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ON LEFT r-CLEAN BIMODULES

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ABSTRACT. Let R be an associative ring with identity and M an R-bimodule. We introduce the generalization of r-clean rings called left r-clean R-bimodules, defined without their endomorphism rings. An R-bimodule M is said to be left r-clean if each element is the sum of a left idempotent and a left regular element of M. We present some properties of the left r-clean R-bimodule. At the end of this paper, we give the sufficient and necessary condition for an R-bimodule to form a left r-clean R-bimodule.

1. INTRODUCTION

Research on clean rings has been introduced in [16]. A ring with identity is considered clean if each element can be written as the sum of an idempotent and a unit. Some follow-up studies regarding clean rings are presented in [9, 10, 13]. Research related to clean rings that have been done previously is applied to an associative ring with identity (not necessarily commutative). In [3, 15], we find results related to clean rings in the case of commutative rings. Research related to this clean ring has constantly been evolving until now. Several researchers have generalized the definition of a clean ring as presented in [1, 5, 19, 20].

A study about the ring endomorphism of the module, which is a clean ring, has been done in [17]. The concept of a clean endomorphism ring by Nicholson, et al [17] is used to define the idea of a clean module [8]. An R-module M is said to be clean if the endomorphism ring of M is clean. Furthermore, some concepts related to the generalization

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of clean modules are always defined through the endomorphism ring of the module, as presented in [7, 14, 18, 23].

One of the generalizations of the clean ring is the r-clean ring. According to [5], a ring R is called an r-clean ring if each element can be written as the sum of an idempotent and a von Neumann regular element. Referring to [12], an element $a \in R$ is said to be von Neumann regular if $a = ara$ for an element $r \in R$. Every unit of R is a von Neumann regular element. Thus, the r-clean ring concept is a generalization of the clean ring concept.

On the other hand, in a ring, every idempotent is a regular element. So, the definition of the r-clean element in [5] can be viewed as the sum of two regular elements in the ring. Furthermore, in [5], the author also presents some relationships between r-clean and clean rings and related properties of r-clean rings. The author continues this research on the r-clean ring in [6], showing that the clean ring and the r-clean are equivalent in the Abelian ring. We recall that a ring R is considered Abelian if all idempotents are central.

Some definitions of the ring structure have been brought into the module structure. According to [2], an element $m \in M$ is called an idempotent (resp. von Neumann regular) element if there exists $a \in (Ra :_R M)$ (resp. $a \in (Ra :_R M)$ and $r \in R$) such that $m = am$ (resp. $m = arm$). This definition of idempotent and von Neumann regular elements in [2] applies to the left unitary module of a commutative ring. This commutative property is used when a module's definition of idempotent and regular elements is returned to the ring R as R-modules. The definition of idempotent and von Neumann's regular elements in a module can be seen as a generalization of the definition of idempotent and von Neumann's regular elements in a ring.

This paper introduces the generalization of r-clean rings called left r-clean R-bimodule. We give a different approach to define them without using their endomorphism ring. Therefore, some of the properties in the r-clean ring in [5, 6] can be carried over to this R-bimodule structure. In Section 2 of this paper, we study left r-clean R-bimodules. An R-bimodule M is said to be left r-clean if each element is the sum of a left idempotent and a left von Neumann regular element of M. According to [2], we define an element $m \in M$ as a left idempotent (resp. left von Neumann regular) element if there exists an $a \in (RmR :_R M)_l$ (resp. $a \in (RmR :_R M)_l$ and $r \in R$) such that $m = am$ (resp. $m = arm$). We show that each left idempotent and left von Neumann regular element of R-bimodule M coincide. So, the definition of the left r-clean element of R-bimodule M can be seen as the sum of the two left von Neumann regular elements (or two left idempotents) of M. Moreover, we also present some properties of the left r-clean R-bimodule and give some examples.

Since there exists an R-bimodule which is not r-clean but has some left r-clean elements, in Section 3 we study left r-clean R-bisubmodules. An R-bisubmodule P of M is said to be left r-clean if every element $a \in P$ can be written as the sum of two left von Neumann regular elements (or two left idempotents) of M. We present some left r-clean bisubmodules properties as the generalization of clean ideal in [11] and r-clean ideal in [22]. We give the sufficient condition that infinite sums and unions of the family of sets left r-clean bisubmodule to be left r-clean. Moreover, we give the sufficient condition that every proper left r-clean bisubmodule is contained in the maximal left r-clean bisubmodule. At the end of this paper, we give the sufficient and necessary condition for an R-bimodule to be left r-clean.

Throughout this article, we assume R to be an associative ring with identity and M an R-bimodule unless otherwise stated. Furthermore, a left von Neumann regular element of R-bimodule M is simply called a left regular element. Consider Λ as the index set with $\Lambda = \mathbb{N}$, $Id_R(M)_l$ as the set of all left idempotents of M, $Reg_R(M)_l$ as the set of all left regular elements of M, and $(X :_R M)_l$ as the set left annihilator of an R-bimodule M/X.

2. Left r-Clean Bimodules

Let M be an R-bimodule and element $m \in M$. We define an R-bisubmodule generated by m as the set

$$RmR := \Big\{ \sum_{i=1}^{n} r_i m s_i \mid n \in \mathbb{N}, r_i, s_i \in R \; \forall a \leq i \leq n \Big\}.$$

So, we have the set left annihilator of an R-bimodule M/RmR is

$$(RmR :_R M)_l := \{a \in R \mid aM \subseteq RmR\}.$$

According to [2], when R is a commutative ring with identity, and M left unitary R-module, the element $a \in M$ is called an idempotent if it satisfies $a = xa$ for an element $x \in (Rx :_R M)_l$. Using this definition, we define the left idempotent of an R-bimodule.

Definition 2.1. Element $m \in M$ is called a left idempotent if there exists an element $a \in (RmR :_R M)_l$ such that $m = am$.

When the ring R is viewed as an R-bimodule, we can show that the left idempotent of R-bimodule R is a generalization of the idempotent of R. Moreover, let e be an idempotent of R. Then, $-e$ is not necessarily an idempotent of R. However, this property holds in R-bimodules.

Proposition 2.2. *Element $e \in M$ is a left idempotent if and only if $-e$ is also a left idempotent of M.*

Proof. Let $e \in M$ be a left idempotent of M. It means that there exists $r \in (ReR :_R M)_l$ such that $e = re$. Let $-1_R \in R$, we obtain $-e = r(-e)$. Since $ReR = R(-e)R$, $r \in (R(-e)R :_R M)_l$. So, $-e$ is a left idempotent of M. Conversely, let $-e \in M$ be a left idempotent of M. It means that there exists $r \in (R(-e)R :_R M)_l$ such that $-e = r(-e)$. Let $-1_R \in R$, we have $e = re$. Since $ReR = R(-e)R$, $r \in (ReR :_R M)_l$. So, e is a left idempotent of M. □

According to [2], for any left unitary module M over a commutative ring R, $a \in M$ is called a von Neumann regular element if there exists $x \in (Rx :_R M)_l$ and $y \in R$ such that $a = xya$. We define the left von Neumann regular element of an R-bimodule using this definition.

Definition 2.3. Element $m \in M$ is called a left von Neumann regular element if there exists an element $a \in (RmR :_R M)_l$ and $b \in R$ such that $m = abm$.

Moreover, throughout this article, a left von Neumann regular element of R-bimodule M is simply called a left regular element. When the ring R is viewed as an R-bimodule, we can show that the left regular element in an R-bimodule R is a generalization of the regular element of R.

In a ring, we know that every idempotent is a regular. However, in an R-bimodule, it turns out that every left idempotent and regular element coincides.

Proposition 2.4. *Element $e \in M$ is a left idempotent if and only if e is a left regular element of M.*

Proof. Let $e \in M$ be a left idempotent. Then, there exists an element $r \in (ReR :_R M)_l$ such that $e = re$. From $e = re$, we obtain $e = re = r(re) = rre$. Thus, e is the left regular element of M. Conversely, let e be a left regular element of M. Then, there exists an element $b \in R$ and $a \in (ReR :_R M)_l$ such that $e = abe$. From $a \in (ReR :_R M)_l$, we have $aM \subseteq ReR$. As a result, $a(bM) \subseteq aM \subseteq ReR$. So, we get $ab \in (ReR :_R M)_l$. Hence, e is the left idempotent of M. □

Now, we define a left r-clean element of an R-bimodule.

Definition 2.5. An element x of an R-bimodule M is said to be left r-clean if it can be written as the sum of a left idempotent and a left regular element in M.

Since the left idempotent of R-bimodule coincides with the left regular element, the definition of the left r-clean element of R-bimodule M can be viewed as the sum of the two left regular elements (or two left idempotents) of M.

Example 2.6. The zero element of an R-bimodule M, i.e. 0_M, is a left r-clean element.

Example 2.7. Every left idempotent and left regular element of an R-bimodule is a left r-clean element.

Example 2.8. Let $M_2(\mathbb{Z}_2)$ be an $M_2(\mathbb{Z})$-bimodule. The element $\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{bmatrix}$ is a left r-clean element of $M_2(\mathbb{Z}_2)$.

Next, we define left r-clean R-bimodules.

Definition 2.9. An R-bimodule M is said to be left r-clean if every element of M is a left r-clean element.

Example 2.10. Let $\mathbb{Z}_6$ be an $\mathbb{Z}$-bimodule. Consider the set $Id_{\mathbb{Z}}(\mathbb{Z}_6)_l = Reg_{\mathbb{Z}}(\mathbb{Z}_6)_l = \mathbb{Z}_6$. Thus, every element of $\bar{x} \in \mathbb{Z}_6$ can be written as $\bar{x} = \bar{0} + \bar{x}$ with $\bar{0}, \bar{x} \in Reg_{\mathbb{Z}}(\mathbb{Z}_6)_l$. So we have $\mathbb{Z}_6$ is a left r-clean $\mathbb{Z}$-bimodule.

Example 2.11. For every prime number $p \in \mathbb{N}$, the $\mathbb{Z}$-bimodule $\mathbb{Z}_p$ is left r-clean $\mathbb{Z}$-bimodules.

Example 2.12. Let $\mathbb{Z}$ be an $\mathbb{Z}$-bimodule. The r-clean element of $\mathbb{Z}$ is only the element -2, -1, 0, 1, and 2. Thus, $\mathbb{Z}$ is not a left r-clean $\mathbb{Z}$-bimodule.

Example 2.13. Let $M_2(\mathbb{Z}_2)$ be an $M_2(\mathbb{Z})$-bimodule. Every element of $M_2(\mathbb{Z}_2)$ is a left regular element. Thus, $M_2(\mathbb{Z}_2)$ is a left r-clean $M_2(\mathbb{Z})$-bimodule.

In the ring R, we know that if $a \in R$ is clean, then $-a$ is not necessarily clean. However, in R-bimodule if $a \in M$ is left r-clean, then $-a \in M$ is also left r-clean.

Proposition 2.14. *Let M be an R-bimodule, and $a \in M$ the left r-clean element. Then, $-a$ is also a left r-clean.*

Proof. Let a be a left r-clean element. It means $a = x + y$ with $x, y \in Reg_R(M)_l$. So, $-x$ and $-y$ are also left regular elements, respectively. Let $-1_R \in R$. Then, we have $-a = -x + (-y)$ with $-x, -y \in Reg_R(M)_l$. Hence, $-a$ is a left r-clean element of M. $\square$

Let M be a cyclic R-bimodule. We have M is not necessarily a left r-clean R-bimodule. For example, $\mathbb{Z}$ is a cyclic $\mathbb{Z}$-bimodules, but it is not left r-clean. However, $\mathbb{Z}$ contain a left r-clean element. The following property describes every cyclic R-bimodule containing a left r-clean element.

Proposition 2.15. *Every R-cyclic bimodule contains a left r-clean element.*

Proof. Suppose that M is a cyclic R-bimodule generated by element $a \in M$. Then $M = RaR$. Thus, a is the left regular element of M. So, we have a a left r-clean element. Thus, every cyclic R-bimodule contains a left r-clean element. $\square$

When R is a simple commutative ring, every cyclic R-bimodule is a left r-clean R-bimodule.

Proposition 2.16. *Let R be a simple commutative ring. Then, every cyclic R-bimodule M is a left r-clean R-bimodule.*

Proof. Assume that $M = RpR$, for an element $p \in M$. Let non-zero element $a \in M$, we have $a = rps$ for an $r, s \in R$. Since $a \neq 0_M$ and R is simple, we have both r and s are generator of R. Thus, $RrR = R$ and $RsR = R$. For the element $1_R \in R$, there exist $x, y, w, z \in R$ such that satisfy $1_R = xry$ and $1_R = wsz$. From $a = 1_R a = xrya$, we have $rps = xrya$, so $rpswz = xryawz$. Since R is commutative, $rp = rp(wsz) = (xry)awz = awz$. Note that $rM = rRpR = R(rp)R = R(awz)R = Ra(wzR) \subseteq RaR$, so $r \in (RaR :_R M)_l$. Consequently, $xr \in (RaR :_R M)_l$. From the equation $a = xrya = (xr)ya$, a is a left regular element of M. Hence, a is a left r-clean element of M. Thus, M is a left r-clean R-bimodule. $\square$

Proposition 2.17. *Let R be an Abelian ring with identity, and M a cyclic R-bimodule with $M = RmR$ for an element $m \in M$. Every element $x \in M$, with $x = rm$ for an idempotent $r \in R$, is a left r-clean element of M.*

Let R be an assosiative ring with identity 1_R. Clearly, 1_R is an idempotent of R. We know that $R = R1_R$, so 1_R can be viewed as the generator element of R as an R-module. If $e \in R$ is an idempotent of R, $1_R - e$ is also an idempotent of R. This property is not necessarily true for arbitrary R-bimodule structures unless they are cyclic. We have the following proposition since every left idempotent of R-bimodules is left r-clean.

Proposition 2.18. *Let R be an Abelian ring with identity, and M a cyclic R-bimodule with $M = RmR$ for an element $m \in M$. If $n \in M$*

is a left r-clean element of M with $n = em$ for an idempotent $e \in R$, then $m - n$ is also a left r-clean element of M.

We recall that an R-module M is simple if $M \neq 0_M$ and M has no non-zero proper submodules. This simple notion can be carried over to the R-bimodule structure.

Definition 2.19. An R-bimodule M is said to be simple if $M \neq 0_M$ and M has no non-zero proper bisubmodules, i.e., the bisubmodules of M are just $\{0_M\}$ and M itself.

According to [4], every R-module M is a simple module if and only if $M \neq 0_M$ and all non-zero elements of M are the generators of M. We can carry this property into the R-bimodule structure, i.e., each R-bimodule M is simple if and only if $M \neq 0_M$ and all non-zero elements of M are the generators of M. Furthermore, the following shows that every simple R-bimodule is a left r-clean bimodule.

Proposition 2.20. *Every simple R-bimodule is a left r-clean bimodule.*

Proof. Assuming that M is a simple R-bimodule, every non-zero element of M generates M. Let a non-zero element $a \in M$. Thus, we have $M = RaR$. So, a is a left regular element of M. Therefore, a is a left r-clean element of M. Thus, M is a left r-clean R-bimodule. $\square$

The following property explains that the R-bimodule epimorphism preserves the left r-clean property in an R-bimodule.

Proposition 2.21. *Let M and N be R-bimodules, $f : M \to N$ the R-bimodule epimorphism, and M a left r-clean R-bimodule. Then, N is also a left r-clean R-bimodule.*

Proof. Let $y \in N$. Since f is an R-bimodule epimorphism, there exists $m \in M$ such that $y = f(m)$. Since M is a left r-clean bimodule, for each $m \in M$ can be written as $m = a+d$ with $a, d \in Reg_R(M)_l$. As f is a homomorphism, we obtain $f(m) = f(a+d) = f(a)+f(d)$. Since a is a left regular element of M, $a = cba$ for an $c \in (RaR :_R M)_l$ and $b \in R$. Since f is a homomorphism, we get $f(a) = f(cba) = cbf(a)$. Since $c \in (RaR :_R M)_l$, $cM \subseteq RaR$. As f is an epimorphism, from $f(cM) \subseteq f(RaR)$ we obtain $cN = cf(M) \subseteq Rf(a)R$. So, $c \in (Rf(a)R :_R N)_l$. Thus $f(a)$ is the left regular element of N. Similarly, $f(d)$ is a left regular element of N. Hence, N is a left r-clean R-bimodule. $\square$

Proposition 2.22. *Let M and N be R-bimodules, $f : M \to N$ the R-bimodule epimorphism, and N a left r-clean bimodule. Then, $M/Ker(f)$ is a left r-clean R-bimodule.*

Proof. Since $f : M \to N$ is an epimorphism of R-bimodule, we have $M/Ker(f) \cong N$. Since N is a left r-clean R-bimodule, $M/Ker(f)$ is also a left r-clean R-bimodule. □

Proposition 2.23. *Let M be a left r-clean R-bimodule, and N a bisubmodule of M. Then, M/N is also a left r-clean R-bimodule.*

Proof. Let the R-bimodule epimorphism $f : M \to M/N$ with $f(m) = m+N$ for every $m \in M$. Since M is a left r-clean R-bimodule, referring to Proposition 2.21 we have M/N is also a left r-clean R-bimodule. □

Proposition 2.24. *Let N and P be bisubmodules of an R-bimodule M.*

(1) *If $N + P$ is a left r-clean R-bimodule, then $N/(N \cap P)$ is a left r-clean R-bimodule.*
(2) *If $(N + P)/P$ is a left r-clean R-bimodule, then $N/(N \cap P)$ is a left r-clean R-bimodule.*
(3) *If $P \subseteq N$ and M/P is a left r-clean R-bimodule, then M/N is a left r-clean R-bimodule.*

Proposition 2.25. *Let $\{M_i\}_{i\in\Lambda}$ be the family of R-bimodules, and $\prod_{i\in\Lambda} M_i$ a left r-clean R-bimodule. Then, M_i is also a left r-clean R-bimodule for each $i \in \Lambda$.*

Proof. Let $j \in \Lambda$ and the fuction $f : \prod_{i\in\Lambda} M_i \to M_j$ with $f((m_i)_{i\in\Lambda}) = m_j$ for each $(m_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} M_i$. It is clear that f is an R-bimodule epimorphism. Referring to Proposition 2.21, since $\prod_{\alpha\in\Lambda} M_\alpha$ is a left r-clean R-bimodule, M_j is also a left r-clean R-bimodule. This is shows that M_i is a left r-clean R-bimodule for each $i \in \Lambda$. □

Proposition 2.26. *Let M_i be an R_i-bimodule for each $i \in \Lambda$, and M_i a left r-clean R_i-bimodule for each $i \in \Lambda$. Then, $\prod_{i\in\Lambda} M_i$ is also a left r-clean $\prod_{i\in\Lambda} R_i$-bimodule.*

Proof. Let $(m_i)_{i\in\Lambda} \in \prod_{i\in\Lambda} M_i$ and $j \in \Lambda$. For any $m_j \in M_j$, m_j is a left r-clean element. Hence, $m_j = x_j + e_j$ with $x_j, e_j \in Reg_{R_j}(M_j)_l$. In general, we obtain $(m_i)_{i\in\Lambda} = (x_i + e_i)_{i\in\Lambda} = (x_i)_{i\in\Lambda} + (e_i)_{i\in\Lambda}$. As x_i is a left regular element of M_i for each $i \in \Lambda$, $x_i = a_i b_i x_i$ for an $a_i \in (R_i x_i R_i :_{R_i} M_i)_l$ and $b_i \in R_i$. In general, we get $(x_i)_{i\in\Lambda} =$

$(a_ib_ix_i)_{i\in\Lambda} = (a_i)_{i\in\Lambda}(b_i)_{i\in\Lambda}(x_i)_{i\in\Lambda}$. Clearly that $(b_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} R_i$. Since for each $i \in \Lambda$ satisfy $a_iM_i \subseteq R_ix_iR_i$, we obtain

$$(a_i)_{i\in\Lambda}\prod_{i\in\Lambda} M_i \subseteq \prod_{i\in\Lambda} R_i(x_i)_{i\in\Lambda}\prod_{i\in\Lambda} R_i.$$

Hence, we have $(a_i)_{i\in\Lambda} \in (\prod\limits_{i\in\Lambda} R_i(x_i)_{i\in\Lambda}\prod\limits_{i\in\Lambda} R_i :_{\prod\limits_{i\in\Lambda} R_i} \prod\limits_{i\in\Lambda} M_i)_l$. Thus, $(x_i)_{i\in\Lambda}$ is a left regular element of $\prod\limits_{i\in\Lambda} M_i$. Similarly, $(e_i)_{i\in\Lambda}$ is a left regular element of $\prod\limits_{i\in\Lambda} M_i$. Thus, this shows that $\prod\limits_{i\in\Lambda} M_i$ is also a left r-clean $\prod\limits_{i\in\Lambda} R_i$-bimodule. □

We need this following lemma for the converse of the Proposition 2.26.

Lemma 2.27. *Let M_i be an R_i-bimodule for each $i \in \Lambda$, and $\prod\limits_{i\in\Lambda} M_i$ an $\prod\limits_{i\in\Lambda} R_i$-bimodule. For any non-zero element $(a_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} M_i$, we obtain*

$$\prod_{i\in\Lambda} R_i(a_i)_{i\in\Lambda}\prod_{i\in\Lambda} R_i = \prod_{i\in\Lambda} R_ia_iR_i.$$

Proposition 2.28. *Let M_i be an R_i-bimodule for each $i \in \Lambda$, and $\prod\limits_{i\in\Lambda} M_i$ a left r-clean $\prod\limits_{i\in\Lambda} R_i$-bimodule. Then, M_i is also a left r-clean R_i-bimodule for each $i \in \Lambda$.*

Proof. Let $(m_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} M_i$. As $\prod\limits_{i\in\Lambda} M_i$ is a left r-clean $\prod\limits_{i\in\Lambda} R_i$-bimodule, we get

$$(m_i)_{i\in\Lambda} = (e_i)_{i\in\Lambda} + (r_i)_{i\in\Lambda},$$

with $(e_i)_{i\in\Lambda}, (r_i)_{i\in\Lambda} \in Reg_{\prod\limits_{i\in\Lambda} R_i}(\prod\limits_{i\in\Lambda} M_i)_l$. Hence, $(m_i)_{i\in\Lambda} = (e_i + r_i)_{i\in\Lambda}$. Moreover, let $j \in \Lambda$. Then, $m_j = e_j + r_j$. Since $(r_i)_{i\in\Lambda}$ is a left regular element of $\prod\limits_{i\in\Lambda} M_i$,

$$(r_i)_{i\in\Lambda} = (a_i)_{i\in\Lambda}(b_i)_{i\in I}(r_i)_{i\in\Lambda} = (a_ib_ir_i)_{i\in\Lambda},$$

where $(a_i)_{i\in\Lambda} \in (\prod\limits_{i\in\Lambda} R_i(r_i)_{i\in\Lambda}\prod\limits_{i\in\Lambda} R_i :_{\prod\limits_{i\in\Lambda} R_i} \prod\limits_{i\in\Lambda} M_i)$ and $(b_i)_{i\in\Lambda} \in \prod\limits_{i\in\Lambda} R_i$. As a result, we get

$$(a_i)_{i\in\Lambda}\prod_{i\in\Lambda} M_i \subseteq \prod_{i\in\Lambda} R_i(r_i)_{i\in\Lambda}\prod_{i\in\Lambda} R_i.$$

Referring to Lemma 2.27, we obtain $\prod_{i\in\Lambda} R_i(r_i)_{i\in\Lambda}\prod_{i\in\Lambda} R_i = \prod_{i\in\Lambda} R_ir_iR_i$. So $(a_i)_{i\in\Lambda}\prod_{i\in\Lambda} M_i = \prod_{i\in\Lambda} a_iM_i \subseteq \prod_{i\in\Lambda} R_ir_iR_i$. For index $j \in \Lambda$, we get $a_jM_j \subseteq R_jr_jR_j$, so $a_j \in (R_jr_jR_j :_{R_j} M_j)_l$. Since $(r_i)_{i\in\Lambda} = (a_ib_ir_i)_{i\in\Lambda}$, we obtain $r_j = a_jb_jr_j$ with $a_j \in (R_jr_jR_j :_{R_j} M_j)_l$. Thus, a_j is a left r-clean element of M_j. Similarly, e_j is a left regular element of M_j. Hence, M_j is a left r-clean R_j-bimodule. Thus, M_i is a left r-clean R_i-bimodule for every $i \in \Lambda$. □

Let R be any ring and M a left R-module. Based on [21], the formal power series is an infinite series $\sum_{i=0}^{\infty} a_ix^i = a_0 + a_1x + \cdots + a_nx^n + \cdots$, with $a_i \in M$ for each $i = 0, 1, \cdots$. We can generalize the definition of a formal power series in a module into an R-bimodule as follows.

Definition 2.29. Let R be an arbitrary ring and M an R-bimodule. The formal power series $p(x)$ is an infinite series

$$\sum_{i=0}^{\infty} a_ix^i = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

with $a_i \in M$ for each $i = 0, 1, 2, \cdots$.

Furthermore, the element a_i is the coefficient of the formal power series $p(x)$. The element x is called the indeterminate, and the form a_ix^i is called the ith term of the formal power series $p(x)$.

Proposition 2.30. *The set of all formal power series with indeterminate x whose coefficients are elements of R-bimodule M, i.e., the set $M[[x]] = \left\{ \sum_{i=0}^{\infty} a_ix^i \mid a_i \in M \ \forall i = 0, 1, 2, \cdots \right\}$, is an R-bimodule with scalar addition and multiplication operations as follows:*

(1) $\sum_{i=0}^{\infty} a_ix^i + \sum_{i=0}^{\infty} b_ix^i = \sum_{i=0}^{\infty} (a_i + b_i)x^i$, *for each* $\sum_{i=0}^{\infty} a_ix^i, \sum_{i=0}^{\infty} b_ix^i \in M[[x]]$.

(2) $r \cdot \sum_{j=0}^{\infty} a_jx^j = \sum_{j=0}^{\infty} (ra_j)x^j$, *for each* $\sum_{j=0}^{\infty} a_jx^j \in M[[x]]$ *and* $r \in R$.

(3) $\sum_{j=0}^{\infty} a_jx^j \cdot r = \sum_{j=0}^{\infty} (a_jr)x^j$, *for each* $\sum_{j=0}^{\infty} a_jx^j \in M[[x]]$ *and* $r \in R$.

Moreover, it is clear that $\prod_{i\in\Lambda} M$ is isomorphic to $M[[x]]$.

Proposition 2.31. *Let $M_i = M$ be R-bimodules for each $i \in \Lambda$, $\prod_{i\in\Lambda} M_i$ an R-bimodule, and $M[[x]]$ an R-bimodule. Then, $M[[x]] \cong \prod_{i\in\Lambda} M_i$.*

Next, we give the necessary conditions for an R-bimodule $M[[x]]$ to be left r-clean.

Proposition 2.32. *Let M be an R-bimodule, and $M[[x]]$ a left r-clean R-bimodule. Then, M is also a left r-clean R-bimodule.*

Proof. Referring to Proposition 2.31, we have $M[[x]] \cong \prod_{i\in\Lambda} M_i$ where $M_i = M$ for each $i \in \Lambda$. By using Proposition 2.25, since $M[[x]]$ is a left r-clean R-bimodule, M is also a left r-clean R-bimodule. $\square$

In the following, we give the necessary conditions for an R-bimodule $M_n(R)$ to be left r-clean.

Proposition 2.33. *Let $M_n(R)$ be a left r-clean R-bimodule. Then, R is a left r-clean R-bimodule.*

Proof. Let the R-bimodule epimorphism

$$\begin{aligned} f : M_n(R) &\rightarrow R \\ (a_{ij}) &\mapsto f((a_{ij})) = \sum_{i,j=1}^{n} a_{ij}, \text{ for all } (a_{ij}) \in M_n(R). \end{aligned}$$

Since $M_n(R)$ is a left r-clean R-bimodule, we obtain $f(M_n(R)) = R$ is a left r-clean R-bimodule. $\square$

Proposition 2.34. *Let A and B be R-bimodule, and $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ a left r-clean R-bimodule. Then, both A and B are left r-clean R-bimodules.*

The converse of the above proposition holds if R is a commutative ring.

Proposition 2.35. *Let R be a commutative ring with identity, A and B R-bimodules, and $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ an R-bimodule. If both A and B are left r-clean R-bimodules, then T is a left r-clean R-bimodule.*

Proof. Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in T$. Since both A and B are left r-clean R-bimodules, we have $a = e_1 + f_1$ and $b = e_2 + f_2$ with $e_1, f_1 \in Id_R(A)_l$

and $e_2, f_2 \in Id_R(B)_l$. Considering that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} e_1 + f_1 & 0 \\ 0 & e_2 + f_2 \end{bmatrix} = \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} + \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}.$$

Since $e_1 \in Id_R(A)_l$ and $e_2 \in Id_R(B)_l$, there exists $r_1 \in (Re_1R :_R A)_l$ and $r_2 \in (Re_2R :_R B)_l$ such that $e_1 = r_1e_1$ and $e_2 = r_2e_2$. Consequently, we obtain $r_1A \subseteq Re_1R$ and $r_2B \subseteq Re_2R$. As R is commutative, $r_1r_2A \subseteq Re_1R$ and $r_1r_2B \subseteq Re_2R$. Thus, we get

$$r_1r_2T = \begin{bmatrix} r_1r_2A & 0 \\ 0 & r_1r_2B \end{bmatrix} \subseteq \begin{bmatrix} Re_1R & 0 \\ 0 & Re_2R \end{bmatrix} = R \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R.$$

Hence, we have $r_1r_2 \in \left(R \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R :_R T \right)_l$, so that we obtain

$$\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} = r_1r_2 \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix},$$

for some $r_1r_2 \in \left(R \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} R :_R T \right)_l$. Thus, $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix} \in Id_R(T)_l$. Similarly, $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} \in Id_R(T)_l$. Hence, T is a left r-clean R-bimodule.

□

3. Left r-Clean Bisubmodule

Let $\mathbb{Z}$ be an $\mathbb{Z}$-bimodule. We know that the set of left r-clean elements of $\mathbb{Z}$ is $K = \{-2, -1, 0, 1, 2\}$. Clearly, K is not a bisubmodule of $\mathbb{Z}$. This is the background for the emergence of research on left r-clean R-bisubmodule. This section presents the definition of the left r-clean R-bisubmodule and some of its properties.

Definition 3.1. A bisubmodule N is said to be left r-clean if every element $n \in N$ can be written as the sum of two left regular elements (or two left idempotents) of M.

Every bisubmodule of a left r-clean R-bimodule is also a left r-clean.

Example 3.2. Let $\mathbb{Z}_6$ be a $\mathbb{Z}$-bimodule. Based on Example 2.10 we have $\mathbb{Z}_6$ is a left r-clean $\mathbb{Z}$-bimodule. Thus, $\mathbb{Z}$-bisubmodule $H = \{\bar{0}, \bar{3}\}$ and $K = \{\bar{0}, \bar{2}, \bar{4}\}$ are left r-clean $\mathbb{Z}$-bisubmodule of $\mathbb{Z}_6$, respectively.

Example 3.3. Let M be an R-bimodule, and $\{N_i\}_{i\in\Lambda}$ the family of bisubmodules of M. If M is a left r-clean bimodule, then

(1) $\bigcap\limits_{i\in\Lambda} N_i$ is a left r-clean bisubmodule of M.

(2) $\sum_{i\in\Lambda} N_i$ is a left r-clean bisubmodule of M.

Example 3.4. Let M and N be R-bimodules and $f : M \to N$ the R-bimodule homomorphism. If M is a left r-clean bimodule, then

(1) $Ker(f)$ is a left r-clean bisubmodule of M.
(2) $f^{-1}(N)$ is a left r-clean bisubmodule of M.

It turns out that there exists an R-bimodule that is not a left r-clean but contains a left r-clean bisubmodule.

Example 3.5. Let M_1 be an R_1-bimodule and M_2 R_2-bimodule, M_1 is a left r-clean R-bimodule, and M_2 is not a left r-clean R-bimodule. We form a ring $R = R_1 \times R_2$ and an R-bimodule $M = M_1 \times M_2$. It is clear that M is not left r-clean. Let R-bisubmodule $P = M_1 \times \{0_{M_2}\}$ of M. We will show that P is left r-clean. Let $(x, 0_{M_2}) \in P$, so $x \in M_1$. Since M_1 is a left r-clean R_1-bimodule, we have $x = e + a$ with $e, a \in Reg_{R_1}(M_1)_l$. As a result, we get

$$(x, 0_{M_2}) = (e + a, 0_{M_2}) = (e, 0_{M_2}) + (a, 0_{M_2}).$$

Since $a \in Reg_{R_1}(M_1)_l$, $a = cda$ with $c \in (R_1aR_1 :_{R_1} M_1)_l$ and $d \in R_1$. So $cM_1 \subseteq R_1aR_1$. Notice that

$$(a, 0_{M_2}) = (cda, 0_{M_2}) = (c, 0_{R_2})(d, 0_{R_2})(a, 0_{M_2}).$$

Obviously $(d, 0_{R_2}) \in R$. Then, notice that

$$(c, 0_{R_2})M = (cM_1, 0_{M_2}) \subseteq (R_1aR_1, 0_{M_2}) = R(a, 0_{M_2})R.$$

Thus, we obtain $(c, 0_{R_2}) \in (R(a, 0_{M_2})R :_R M)_l$, so $(a, 0_{M_2})$ is a left regular element of M. Similarly, $(e, 0_{M_2})$ is a left regular element of M. Thus P is a left r-clean bisubmodule of M.

Furthermore, we give some properties of a left r-clean bisubmodule over arbitrary R-bimodules.

Proposition 3.6. *Let M be an R-bimodule, and $\{N_i\}_{i\in\Lambda}$ the family of R-bisubmodules of M. If N_i is a left r-clean R-bisubmodule for each $i \in \Lambda$, then $\bigcap_{i\in\Lambda} N_i$ is a left r-clean R-bisubmodule of M.*

Proof. Since $\{N_i\}_{i\in\Lambda}$ is the family of bisubmodules of M, $\bigcap_{i\in\Lambda} N_i$ is a bisubmodule of M. Let $a \in \bigcap_{i\in\Lambda} N_i$, then $a \in N_i$ for every $i \in \Lambda$. Since N_i is a left r-clean bisubmodule for each $i \in \Lambda$, we obtain $a = x + y$ with $x, y \in Reg_R(M)_l$. Thus, $\bigcap_{i\in\Lambda} N_i$ is a left r-clean bisubmodule of M. □

Next, we define the ascending chain condition on R-bisubmodules and the Noetherian R-bimodule.

Definition 3.7. An R-bimodule M satisfies the ascending chain condition on bisubmodules if, for any chain $M_1 \subseteq M_2 \subseteq M_2 \subseteq \cdots$ of bisubmodules of M, there exists a natural number n such that $M_k = M_n$ for all $k \geq n$, i.e., the chain is eventually constant.

Definition 3.8. An R-bimodule M is Noetherian if M satisfies the ascending chain condition on bisubmodules of M.

Let $\{N_i\}_{i\in\Lambda}$ be the family of bisubmodules of M. Considering the set $\sum\limits_{N_i\in\Lambda} N_i = \left\{ \sum\limits_{n_i\in N_i} n_i \mid \ n_i = 0 \text{ except for finitely many indices } i \in \Lambda \right\}$. Obviously $\sum\limits_{i\in\Lambda} N_i$ is also a bisubmodule of M. However, if N_i is a left r-clean bisubmodule of M for all $i \in \Lambda$, $\sum\limits_{i\in\Lambda} N_i$ is not necessarily a left r-clean bisubmodule of M unless we give an additional condition as follows.

Proposition 3.9. *Let M be a Noetherian R-bimodule, and $\mathcal{J} = \{N_i\}_{i\in\Lambda}$ the family of bisubmodules of M that satisfies the ascending chain condition. If N_i is a left r-clean bisubmodule for each $N_i \in \mathcal{J}$, then $\sum\limits_{N_i\in\mathcal{J}} N_i$ is also a left r-clean bisubmodule of M.*

Proof. Clearly $\sum\limits_{N_i\in\mathcal{J}} N_i$ is a bisubmodule of M. Since every bisubmodule of M in $\mathcal{J}$ satisfies the ascending chain condition, and M is Noetherian, there exists a natural number k such that $N_i = N_k$ for each $i \geq k$. Consequently we have $\sum\limits_{N_i\in\mathcal{J}} N_i = N_k$. Moreover, since N_k is a left r-clean bisubmodule of M, $\sum\limits_{N_i\in\mathcal{J}} N_i$ is a left r-clean bisubmodule of M. □

Let $\{N_i\}_{i\in\Lambda}$ be the family of left r-clean bisubmodules of M. Then, $\bigcup\limits_{i\in\Lambda} N_i$ is not necessarily left r-clean bisubmodules of M. However, $\bigcup\limits_{i\in\Lambda} N_i$ forms a left r-clean bisubmodule if we give an additional condition as follows.

Proposition 3.10. *Let M be a Noetherian R-bimodule, and $\mathcal{J} = \{N_i\}_{i\in\Lambda}$ the family of bisubmodules of M that satisfies the ascending* *chain condition. If N_i is a left r-clean bisubmodule for each $N_i \in \mathcal{J}$, then $\bigcup\limits_{N_i \in \mathcal{J}} N_i$ is also a left r-clean bisubmodule of M.*

Proof. Since $\mathcal{J} = \{N_i\}_{i\in\Lambda}$ is the family of bisubmodules of M satisfies the ascending chain condition and M is Noetherian, there exists a natural number k such that $N_i = N_k$ for each $i \geq k$. Consequently, we have $\bigcup\limits_{N_i \in \mathcal{J}} N_i = N_k$. Moreover, since N_k is a left r-clean bisubmodule of M, $\bigcup\limits_{N_i \in \mathcal{J}} N_i$ is also a left r-clean bisubmodule of M. □

The following proposition explains that the R-bimodule homomorphism preserves the image of the left r-clean bisubmodule.

Proposition 3.11. *Let M and K be R-bimodules, and N a bisubmodule of M. If $\alpha : M \to K$ is an R-bimodule homomorphism and N is a left r-clean bisubmodule of M, then $\alpha(N)$ is a left r-clean bisubmodule of $\alpha(M)$.*

Proof. Let $y \in \alpha(N)$. There exists $n \in N$ such that $y = \alpha(n)$. Considering that N is a left r-clean bisubmodule of M, we obtain $n = a + b$ with $a, b \in Reg_R(M)_l$. Moreover, since α is a homomorphism, we have $y = \alpha(n) = \alpha(a + b) = \alpha(a) + \alpha(b)$. Since $a, b \in Reg_R(M)_l$, it is clear that $\alpha(a), \alpha(b) \in Reg_R(\alpha(M))_l$. So, y is a left r-clean bisubmodule of $\alpha(M)$. Hence, $\alpha(N)$ is a left r-clean bisubmodule of $\alpha(M)$. □

Proposition 3.12. *Let M and N be R-bimodules, and $f : M \to N$ the R-bimodule epimorphism. If P is a left r-clean bisubmodule of M, then $f(P)$ is also a left r-clean bisubmodule of N.*

Below we give some properties derived from the Proposition 3.12.

Proposition 3.13. *Let P be an R-bisubmodule of M, N a bisubmodule of M that contained in P, and $f : M \to M/N$ the R-bimodul epimorphism. If P is a left r-clean bisubmodule of M, then P/N is also a left r-clean bisubmodule of M/N.*

Proposition 3.14. *Let M_i be an R-bimodule, and P_i a bisubmodule of M_i for each $i \in \Lambda$. If $\prod_{i\in\Lambda} P_i$ is a left r-clean bisubmodule of $\prod_{i\in\Lambda} M_i$, then P_i is a left r-clean bisubmodul of M_i for each $i \in \Lambda$.*

The converse of Proposition 3.14 is not necessarily true. See the following example.

Example 3.15. Let $\mathbb{Z}_6$ be a $\mathbb{Z}$-bimodule, $H = \{\bar{0}, \bar{2}, \bar{4}\}$ and $K = \{\bar{0}, \bar{3}\}$ bisubmodules of $\mathbb{Z}_6$. Since $\mathbb{Z}_6$ is a left r-clean bimodule, the bisubmodules H and K are also left r-clean. However, $H \times K$ is not a left r-clean bisubmodule of $\mathbb{Z}_6 \times \mathbb{Z}_6$. Let $(\bar{0}, \bar{3}) \in H \times K$, we get

$$(\bar{0}, \bar{3}) = (\bar{0}, \bar{0}) + (\bar{0}, \bar{3}).$$

Clearly, $(\bar{0}, \bar{0})$ is a left regular element of $\mathbb{Z}_6 \times \mathbb{Z}_6$. Consider that $(\mathbb{Z}(\bar{0}, \bar{3})\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}_6 \times \mathbb{Z}_6)_l = \{0\}$, so it cannot be found elements $x \in (\mathbb{Z}(\bar{0}, \bar{3})\mathbb{Z} :_{\mathbb{Z}} \mathbb{Z}_6 \times \mathbb{Z}_6)_l$ and $y \in \mathbb{Z}$ such that satisfy $(\bar{0}, \bar{3}) = xy(\bar{0}, \bar{3})$. So, $(\bar{0}, \bar{3})$ is not a left regular element of $\mathbb{Z}_6 \times \mathbb{Z}_6$. Thus, $H \times K$ is not a left r-clean $\mathbb{Z}$-bisubmodule.

Definition 3.16. A proper left r-clean bisubmodule X of M is maximal if no other proper left r-clean bisubmodule of M contains X.

We can show that every proper left r-clean R-bisubmodule is contained in the maximal left r-clean R-bisubmodule if the R-bimodule satisfies certain conditions.

Proposition 3.17. *Let $M \neq 0_M$ be a Noetherian finitely generated R-bimodule. Every proper left r-clean bisubmodule of M is contained in a maximal left r-clean bisubmodule of M.*

Proof. Assume that P is a left r-clean bisubmodule of M. Let $\mathfrak{J}$ be the set of all proper left r-clean bisubmodules of M containing P. It is clear that $\mathfrak{J} \neq \emptyset$ because $P \in \mathfrak{J}$. Using Zorn's Lemma, we show that $\mathfrak{J}$ has maximum elements. Equivalent to proving that every non-empty chain of $\mathfrak{J}$ has an upper bound in $\mathfrak{J}$. Let non-empty chain $\mathfrak{G} \subseteq \mathfrak{J}$. Assume that $A_1 \in \mathfrak{G}$ is a maximal left r-clean bisubmodule that contains P. Clearly, $\mathfrak{G}$ has an upper bound in $\mathcal{J}$. However, if $A_1 \in \mathfrak{G}$ is not maximal left r-clean bisubmodule that contain P, then there exists $A_2 \in \mathfrak{G}$ such that $A_1 \subseteq A_2$. If A_2 is maximal, then $\mathfrak{G}$ has an upper bound in $\mathcal{J}$. However, if $A_2 \in \mathfrak{G}$ is not maximal left r-clean bisubmodule that contain P, then there exists $A_3 \in \mathfrak{G}$ such that $A_1 \subseteq A_2 \subseteq A_3$. If A_3 is maximal, then $\mathfrak{G}$ has an upper bound in $\mathcal{J}$. However, if $A_3 \in \mathfrak{G}$ is not maximal left r-clean bisubmodule that contain P, then there exists $A_4 \in \mathfrak{G}$ such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$. This process is continued until $\mathfrak{G}$ satisfies the ascending chain condition of bisubmodules in $\mathfrak{G}$, that is

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq \cdots.$$

Next, we form $Q = \bigcup_{A_i \in \mathfrak{G}} K$. Since M is a Noetherian R-bimodule, based on Proposition 3.10, we have Q is a left r-clean bisubmodule of M. Moreover, we can prove that Q is a proper bisubmodule of M. Since M is finitely generated, a finite set X of M generates M. Suppose that Q is not a proper bisubmodule of M. That means $Q = M$. Consequently, there are bisubmodule $A_k, A_{k+1}, \cdots, A_{k+n-1} \in \mathfrak{G}$ such that $X \subseteq \bigcup_{i=k}^{k+n-1} A_i$. Since X is a generating set of M, M can be viewed as the smallest bisubmodule containing X. In other words, if $\bigcup_{i=k}^{k+n-1} A_i$ contain X, then $M \subseteq \bigcup_{i=k}^{k+n-1} A_i$. Since $A_k, A_{k+1}, \cdots, A_{k+n-1} \in \mathfrak{G}$, there exists the largest bisubmodule $A_\alpha \in \mathfrak{G}$ such that $A_k, A_{k+1}, \cdots, A_{k+n-1} \subseteq A_\alpha$. Thus, $M \subseteq \bigcup_{i=k}^{k+n-1} A_i \subseteq A_\alpha$. That means $A_\alpha = M$, so A_α is not a proper bisubmodule of M, a contradiction. Hence, Q is a proper bisubmodule of M. Moreover, since every bisubmodules $A_i \in \mathfrak{G}$ contain P, it is obvious that Q also contains P. Thus, $Q \in \mathfrak{J}$ and Q is an upper bound for $\mathfrak{G}$. So, every non-empty chain of $\mathfrak{J}$ has an upper bound at $\mathfrak{J}$. Therefore, according to Zorn's Lemma, there exists a left r-clean bisubmodule $P^* \in \mathfrak{J}$ which is the maximum among all left r-clean bisubmodules at $\mathfrak{J}$. Thus, the left r-clean bisubmodule P is contained in the maximal left r-clean bisubmodule P^* of M. □

Let M be a left R-module. We recall that an element $u \in M$ is a unit if u is the generator of M, i.e., $M = Ru$. We can bring this definition to R-bimodule structure. An element $u \in M$ is a unit if u is the generator of an R-bimodule M, i.e., $M = RuR$. Every unit of R-bimodule M is a left r-clean element. In the following, we give the sufficient and necessary condition for an R-bimodule to be left r-clean.

Proposition 3.18. *Let M be an R-bimodule. Then, M is a left r-clean R-bimodule if and only if every proper R-bisubmodule of M is a left r-clean R-bisubmodule.*

Proof. Since every bisubmodule of a left r-clean R-bimodule is r-clean, it is clear that every proper R-bisubmodule of M is a left r-clean R-bisubmodule. Conversely, let any $m \in M$. If m is a unit, it is clear that m is a left r-clean element of M. If m is not a unit, RmR is a proper bisubmodule of M. Based on the hypothesis, RmR is a left r-clean R-bisubmodule of M. Since $m \in RmR$, we obtain m is a left r-clean element of M. Thus, M is a left r-clean R-bimodule. □

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