle\frac{\partial\phi_3}{\partial s},\phi_3\rangle_{L_g(t)} = \frac{1}{2}\int_0^\infty g'(s)b(t)\|\nabla\phi_3(s)\|^2\,ds. \tag{20}$$

Recalling (10), we may write

$$-\xi\langle\frac{\partial\phi_4}{\partial\rho},\phi_4\rangle = -\xi\int_0^1\langle\frac{\partial\phi_4}{\partial\rho},\phi_4\rangle\,d\rho = \frac{\xi}{2}\left(\|\phi_4(0)\|^2 - \|\phi_4(1)\|^2\right) = \frac{\xi}{2}\left(\|\phi_2\|^2 - \|\phi_4(1)\|^2\right). \tag{21}$$

Inserting (17), (18), (19), (20) and (21) in (16), we arrive at

$$\langle\mathscr{A}(t)\Phi,\Phi\rangle \le \frac{b(t)}{2}\int_0^\infty g'(s)\|\nabla\phi_3(s)\|^2\,ds + \left(-\mu_1 + \frac{|\mu_2|}{2} + \frac{\xi}{2}\right)\|\phi_2\|^2 + \left(\frac{|\mu_2|}{2} - \frac{\xi}{2}\right)\|\phi_4(1)\|^2. \tag{22}$$

As g is non-increasing function and by the inequality (15), we conclude that $\mathscr{A}(t)$ is dissipative operator.

Secondly, let show that $I-\mathscr{A}(t)$ is surjective

$$\begin{gathered}\forall F = (f_1, f_2, f_3, f_4) \in \mathscr{H}(t), \ \ \exists\Phi = (\phi_1,\phi_2,\phi_3,\phi_4) \in D(\mathscr{A}(t)), \text{ such that} \\ (I-\mathscr{A}(t))W = F,\end{gathered} \tag{23}$$

which is equivalent to

$$\begin{cases}\phi_1 - \phi_2 = f_1, \\ \phi_2 - (a(t)-g_0b(t))\Delta\phi_1 - \displaystyle\int_0^\infty g(s)b(t)\Delta\phi_3(s)\,ds + \mu_1 w_2 + \mu_2\phi_4(1) = f_2, \\ \phi_3 - \phi_2 + \dfrac{\partial\phi_3}{\partial s} = f_3, \\ \phi_4 + \dfrac{1}{\tau}\dfrac{\partial\phi_4}{\partial\rho} = f_4.\end{cases} \tag{24}$$

From the first equation of (24), we have

$$\phi_2 = \phi_1 - f_1. \tag{25}$$

The third equation of (24) with $\phi_3(0) = 0$ has a unique solution given by

$$\phi_3(s) = (1 - e^{-s})\phi_1 + e^{-s}\int_0^s e^y (f_3(y) - f_1)\,dy, \tag{26}$$

and the fourth equation of (24) with $\phi_4(0) = \phi_2 = \phi_1 - f_1$ has a unique solution

$$\phi_4(\rho) = \left(\phi_1 - f_1 + \tau\int_0^\rho f_4(y)e^{\tau y}\,dy\right)e^{-\tau\rho}, \quad \rho \in (0,1). \tag{27}$$

For $\rho = 1$,

$$\phi_4(1) = \left(\phi_1 - f_1 + \tau\int_0^1 f_4(y)e^{\tau y}\,dy\right)e^{-\tau}.$$

We insert (25) and (26) in the second equation of (24), we get

$$-(a(t) - g_1 b(t))\Delta\phi_1 + \left(I + \mu_1 I + \mu_2 e^{-\tau} I\right)\phi_1 = \tilde{f}, \tag{28}$$

while

$$g_1 = \int_0^\infty e^{-s} g(s)\,ds,$$

and

$$\begin{aligned}\tilde{f} =& f_2 + \left(\mu_1 + \mu_2 e^{-\tau} - 1\right)f_1 - \mu_2\tau e^{-\tau}\int_0^1 f_4(y)e^{\tau y}\,dy \\ &- \int_0^\infty g(s)b(t)e^{-s}\int_0^s e^y\Delta(f_3(y) - f_1)\,dy\,ds.\end{aligned}$$

Let now prove that (28) has a solution $\phi_1 \in V$ then, we find $\Phi \in D(\mathscr{A}(t))$ satisfies (24). Indeed, we have $g_1 < g_0$ then $-(a(t) - g_1 b(t))\Delta$ is positive definite operator. So, we take the duality brackets $\langle\cdot,\cdot\rangle_{V',V}$, with $\varphi \in V$:

$$\left\langle -(a(t) - g_1 b(t))\Delta\phi_1 + (1 + \mu_1 + \mu_2 e^{-\tau})I\phi_1, \varphi\right\rangle_{V',V} = \left\langle \tilde{f}, \varphi\right\rangle_{V',V}.$$

Using Green's formula, we get

$$\langle (a(t) - g_1 b(t))\nabla\phi_1, \nabla\varphi\rangle_{V',V} + \left\langle (1 + \mu_1 + \mu_2 e^{-\tau})I\phi_1, \varphi\right\rangle_{V',V} = \left\langle \tilde{f}, \varphi\right\rangle_{V',V}. \tag{29}$$

The left hand of (29) is bilinear, coercive and continuous form, then

$$\left|\left\langle -(a(t) - g_1 b(t))\Delta\phi_1 + (1 + \mu_1 + \mu_2 e^{-\tau})I\phi_1, \varphi\right\rangle_{V',V}\right| \le C\|\phi_1\|\|\varphi\|.$$

For $\varphi = \phi_1 \in V$

$$\begin{aligned}&\left|\left\langle (a(t) - g_1 b(t))\Delta\phi_1 + (1 + \mu_1 + \mu_2 e^{-\tau})I\phi_1, \phi_1\right\rangle_{V',V}\right| \\ &\qquad = (a(t) - g_1 b(t))\|\nabla\phi_1\|^2 + (1 + \mu_1 + \mu_2 e^{-\tau})\|\phi_1\|^2 \ge C\|\phi_1\|^2.\end{aligned}$$

Applying the Lax-Milgram's theorem, we conclude that (24) has a unique solution $w_1 \in V$, by (26), satisfies

$$-(a(t)-g_0b(t))\Delta\phi_1 + b(t)\int_0^\infty g(s)\Delta\phi_3(s)\,ds + \mu_1\phi_2 \in L^2(\Omega).$$

Thus, $I-\mathscr{A}(t)$ is surjective.

Finally, (22) and (23) mean that $\mathscr{A}(t)$ is maximal monotone operator. Then, using Lummer-Phillips's theorem [40, Theorem I.4.6], we deduce that $\mathscr{A}(t)$ is the infinitesimal generator of a C_0-semi-group of contraction on $\mathscr{H}(t)$.

Lastly, let $\Phi = (\phi_1,\phi_2,\phi_3,\phi_4)^T \in D(\mathscr{A}(0))$, then

$$\frac{d}{dt}\mathscr{A}(t)\Phi = \begin{pmatrix} 0 \\ (a'(t)-g_0b'(t))\Delta\phi_1 + b'(t)\int_0^\infty g(s)\Delta\phi_3\,ds \\ 0 \\ 0 \end{pmatrix}. \tag{30}$$

As $\phi_3 \in L_g(t)$, and by (6). Then

$$\frac{d}{dt}\mathscr{A}(t)\Phi \in L^\infty([0,T],B(D(\mathscr{A}(0)),\mathscr{H}(t)).$$

Thus, the assumptions of Theorem 1 are hold, so system (1) has a unique solution that achieved the proof of Theorem 1. □

3 Technical lemmas

This section is devoted to state some technical lemmas. To begin with, let define the energy functional E associated with problem (7) by

$$\begin{aligned} E(t) = \frac{1}{2}\|U(t)\|^2_{\mathscr{H}(t)} = &\frac{(a(t)-g_0b(t))}{2}\|\nabla u(t)\|^2 + \|u_t(t)\|^2 + \frac{b(t)}{2}\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds \\ &+ \frac{\tau\xi}{2}\int_0^1 \|z(\rho,t)\|^2\,d\rho, \qquad \forall t\in\mathbb{R}_+. \end{aligned} \tag{31}$$

Lemma 1. *For all $t\geq 0$, we have*

$$E'(t) \leq \frac{a'(t)-g_0b'(t)}{2}\|\nabla u(t)\|^2 - \frac{\theta_2+\theta_3}{2}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds. \tag{32}$$

Proof. Multiplying the first equation of (1) by u_t, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 - a(t)\langle\Delta u(t),u_t(t)\rangle + \left\langle\int_0^\infty g(s)b(t)\Delta u(t-s)\,ds, u_t(t)\right\rangle \\ + \mu_1\|u_t(t)\|^2 + \mu_2\langle u_t(t-\tau),u_t(t)\rangle = 0. \end{aligned}$$

Using the Green's formula, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 + a(t)\langle\nabla u(t),\nabla u_t(t)\rangle - \left\langle\int_0^\infty g(s)b(t)\nabla u(t-s)\,ds, \nabla u_t(t)\right\rangle \\ + \mu_1\|u_t(t)\|^2 + \mu_2\langle u_t(t-\tau),u_t(t)\rangle = 0. \end{aligned} \tag{33}$$

From the definitions of η and z, we find

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2+(a(t)-g_0b(t))\langle\nabla u(t),\nabla u_t(t)\rangle+\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla u_t(t)\right\rangle$$
$$+\mu_1\|u_t(t)\|^2+\mu_2\langle z(1),u_t(t)\rangle=0.$$

As $\eta_t(t,s)+\eta_s(t,s)=u_t(t)$, we have

$$\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla u_t(t)\right\rangle=\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla\left(\eta_t(t,s)+\eta_s(t,s)\right)\right\rangle. \tag{34}$$

A simple calculation leads to

$$(a(t)-g_0b(t))\langle\nabla u(t),\nabla u_t(t)\rangle=\frac{1}{2}\frac{d}{dt}\left[(a(t)-g_0b(t))\|\nabla u(t)\|^2\right]-\frac{1}{2}(a'(t)-g_0b'(t))\|\nabla u(t)\|^2, \tag{35}$$

and

$$\begin{aligned}\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla\eta_t(t,s)\right\rangle=&\frac{1}{2}\frac{d}{dt}\left[\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds\right]\\&-\frac{1}{2}\int_0^\infty g(s)b'(t)\|\nabla\eta(t,s)\|^2\,ds.\end{aligned} \tag{36}$$

By integration by parts and using the condition $\lim_{s\to\infty}g(s)=0$ and $\eta(t,0)=0$, we arrive at

$$\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla\eta_s(t,s)\rangle=-\frac{1}{2}\int_0^\infty g'(s)b(t)\|\nabla\eta(t,s)\|^2\,ds. \tag{37}$$

Thus, combining (34), (36) and (37), we get

$$\begin{aligned}\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla u_t(t)\right\rangle=&\frac{1}{2}\frac{d}{dt}\left[\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds\right]-\frac{1}{2}\int_0^\infty g(s)b'(t)\|\nabla\eta(t,s)\|^2\,ds\\&-\frac{1}{2}\int_0^\infty g'(s)b(t)\|\nabla\eta(t,s)\|^2\,ds.\end{aligned} \tag{38}$$

Applying Cauchy-Schwartz's and Young's inequalities, we obtain

$$\mu_2\langle z(1),u_t(t)\rangle\le\frac{|\mu_2|}{2}\left(\|z(1)\|^2+\|u_t(t)\|^2\right). \tag{39}$$

Inserting (35), (38) and (39) in (33), we find

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|u_t(t)\|^2+(a(t)-g_0b(t))\|\nabla u(t)\|^2+b(t)\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds\right]\\&\quad\le\frac{a'(t)-g_0b'(t)}{2}\|\nabla u(t)\|^2+\frac{b(t)}{2}\int_0^\infty g'(s)\|\nabla\eta(t,s)\|^2\,ds\\&\qquad+\frac{b'(t)}{2}\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds\quad+\left(\frac{|\mu_2|}{2}-\mu_1\right)\|u_t(t)\|^2+\frac{|\mu_2|}{2}\|z(1)\|^2.\end{aligned}$$

From the inequality (15), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\|u_t(t)\|^2+(a(t)-g_0b(t))\|\nabla u(t)\|^2+b(t)\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds\right]\\ &\le \frac{a'(t)-g_0b'(t)}{2}\|\nabla u(t)\|^2+\frac{b(t)}{2}\int_0^\infty g'(s)\|\nabla\eta(t,s)\|^2\,ds\\ &\quad+\frac{b'(t)}{2}\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds+\frac{\xi}{2}\left(\|z(1)\|^2-\|u_t(t)\|^2\right).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_0^1\|z(\rho,t)\|^2\,d\rho\right]&=\int_0^1\langle z(\rho,t),z_t(\rho,t)\rangle\,d\rho\\ &=-\frac{1}{\tau}\int_0^1\langle z(\rho,t),z_\rho(\rho,t)\rangle\,d\rho\\ &=\frac{1}{2\tau}\left[\|u_t(t)\|^2-\|z(1)\|^2\right].\end{aligned}\tag{40}$$

Then, the last identity (40) leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\left[\|u_t(t)\|^2+(a(t)-g_0b(t))\|\nabla u(t)\|^2+b(t)\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds\right]\\ &\le\frac{a'(t)-g_0b'(t)}{2}\|\nabla u(t)\|^2+\frac{b(t)}{2}\int_0^\infty g'(s)\|\nabla\eta(t,s)\|^2\,ds\\ &\quad+\frac{b'(t)}{2}\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds-\frac{\tau\xi}{2}\frac{d}{dt}\left[\int_0^1\|z(\rho,t)\|^2\,d\rho\right].\end{aligned}$$

According to (3), (4) and (5), we conclude (32), this completes the proof. □

Lemma 2. *Let u be the solution of (7). Then, the functional I_1*

$$I_1(t)=\langle u_t(t),u(t)\rangle,\tag{41}$$

satisfies, for all $t\ge 0$

$$\begin{aligned}I_1'(t)\le&\left(1+\frac{\mu_1}{2}\right)\|u_t(t)\|^2-\left(a(t)-g_0b(t)-\frac{1+\mu_1c_0}{2}\right)\|\nabla u(t)\|^2\\ &+\frac{g_0b(t)}{2}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds-\mu_2\langle z(1),u(t)\rangle.\end{aligned}\tag{42}$$

Proof. By differentiating (41) with respect to t, we obtain

$$I_1'(t)=\|u_t(t)\|^2+\langle u_{tt}(t),u(t)\rangle.\tag{43}$$

We multiply the first equation of (1) by u and use the Green's formula with the definitions (8), we obtain

$$\begin{aligned}\langle u_{tt}(t),u(t)\rangle+(a(t)-g_0b(t))\|\nabla u(t)\|^2+\left\langle\int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds,\nabla u(t)\right\rangle\\ +\mu_1\langle u_t(t),u(t)\rangle+\mu_2\langle z(1),u(t)\rangle=0.\end{aligned}\tag{44}$$

Exploiting (43) and (44), we deduce

$$\begin{aligned} I_1'(t) &= \|u_t(t)\|^2 - (a(t) - g_0 b(t)))\|\nabla u(t)\|^2 - \mu_1 \langle u_t(t), u(t)\rangle \\ &- \left\langle \int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds, \nabla u(t)\right\rangle - \mu_2 \langle z(1), u(t)\rangle . \end{aligned} \tag{45}$$

Using the Cauchy-Schwartz's, Young's and Poincary's inequalities on the two last terms give

$$-\mu_1 \langle u_t(t), u(t)\rangle \le \frac{\mu_1}{2}\|u_t(t)\|^2 + \frac{\mu_1 c_0}{2}\|\nabla u(t)\|^2, \tag{46}$$

and

$$-\langle \int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds, \nabla u(t)\rangle \le \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\left\| \int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds\right\|^2 . \tag{47}$$

The last term of (47) can be written as

$$\begin{aligned} \left\| \int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds\right\|^2 &\le \left(\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|\,ds\right)^2 \\ &\le \left(\int_0^\infty \sqrt{g(s)b(t)}\sqrt{g(s)b(t)}\|\nabla\eta(t,s)\|\,ds\right)^2 \\ &\le \left(\int_0^\infty g(s)b(t)\,ds\right)\left(\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds\right) \\ &= g_0 b(t)\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds. \end{aligned}$$

Therefore

$$-\langle \int_0^\infty g(s)b(t)\nabla\eta(t,s)\,ds, \nabla u(t)\rangle \le \frac{1}{2}\|\nabla u(t)\|^2 + \frac{g_0 b(t)}{2}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds. \tag{48}$$

We substitute (46) and (48) in (45), we conclude (42). □

Lemma 3. *The function defined by*

$$I_2(t) = -\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\rangle, \tag{49}$$

satisfies the following inequality

$$\begin{aligned} I_2'(t) &\le \left(\frac{\mu_1+1}{2} - g_0\right)\|u_t(t)\|^2 + \left(\frac{a(t)-g_0 b(t)}{2}\right)\|\nabla u(t)\|^2 + \mu_2\left\langle z(1), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle \\ &+ g_0\left(\frac{a(t)-g_0 b(t)}{2b(t)} + c_0\frac{\mu_1+\theta_1}{2b(t)} + 1\right)\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2 ds, \qquad \textit{for all} \quad t \ge 0. \end{aligned} \tag{50}$$

Proof. Let multiply the first equation of (1) by $\int_0^\infty g(s)\eta(t,s)\,ds$, we get

$$\begin{aligned} \left\langle u_{tt}(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle &+ (a(t)-g_0 b(t))\left\langle \nabla u(t), \int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\rangle \\ &+ b(t)\left\|\int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\|^2 + \mu_1\left\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle \\ &+ \mu_2\left\langle z(1), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle = 0. \end{aligned} \tag{51}$$

As $u_t(t) = \eta_t(t,s) + \eta_s(t,s)$, then

$$\begin{aligned}\langle u_{tt}(t), \int_0^\infty g(s)\eta(t,s)\,ds\rangle &= \frac{d}{dt}\left\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle - \left\langle u_t(t), \int_0^\infty g(s)\eta_t(t,s)\,ds\right\rangle \\ &= -I_2'(t) - g_0\|u_t(t)\|^2 + \left\langle u_t(t), \int_0^\infty g(s)\eta_s(t,s)\,ds\right\rangle.\end{aligned}$$

We integrate by part relating to s, so, we get

$$\left\langle u_{tt}(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle = -I_2'(t) - g_0\|u_t(t)\|^2 - \left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)\,ds\right\rangle. \tag{52}$$

Inserting (52) in (51), we obtain

$$\begin{aligned}I_2'(t) =& - g_0\|u_t(t)\|^2 + b(t)\left\|\int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\|^2 + (a(t) - g_0 b(t))\left\langle \nabla u(t), \int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\rangle \\ &+ \mu_1\left\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle - \left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)\,ds\right\rangle \\ &+ \mu_2\left\langle z(1), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle.\end{aligned} \tag{53}$$

Using Cauchy-Schwartz's, and Young's inequalities, we get

$$\begin{aligned}\left\|\int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\|^2 &\le \left(\int_0^\infty g(s)\|\nabla\eta(t,s)\|\,ds\right)^2 \\ &\le \left(\int_0^\infty \sqrt{g(s)}\sqrt{g(s)}\|\nabla\eta(t,s)\|\,ds\right)^2 \\ &\le g_0\int_0^\infty g(s)\|\nabla\eta(t,s)\|^2\,ds,\end{aligned} \tag{54}$$

and

$$\begin{aligned}\left\langle \nabla u(t), \int_0^\infty g(s)\nabla\eta(t,s)\,ds\right\rangle &\le \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\int_0^\infty g(s)\nabla\eta(t,s)\,ds\|^2 \\ &\le \frac{1}{2}\|\nabla u(t)\|^2 + \frac{g_0}{2b(t)}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds.\end{aligned} \tag{55}$$

Poincary's inequality leads to

$$\left\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle \le \frac{1}{2}\|u_t(t)\|^2 + \frac{g_0 c_0}{2b(t)}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds, \tag{56}$$

and according to (4), we get

$$\begin{aligned}-\left\langle u_t(t), \int_0^\infty g'(s)\eta(t,s)\,ds\right\rangle &\le \theta_1\left\langle u_t(t), \int_0^\infty g(s)\eta(t,s)\,ds\right\rangle \\ &\le \frac{1}{2}\|u_t(t)\|^2 + g_0\frac{c_0\theta_1}{2b(t)}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds.\end{aligned} \tag{57}$$

Then, we substitute (54), (55), (56) and (57) in (53) to hold (50). □

Lemma 4 ([10]). *The function*

$$I_3(t) = \tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \|z(\rho,t)\|^2 \, d\rho, \tag{58}$$

satisfies, for all $t \geq 0$

$$I_3'(t) \leq -2\tau \int_0^1 \|z(\rho,t)\|^2 \, d\rho + e^{2\tau} \|u_t(t)\|^2 - \|z(1)\|^2. \tag{59}$$

4 Asymptotic behavior

This section is dedicated to investigate the asymptotic behavior of the solution. To state our main results, we introduce a suitable Lyapunov functional which constructed as below:

$$\mathscr{L}(t) = M(t)E(t) + M_1 I_1(t) + M_2 I_2(t) + I_3(t), \tag{60}$$

where $M_1, M_2 \in \mathbb{R}_+$ and M is a differentiable function from $\mathbb{R}_+$ to itself.

Proposition 1. *There exist differentiable functions* $\lambda_1, \lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+^*$, *such that*

$$\lambda_1(t)E(t) \leq \mathscr{L}(t) \leq \lambda_2(t)E(t). \tag{61}$$

Proof. By Cauchy-Schwartz's, Young's and Poincary's inequalities, we have

$$|I_1(t)| \leq \frac{1}{2}\|u_t(t)\|^2 + \frac{c_0}{2}\|\nabla u(t)\|^2 \leq \max\left\{1, \frac{c_0}{a(t) - g_0 b(t)}\right\} E(t), \tag{62}$$

and

$$|I_2(t)| \leq \frac{1}{2}\|u_t(t)\|^2 + \frac{g_0 c_0}{2b(t)} \int_0^\infty g(s) b(t) \|\nabla \eta(t,s)\|^2 \, ds \leq \max\left\{1, \frac{g_0 c_0}{b(t)}\right\} E(t). \tag{63}$$

Also, we have

$$|I_3(t)| \leq \tau e^{2\tau} \int_0^1 \|z(\rho,t)\|^2 \, d\rho \leq \frac{e^{2\tau}}{\xi} E(t). \tag{64}$$

From (62), (63) and (64), we obtain

$$|\mathscr{L}(t) - M(t)E(t)| \leq C(t)E(t),$$

where

$$C(t) = M_1 \max\left\{1, \frac{c_0}{a(t) - g_0 b(t)}\right\} + M_2 \max\left\{1, \frac{g_0 c_0}{b(t)}\right\} + \frac{e^{2\tau}}{\xi}. \tag{65}$$

Consequently, by choosing $M(t)$, for all $t \geq 0$ so large, we conclude $\mathscr{L} \sim E$. □

In the following theorem, we state the stability result of the solution.

Theorem 2. *Assume that the assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$ hold. For any $U_0 \in \mathscr{H}(0)$ there exists a positive constant δ such that under a very small choice of μ_2, the solution of (7) satisfies*

$$\|U(t)\|_{\mathscr{H}(t)} \leq \frac{\delta e^{\tilde{\varepsilon}(t)}}{\lambda_1(t)}, \tag{66}$$

where δ and $\tilde{\varepsilon}$ are defined later.

Proof. Suppose that $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ hold, then all the estimates are explained. Let start the demonstration by estimating the derivative of Lyapunov functional. From (32), (42), (50), and (59), we get

$$\begin{aligned}\mathscr{L}'(t) \leq\ & M'(t)E(t) - \left[M_2\big(g_0 - \frac{\mu_1+1}{2}\big) - M_1(1+\frac{\mu_1}{2}) - e^{2\tau}\right]\|u_t(t)\|^2 \\ & - \left[M_1\big(a(t) - g_0 b(t) - \frac{1+\mu_1 c_0}{2}\big) - \frac{a'(t)-g_0 b'(t)}{2}M(t) - M_2\frac{a(t)-g_0 b(t)}{2}\right]\|\nabla u(t)\|^2 \\ & - \left[\frac{\theta_2+\theta_3}{2}M(t) - M_1\frac{g_0 b(t)}{2} - M_2 g_0\big(\frac{a(t)-g_0 b(t)}{2b(t)} + c_0\frac{\mu_1+\theta_1}{2b(t)} + 1\big)\right]\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds \\ & - 2\tau\int_0^1 \|z(\rho,t)\|^2\,d\rho + \mu_2\left\langle z(1), M_2\int_0^\infty g(s)\eta(t,s)\,ds - M_1 u(t)\right\rangle - \|z(1)\|^2.\end{aligned} \tag{67}$$

Using Cauchy-Schwartz's, Young's and the Poincary's inequalities, we have

$$\begin{aligned}\mu_2\left\langle z(1), M_2\int_0^\infty g(s)\eta(t,s)ds - M_1 u(t)\right\rangle \leq\ & \|z(1)\|^2 + M_2^2 c_0\frac{g_0\mu_2^2}{2b(t)}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds \\ & + M_1^2\frac{c_0\mu_2^2}{2}\|\nabla u(t)\|^2.\end{aligned} \tag{68}$$

Combining (67) and (68) lead to

$$\begin{aligned}\mathscr{L}'(t) \leq\ & M'(t)E(t) - \left[M_2\left(g_0 - \frac{\mu_1+1}{2}\right) - M_1\left(1+\frac{\mu_1}{2}\right) - e^{2\tau}\right]\|u_t(t)\|^2 \\ & - \left[M_1\left(a(t) - g_0 b(t) - \frac{1+\mu_1 c_0 + \mu_2^2 M_1 c_0}{2}\right) - M_2\frac{a(t)-g_0 b(t)}{2} - M(t)\frac{a'(t)-g_0 b'(t)}{2}\right]\|\nabla u(t)\|^2 \\ & - \left[\frac{\theta_2+\theta_3}{2}M(t) - M_1\frac{g_0 b(t)}{2} - M_2 g_0\left(\frac{a(t)-g_0 b(t)}{2b(t)} + c_0\frac{\mu_1+\theta_1+\mu_2^2 M_2}{2b(t)} + 1\right)\right]\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2\,ds \\ & - 2\tau\int_0^1 \|z(\rho,t)\|^2\,d\rho.\end{aligned} \tag{69}$$

Now, let fix $M_1 = M_2 = 1$, such that

$$g_0 - \mu_1 - e^{2\tau} > \frac{5}{2}, \tag{70}$$

and, let choose $M(t)$, as follow

$$M(t) > \frac{2}{\theta_2+\theta_3}M_3(t),$$

while

$$M_3(t) = \frac{g_0 b(t)}{2} + g_0\left(\frac{a(t)-g_0 b(t)}{2b(t)} + c_0\frac{\mu_1+\theta_1+\mu_2^2}{2b(t)} + 1\right) + \frac{a(t)-g_0 b(t)}{4}. \tag{71}$$

From (6) and using the fact M_3 does not depend on a' and b', we can suppose that

$$\left(a'(t)-g_0b'(t)\right)M(t)\leq \frac{(a(t)-g_0b(t))}{2}. \tag{72}$$

So, we arrive at

$$\begin{aligned}\mathscr{L}'(t)\leq & M'(t)E(t)-\|u_t(t)\|^2-\left(\frac{a(t)-g_0b(t)}{4}-\frac{1+\mu_1c_0+c_0\mu_2^2}{2}\right)\|\nabla u(t)\|^2\\ & -\frac{a(t)-g_0b(t)}{4}\int_0^\infty g(s)b(t)\|\nabla\eta(t,s)\|^2ds-2\tau\int_0^1\|z(\rho,t)\|^2d\rho.\end{aligned}$$

Also, let choose μ_1 and μ_2 be very small, such that (70) is satisfied and $1+\mu_1c_0+\mu_2^2c_0$ is small enough. Using (31), we get

$$\mathscr{L}'(t)\leq\left(M'(t)-G(t)\right)E(t),$$

where $G(t)=\max\left\{1,\dfrac{a(t)-g_0b(t)}{4}\right\}$. Then, by (61), we obtain

$$\mathscr{L}'(t)\leq\varepsilon(t)\mathscr{L}(t), \tag{73}$$

where $\varepsilon(t)=\dfrac{M'(t)-G(t)}{\lambda_1(t)}$. We integrate (73), so, we deduce

$$\mathscr{L}(t)\leq\mathscr{L}(0)e^{\tilde{\varepsilon}(t)},$$

where $\tilde{\varepsilon}(t)=\int_0^t\varepsilon(s)\,ds$. Exploiting (61) results

$$E(t)\leq\frac{\mathscr{L}(0)}{\lambda_1(t)}e^{\tilde{\varepsilon}(t)}.$$

Therefore, we conclude (66), with $\delta=2\mathscr{L}(0)$. Thus the proof of Theorem 2 is completed. □

Remark 1. *If the functions C and M_3 are bounded, hence we can choose M as a constant, such that*

$$M>\frac{2}{\theta_2+\theta_3}\|M_3(t)\|_{L^\infty(\mathbb{R}_+)}.$$

Then, we get

$$\varepsilon(t)=-\frac{G(t)}{M-\|C(t)\|_{L^\infty(\mathbb{R}_+)}}. \tag{74}$$

Therefore, (66) implies that

$$\exists\ c_1,c_2,\quad \text{such that}\quad \|U(t)\|^2_{\mathscr{H}(0)}\leq c_2e^{-c_1\int_0^t(a(s)-g_0b(s))\,ds}. \tag{75}$$

Thanks to (65) and (71) , we notice that C and M_3 are bounded functions if and only if

$$\|b\|_{L^\infty(\mathbb{R}_+)}<\infty,\qquad g_0<\frac{\|a\|_{L^\infty(\mathbb{R}_+)}}{\|b\|_{L^\infty(\mathbb{R}_+)}}. \tag{76}$$

So, we obtain the exponential stability estimate

$$\|U(t)\|^2_{\mathscr{H}(0)}\leq c_2e^{-c_3t}, \tag{77}$$

where $c_3=c_1(\|a\|_{L^\infty(\mathbb{R}_+)}-g_0\|b\|_{L^\infty(\mathbb{R}_+)})$.

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