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SOME RESULTS OF THE MINIMUM EDGE DOMINATING ENERGY OF THE CAYLEY GRAPHS FOR THE FINITE GROUP $\mathbf{S_n}$

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Abstract. Let Γ be a finite group and S be a non-empty subset of Γ. A Cayley graph of the group Γ, denoted by $Cay(\Gamma, S)$ is defined as a simple graph that its vertices are the elements of Γ and two vertices u and v are adjacent if $uv^{-1} \in \Gamma$.
The minimum edge dominating energy of Cayley graph $Cay(\Gamma, S)$ is equal to the sum of the absolute values of eigenvalues of the minimum edge dominating matrix of graph $Cay(\Gamma, S)$. In this paper, we estimate the minimum edge dominating energy of the Cayley graphs for the finite group S_n.

1. Introduction

There has been a close relationship between group theory and graph theory such that combinatorial properties of graphs have been employed extensively to investigate the theoretic algebraic properties of groups and vice versa. In 1878, Arthur Cayley was considered the first to associate graphs called the Cayley graph to finite groups [7]. Suppose that Γ is a finite group and $S \subseteq \Gamma \setminus \{e\}$ such that $S = S^{-1} = \{s^{-1} : s \in S\}$. The graph $G(V, E) = Cay(\Gamma, S)$ is an undirected and simple graph defined by the vertices $V(G) = \Gamma$ and $E(G) = \{(x, y) \,|\, xy^{-1} \in S\}$. A Cayley graph G is connected if and only if $\Gamma =< S >$ where $< S >$ is generating a subset of Γ [5]. The Cayley graphs have many applications in algebra, computer science, biological sciences and chemistry.

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Eigenvalues of Cayley graphs have been studied due to their roles in algebraic graph theory and applications in chemical graph theory, quantum computing, biology, etc [19]. One of the concepts of chemical graph theory that is associated with the eigenvalues of a graph is the graph energy proposed by Gutman in 1978 [16].

Let $G = (V, E)$ be a simple graph with the vertex set and edge set $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, respectively. For a graph G, the neighborhood of vertex $u \in V$ is defined $N_G(u) = \{v \in V | uv \in E\}$. The number of edges incident to vertex u in G is denoted $deg_G(u)$. The graph G is r-regular if the degree of all vertices is r.

The adjacent matrix $A(G) = (a_{ij})$ of G is an $n \times n$ matrix, where $a_{ij} = 1$ if $v_iv_j \in E$ and $a_{ij} = 0$ otherwise. The eigenvalues of the matrix $A(G)$, are called the eigenvalues of graph G [18]. Let λ_i be the eigenvalue of a graph G with multiplicity m_i for $1 \leq i \leq t$. The spectrum of the graph G is defined as follows.

$$Spec(G) = \left(\begin{array}{ccc} \lambda_1 & \ldots & \lambda_t \\ m_1 & \ldots & m_t \end{array} \right).$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A(G)$. The graph energy $E(G)$ of G, is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$ [16]. There are many kinds of graph energies that are proposed and investigated [1, 3, 12, 13]. The edge energy of a graph G, denoted by $EE(G)$, is defined as the sum of the absolute values of eigenvalues of $A(L_G)$ where L_G is the line graph of G [6]. The line graph L_G of G is the graph that each vertex of it represents an edge of G and two vertices of L_G are adjacent if and only if their corresponding edges are incident in G [18].

A subset $D \subseteq V$ is a dominating set of a graph G if every vertex of $V \setminus D$ is adjacent to some vertices in D [18]. For a graph G, any dominating set with minimum cardinality is called a minimum dominating set of G. The minimum dominating energy of a graph G, by denoted $E_D(G)$, is defined as the sum of the absolute values of eigenvalues of the matrix $A_D(G)$ in which the minimum dominating matrix $A_D(G)$ is as following [23].

$$A_D(G) := (a_{ij}) = \begin{cases} 1 & \text{if } v_iv_j \in E, \\ 1 & \text{if } i = j \, and \, v_i \in D, \\ 0 & otherwise. \end{cases}$$

A subset F of edges in a graph G is the edge dominating set if every edge e in $E \setminus F$ is adjacent to at least one edge in F. The edge domination number, denoted by γ' is the minimum of the cardinality of an edge dominating set of G. The concept of edge domination number was studied by Gupta [17]. Note that the edge dominating set of graph

G is a dominating set for its line graph and vice versa. Let G be a simple graph with edge set $\{e_1, e_2, \ldots, e_m\}$ and $F \subseteq E$ be the minimum edge dominating set of graph G or the minimum dominating set of graph $L(G)$. In [2], the minimum edge dominating matrix of graph G is defined as an $m \times m$ matrix as follows

$$A_F(G) := (a_{ij}) = \begin{cases} 1 & \text{if } e_i \, and \, e_j \, are \, adjacent, \\ 1 & \text{if } i = j \, and \, e_i \in F, \\ 0 & otherwise. \end{cases}$$

The minimum edge dominating energy of G is introduced and studied in [2] as $EE_F(G) = \sum_{i=1}^{m} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $A_F(G)$. The minimum edge dominating energy of graphs is obtained and studied in [2, 9, 10, 11, 20, 21, 22].
In this paper, we investigate the minimum edge dominating energy of some Cayley graphs on finite group S_n.
We use K_n and C_n to denote a complete graph and a cycle graph of order n, respectively. Two graphs G_1 and G_2 are called isomorphic, denoted by $G_1 \simeq G_2$ if there is a bijective correspondence between their vertices and edges.

2. Preliminaries

In this section, we state some results that will be used in the next section.

Lemma 2.1. [2] *Let C_n be the cycle graph of order n for $n \geq 3$. Then,* $EE_F(C_n) = E_D(C_n)$.

Lemma 2.2. [20] *Let G be a graph of order n with m edges. If F is the minimum edge dominating set of G with cardinality k, then $EE_F(G) \leq 4m - 2n + k$.*

Lemma 2.3. [20] *Let G be a graph of order n with $m \geq n$ edges. If F is the minimum edge dominating set of G, then $EE_F(G) \geq 4(m - n + s) + 2p$, where p and s are the number of pendant and isolated vertices in G, respectively.*

Lemma 2.4. [20] *Let G be a regular graph of degree $r \geq 2$ with n vertices and $m = \frac{rn}{2}$ edges. If F is the minimum edge dominating set with cardinality k, then*

(i) *If $r = 2$, then $EE_F(G) \leq E(G) + k$,*
(ii) *If $r > 2$, then $EE_F(G) < E(G) + k + 2n(r - 2)$.*

Lemma 2.5. [4] *Let G be a graph of order n and γ' be the minimum edge domination number of G. Then, $\gamma' \leq \lfloor \frac{n}{2} \rfloor$.*

Lemma 2.6. [20] *Let G be a connected graph of order n. If v^+ is the number of the positive eigenvalues of the matrix $A(G)$, then $EE_F(G) \geq 2E(G) - 4v^+$.*

3. Main Results

In this section, we compute the minimum edge dominating energy of the Cayley graphs $G = Cay(\Gamma, S)$ where Γ is the symmetric group S_n and S is a certain subset of these group.
We first consider the finite symmetric group S_n for $n = 4$. The symmetric group S_n is a group of all permutations of n symbols of the order $n!$ [15]. The energy of the Cayley graph of group S_4 on two different subsets S of S_4 are obtained in [15]. We compute the minimum edge dominating energy of $Cay(S_4, S)$ where S is the given subsets in [15]. To do it, we need some previous results of the Cayley graph on symmetric group S_4.

Lemma 3.1. [15] *Let S_4 be the symmetric group and $S \subseteq S_4$ with the condition $|S| = 1$. Then, $Cay(S_4, S) = \bigcup_{i=1}^{12} K_2$.*

Lemma 3.2. [15] *Let S_4 be the symmetric group and $S \subseteq S_4$ with the condition $|S| = 2$. Then, the Cayley graph of group S_4 on subset S is as follows*

$$Cay(S_4, S) = \begin{cases} \bigcup_{i=1}^{4} C_6 & \text{if } S = \{(ij), (kl)\}, \\ \bigcup_{i=1}^{3} C_8 & \text{if } S = \{(ij), (kl)(mn)\}, \\ \bigcup_{i=1}^{8} C_3 & \text{if } S = \{(ijk), (lmn)\}, \\ \bigcup_{i=1}^{6} C_4 & \text{if } S = \big\{\{(ij), (ij)(kl)\}, \{(ij)(kl), (mn)(pq)\} \\ & \qquad\qquad , \{(ijkl), (mnpq)\}\big\}. \end{cases}$$

Theorem 3.3. *Let F be the minimum edge dominating set of Cayley graph $Cay(S_4, S)$ where S_4 is the symmetric group of order 24 and $S \subset S_4$ where $|S| = 1$. Then*

$$EE_F\big(Cay(S_4, S)\big) = 12.$$

Proof. Let G be the Cayley graph of symmetric group S_4 on subset S with condition $|S| = 1$. Using Lemma 3.1, $G = \bigcup_{i=1}^{12} K_2$. On the other hand, the line graph of K_2 is K_1. That is, $L_G = \bigcup_{i=1}^{12} K_1$.
Therefore, the spectrum of the minimum edge dominating matrix $A_F(G)$

consists of 1 with multiplicity 12. Thus, we have

$$\begin{aligned} EE_F\big(Cay(S_4,S)\big) &= EE_F(G) \\ &= \sum_{i=1}^{12} \big|\lambda_i(A_F(G))\big| \\ &= 12 \times (1) = 12. \end{aligned}$$

□

Theorem 3.4. *Let F be the minimum edge dominating set of Cayley graph $Cay(S_4,S)$ where S_4 is the symmetric group of order 24 and $S \subset S_4$ where $|S| = 2$. Then*

$$EE_F\big(Cay(S_4,S)\big) = \begin{cases} 33.1701 & \text{if } S = \{(ij),(kl)\}, \\ 32.6796 & \text{if } S = \{(ij),(kl)(mn)\}, \\ 30.6274 & \text{if } S = \{(ijk),(lmn)\}, \\ 31.4164 & \text{if } S = \big\{\{(ij),(ij)(kl)\},\{(ij)(kl),(mn)(pq)\} \\ & \qquad ,\{(ijkl),(mnpq)\}\big\}. \end{cases}$$

Proof. Let G be the Cayley graph of the symmetric group S_4 on subset S with condition $|S| = 2$. According to Lemma 3.2, we consider the following cases. First, suppose that C_i is a cycle of order i in which the vertices are labeled as $1, 2, \ldots, i$.

Case 1: Assume that $S = \{(ij),\ (kl)\}$. By applying Lemma 3.2, $G = \bigcup_{i=1}^{4} C_6$. Since the line graph of C_n is the cycle C_n, the minimum edge dominating set and the minimum dominating set of C_n are the same. Therefore, using Lemma 2.1, it is sufficient to obtain the minimum dominating energy of graph C_6. According to the structure of cycle C_6, one can consider the dominating set as $D = \{1, 4\}$. The eigenvalues of matrix $A_D(C_6)$ are $\{1 \pm \sqrt{2}, \pm\sqrt{3}, \pm 1\}$. Therefore, we get

$$\begin{aligned} E_D(C_6) &= \sum_{i=1}^{6} \big|\lambda_i\big| \\ &= \big|1+\sqrt{2}\big| + \big|1-\sqrt{2}\big| + \big|\sqrt{3}\big| \\ &+ \big|-\sqrt{3}\big| + \big|1\big| + \big|-1\big| \\ &= 2(1+\sqrt{2}+\sqrt{3}) \simeq 8.2925. \end{aligned}$$

Consequently, we have

$$\begin{aligned} EE_F\big(Cay(S_4,S)\big) &= EE_F(G) \\ &= 4 \times E_D(C_6) \\ &\simeq 4 \times 8.2925 \simeq 33.1701. \end{aligned}$$

Case 2: Suppose that $S = \{(ij),\ (kl)(mn)\}$. Similar to the proof of Case 1 and since $G = \bigcup_{i=1}^{3} C_8$, one can consider $D = \{1, 4, 6\}$ as the minimum edge dominating set of the cycle C_8. So, the eigenvalues λ_i, where $1 \leq i \leq 8$, for $A_D(C_8)$ are $\{2.4605, 2, 1.8019, -1.6996, -1.2469, -1, 0.4450, 0.2391\}$. Therefore, $E_D(C_8) = \sum_{i=1}^{8} \big|\lambda_i\big| \simeq 10.8930$.
For the minimum edge dominating energy of the graph $Cay(S_4, S)$, we have

$$\begin{aligned} EE_F(G) &= 3 \times E_D(C_8) \\ &\simeq 3 \times 10.8930 \simeq 32.6790. \end{aligned}$$

Case 3: For $S = \{(ijk),\ (lmn)\}$, using Lemma 3.2 we have $G = \bigcup_{i=1}^{8} C_3$. Similar to the discussion in Case 1, we obtain the minimum dominating energy of the cycle C_3. To do it, we consider $D = \{1\}$ as the minimum dominating set of C_3 and the eigenvalues of $A_D(C_3)$ are $\{1 \pm \sqrt{2}, -1\}$.
So, $E_D(C_3) = \big|1 + \sqrt{2}\big| + \big|1 - \sqrt{2}\big| + \big| - 1\big| \simeq 3.8284$. Therefore, we get

$$\begin{aligned} EE_F(G) &= 8 \times E_D(C_3) \\ &\simeq 8 \times 3.8284 \simeq 30.6272. \end{aligned}$$

Case 4: Assume that S is one of the subsets $\{(ij), (ij)(kl)\}$, $\{(ij)(kl), (mn)(pq)\}$ and $\{(ijkl), (mnpq)\}$ of group S_4 where $|S| = 2$. Therefore, by applying Lemma 3.2, we have $G = \bigcup_{i=1}^{6} C_4$. Similar to the proof of previous cases, by considering the minimum dominating set $D = \{1, 2\}$ in cycle C_4, we obtain the minimum dominating energy of C_4. Since the eigenvalues of $A_D(C_4)$ are $\{\frac{1}{2}\big(3 \pm \sqrt{5}\big), \frac{1}{2}\big(-1 \pm \sqrt{5}\big)\}$ thus, $E_D(C_4) = 3 + \sqrt{5} \simeq 5.2361$. Therefore for the minimum edge dominating energy of the Cayley graph $Cay(S_4, S)$, we have

$$EE_F\big(Cay(S_4, S)\big) = 6 \times E_D(C_4) \simeq 31.4166.$$

Therefore, the result is complete. □

In the following theorem, we obtain the minimum edge dominating energy of the Cayley graph of finite symmetric group S_n for $n \geq 2$ on subset $S = \{(12), (13), \ldots, (1n)\}$ of S_n. Note that the Cayley graph $Cay(S_n, S)$ is the star Cayley graph of degree $n - 1$. This graph is a connected $(n - 1)$-regular graph of the order $n!$ [19].

Theorem 3.5. *Let S_n be the symmetric group, where $n \geq 2$ and $S = \{(12), (13), \ldots, (1n)\}$ be the subset of S_n. Let F be the minimum edge dominating set of the Cayley graph $Cay(S_n, S)$.*
i) If $n = 2$, then $EE_F(Cay(S_n, S)) = 1$.
ii) If $n \geq 3$, then $2n!(n - 3) \leq EE_F\big(Cay(S_n, S)\big) \leq n!\big(2n - \frac{7}{2}\big)$.

Proof. The Cayley graph of group S_n on the subset $S = \{(12), (13), \dots, (1n)\}$ of S_n for $n \geq 2$ is the star Cayley graph of order $n!$ where the degree all vertices is $n-1$. Assume that $G = Cay(S_n, S)$. For $n = 2$, it is clear to see that $G \simeq K_2$. Therefore $EE_F(G) = 1$.
For $n \geq 3$, the number of edges of G is $\frac{(n-1)n!}{2}$. Using Lemmas 2.3 and 2.5, we get

$$\begin{aligned} EE_F\big(Cay(S_n, S)\big) &\leq 4\Big(\frac{(n-1)n!}{2}\Big) - 2n! + |F| \\ &\leq 2(n-1)n! - 2n! + \frac{n!}{2} \\ &= n!\big(2n - \frac{7}{2}\big). \end{aligned}$$

The graph G is $(n-1)$-regular graph. Thus, the number of pendant and isolated vertices are 0. For the lower bound, we apply Lemma 2.3 and have

$$\begin{aligned} EE_F\big(Cay(S_n, S)\big) &\geq 4(m - n) \\ &\geq 4\Big(\frac{(n-1)n!}{2} - n!\Big) \\ &= 2(n-1)n! - 4n! \\ &= 2n!(n-3). \end{aligned}$$

□

For the symmetric group S_n, we consider the other kind of Cayley graphs as the arrangement graph. The arrangement graph $A(n, k)$ is a graph with all k-permutations of S_n as vertices and two k-permutations are adjacent if they agree in exactly $k-1$ positions. Graph $A(n, k)$ is a $k(n-k)$-regular graph with $\frac{n!}{(n-k)!}$ vertices [14]. We determine the minimum edge dominating energy $A(n, k)$ for $k \leq 4$. We first obtain the graph energy of $A(n, k)$ for $k = 2, 3, 4$.

Theorem 3.6. *Let $A(n, k)$ be the arrangement graph on the symmetric group S_n for $n \geq 3$.*
i) If $n = 3$, then $E(A(n, 2)) = 8$,
ii) If $n \geq 4$, then $E(A(n, 2)) = 4(n^2 - 3n + 1)$.

Proof. It is easy to see that $A(3, 2)$ is the cycle C_6. Therefore, the spectrum of C_6 is

$$Spec(C_6) = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}.$$

Thus, $E(A(3,2)) = \sum_{i=1}^{6} |\lambda_i| = 8$.
For $n \geq 4$, we apply Proposition 9 in [8] for the eigenvalues of the arrangement graph $A(n,2)$. So, the spectrum of $A(n,2)$, where $n \geq 4$ is as following

$$Spec(A(n,2)) = \begin{pmatrix} -2 & n-4 & n-2 & 2n-4 \\ n^2-3n+1 & n-1 & n-1 & 1 \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} E(A(n,2)) &= \sum_{i=1}^{n(n-1)} \left|\lambda_i(A(n,2))\right| \\ &= (n^2-3n+1)|-2| + (n-1)|n-4| \\ &+ (n-1)|n-2| + |2n-4| \\ &= 4(n^2-3n+1). \end{aligned}$$

□

Theorem 3.7. *Let $A(n,k)$ be the arrangement graph on symmetric group S_n for $n \geq 4$.*
i) If $n = 4$, then $E(A(n,3)) = 36$,
ii) If $n = 5, 6$, then $E(A(n,3)) = 3n^3 - 8n^2 - 13n + 18$,
iii) If $n \geq 7$, then $E(A(n,3)) = 6(n^3 - 6n^2 + 8n - 1)$.

Proof. According to Theorem 10 in [8], the spectrum of graph $A(n,3)$ for $n \geq 4$ is as following

$$\begin{pmatrix} -3 & n-7 & n-6 & n-4 & n-3 & 2n-9 & 2n-6 & 3n-9 \\ n^3-6n^2+8n-1 & \frac{n^2-3n}{2} & (n-1)(n-2) & n^2-3n & \frac{(n-1)(n-2)}{2} & n-1 & 2n-2 & 1 \end{pmatrix}.$$

Therefore for the graph energy of $A(n,3)$, we have

$$\begin{aligned} E(A(n,3)) &= \sum_{i=1}^{n(n-1)} \left|\lambda_i(A(n,3))\right| \\ &= (n^3-6n^2+8n-1)|-3| + (\frac{n^2-3n}{2})|n-7| \\ &+ ((n-1)(n-2))|n-6| \\ &+ (n^2-3n)|n-4| + (\frac{(n-1)(n-2)}{2})|n-3| + (n-1)|2n-9| \\ &+ (2n-2)|2n-6| + |3n-9|. \end{aligned}$$

The result is obtained for two cases $4 \leq n \leq 6$ and $n \geq 7$ from simplifications of the above relation. □

Theorem 3.8. *Let $A(n,k)$ be the arrangement graph on symmetric group S_n for $n \geq 5$.*
i) If $5 \leq n \leq 9$, the graph energy of $A(n,4)$ is shown in Table 1.
ii) If $n \geq 10$, then $E(A(n,4)) = \frac{1}{3}(25n^4 - 250n^3 + 731n^2 - 638n + 96)$.

Proof. Similar to the proof from Theorems 3.6 and 3.7, we have

$$E(A(n,4)) = \sum_{i=1}^{n(n-1)(n-2)(n-3)} \big|\lambda_i(A(n,4))\big|.$$

We use Mathematica software for computing and by substituting the obtained eigenvalues for $A(n,4)$, where $n \geq 5$, from Theorem 11 in [8], the result is complete. $\square$

TABLE 1. The graph energy of $A(n,4)$ for $5 \leq n \leq 9$.

n	$E(A(n,4))$
5	204
6	878
7	2532
8	5888
9	11876

Now, we investigate the minimum edge dominating energy of the arrangement graph $A(n,k)$ for $k \leq 4$. We first determine the minimum edge dominating energy of $A(n,1)$ for $n \geq 2$.

Theorem 3.9. *Let F be the minimum edge dominating set of graph $A(n,1)$. For $n \geq 2$, the minimum edge dominating energy is as follows*
i) If $n = 2$, then $EE_F(A(n,1)) = 1$,
ii) If $n = 3$, then $EE_F(A(n,1)) = 2\sqrt{2} + 1$,
iii) If $n \geq 4$, then $4(n-2) \leq EE_F(A(n,1)) < \frac{1}{2}(4n^2 - 7n - 4)$.

Proof. Since $A(n,1)$ is the complete graph, thus its eigenvalues are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$. Therefore, $E(A(n,1)) = 2(n-1)$.
For $n = 2,3$, it is easy to see that $A(2,1) \simeq K_2$ and $A(3,1) \simeq C_3$. Clearly, $EE_F(A(2,1)) = 1$. According to the proof from Case 3 in Theorem 3.4, we have $EE_F(A(3,1)) = 1 + 2\sqrt{2}$.
For $n \geq 4$, by applying Lemmas 2.4(ii) and 2.5, we get

$$\begin{aligned} EE_F(A(n,1)) &< E(A(n,1)) + |F| + 2n(r-2) \\ &\leq 2(n-1) + \lfloor \frac{n}{2} \rfloor + 2n(n-3) \\ &\leq (2n^2 - 4n - 2) + \frac{n}{2}. \end{aligned}$$

By simplifying of the above relation, the result is complete for the upper bound.
For the lower bound, by setting $E(A(n,1)) = 2(n-1)$ and the number of the positive eigenvalues $v^+ = 1$ in Lemma 2.6, the result holds. $\square$

Theorem 3.10. *Let F be the minimum edge dominating set of graph $A(n,2)$. For $n \geq 3$, the minimum edge dominating energy is as follows*
i) If $n = 3$, then $EE_F(A(n,2)) = 2(1+\sqrt{2}+\sqrt{3})$,
ii) If $n = 4$, then $24 \leq EE_F(A(4,2)) < 74$.
iii) If $n \geq 5$, then $4(2n^2-8n+3) \leq EE_F(A(n,2)) < \frac{1}{2}(8n^3-23n^2-n+8)$.

Proof. The arrangement graph $A(n,2)$ is the $2(n-2)$-regular graph with $n(n-1)$ vertices. We consider two following cases.
Case 1. If $n = 3$, then it is easy to show that $A(3,2) \simeq C_6$. Therefore, using the proof of Case 1 in Theorem 3.4, we have $EE_F(A(3,2)) = 2(1+\sqrt{2}+\sqrt{3})$.
Case 2. Assume that $n \geq 4$. By applying Lemmas 2.4(ii), 2.5 and Theorem 3.6 we get

$$\begin{aligned} EE_F(A(n,2)) &< E(A(n,2)) + |F| + 2(n^2-n)(2(n-2)-2) \\ &\leq 4(n^2-3n+1) + \lfloor \frac{n^2-n}{2} \rfloor + (2n^2-2n)(2n-6) \\ &\leq \frac{1}{2}(8n^3-23n^2-n+8). \end{aligned}$$

For the lower bound, by Lemma 2.6, we have

$$EE_F(A(n,2)) \geq 2E(A(n,2)) - 4v^+$$

in which v^+ is the number of positive eigenvalues of the adjacency matrix of graph $A(n,2)$. Thus, we compute the value v^+ by the obtained eigenvalues in [8].
For $n = 4$, we have $v^+ = 4$ and $E(A(4,2)) = 20$. This completes the result. For $n \geq 5$, $v^+ = 2n-1$ and by putting and simplifications of the relation in Lemma 2.6, the result holds. $\square$

TABLE 2. The lower and upper bounds of $EE_F(A(n,3))$ for $4 \leq n \leq 7$.

n	$EE_F(A(n,3)) \geq$	$EE_F(A(n,3)) <$
4	20	96
5	140	638
6	424	2040
7	880	4929

Theorem 3.11. *Let F be the minimum edge dominating set of graph $A(n,3)$. For $n \geq 4$, the minimum edge dominating energy is as follows*
i) For $4 \leq n \leq 7$, the lower and upper bounds $EE_F(A(n,3))$ are shown in Table 2,
ii) If $n \geq 8$, then

$$12n^3-84n^2+120n-16 \leq EE_F(A(n,3)) < \frac{1}{2}(12n^4-67n^3+81n^2+10n-12).$$

Proof. The arrangement graph $A(n,3)$ is $3(n-3)$-regular graph of order $n(n-1)(n-2)$ where $n \geq 4$. For the minimum edge dominating set of $A(n,3)$, by Lemma 2.5 we have $|F| \leq \lfloor \frac{n(n-1)(n-2)}{2} \rfloor$. Therefore, for the upper bound of $EE_F(A(n,3))$ by Theorem 3.7 and Lemma 2.4(ii), the result holds.
Let $v^+(A(n,3))$ be the number of positive eigenvalues of $A(n,3)$. For the lower bound, we compute the number of positive eigenvalues $A(n,3)$ for $n \geq 4$. Using the spectrum of $A(n,3)$ in [8], $v^+(A(4,3)) = 10$, $v^+(A(5,3)) = 29$, $v^+(A(6,3)) = 44$ and $v^+(A(7,3)) = 92$. For $n \geq 8$, the number of positive eigenvalues of $A(n,k)$ is equal to $3n^2-6n+1$. By applying Lemma 2.6 and Theorem 3.8, the result is complete. □

Table 3. The upper bound of $EE_F(A(n,4))$ for $5 \leq n \leq 9$.

n	$EE_F(A(n,4)) \geq$	$EE_F(A(n,4)) <$
5	228	744
6	1140	5378
7	3352	19752
8	8656	53768
9	18840	122252
10	26256	246488

Theorem 3.12. *Let F be the minimum edge dominating set of graph $A(n,4)$. For $n \geq 5$, the minimum edge dominating energy is as follows*
i) For $5 \leq n \leq 10$, the lower and upper bound of $EE_F(A(n,4))$ is shown in Table 3.
ii) For $n \geq 11$, $\frac{2}{3}(25n^4-274n^3+839n^2-746n+102) \leq EE_F(A(n,4)) < \frac{1}{6}(48n^5-451n^4+1306n^3-1169n^2+2n-192)$.

Proof. Graph $A(n,4)$ is a $4(n-4)$-regular graph with $(n^2-n)(n-2)(n-3)$ vertices. If F is the minimum edge dominating set of $A(n,3)$, then using Lemma 2.5 we have $|F| \leq \lfloor \frac{(n^2-n)(n-2)(n-3)}{2} \rfloor$.
Therefore, the upper bound of $EE_F(A(n,4))$ is obtained using Theorem

3.7 and Lemma 2.4(ii). Mathematica software is used for computing the upper bound of the minimum edge dominating energy of graph $A(n,4)$ for $5 \leq n \leq 9$.
Let $v^+(A(n,4))$ be the number of positive eigenvalues of $A(n,4)$. For the lower bound, we compute the number of positive eigenvalues $A(n,4)$ for $n \geq 5$. Using the spectrum of $A(n,4)$ in [8], $v^+(A(5,4)) = 45$, $v^+(A(6,4)) = 154$, $v^+(A(7,4)) = 428$, $v^+(A(8,4)) = 780$, $v^+(A(9,4)) = 1229$ and $v^+(A(10,4)) = 4572$. For $n \geq 11$, the number of positive eigenvalues of $A(n,k)$ is equal to $4n^3 - 18n^2 + 18n - 1$. By applying Lemma 2.6 and Theorem 3.8, the result is complete. □

In the following theorem, we determine the upper bound for the minimum edge dominating energy of $A(n,k)$ where $n \geq 6$ and $k \geq 5$.

Theorem 3.13. *Let F be the minimum edge dominating set of the arrangement graph $A(n,k)$ where $n \geq 6$ and $k \geq 5$. Then,*

$$E_F(A(n,k)) \leq \frac{n!}{2(n-k)!}\Big(4k(n-k) - 3\Big).$$

Proof. Since graph $A(n,k)$ is a $k(n-k)$-regular graph of order $N = \frac{n!}{(n-k)!}$, the number of edges in $A(n,k)$ is $m = \frac{k(n-k)}{2}\Big(\frac{n!}{(n-k)!}\Big)$. Using Lemmas 2.2 and 2.5, we get

$$\begin{aligned} EE_F(A(n,k)) &\leq 4m - 2N + |F| \\ &= 4\Big(\frac{k(n-k)}{2}\Big)\Big(\frac{n!}{(n-k)!}\Big) - 2\Big(\frac{n!}{(n-k)!}\Big) + \frac{1}{2}\Big\lfloor\frac{n!}{(n-k)!}\Big\rfloor \\ &= \Big(\frac{n!}{(n-k)!}\Big)\big(2k(n-k) - \frac{3}{2}\big). \end{aligned}$$

By simplifying of the above relation, the result is complete. □

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