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ON WEAKLY NIL-SEMICOMMUTATIVE RINGS

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Abstract. We introduce the concept of weakly nil - semicommutative or WNSC rings and provide a condition that establishes the equivalence of WNSC rings to three generalised classes of semicommutative rings. We prove the equivalence between WNSC Laurent polynomial rings and WNSC polynomial rings. We supply examples of these classes of rings by considering Nagata and Dorroh extensions. We also give a characterization for a ring of Morita context with zero pairings to be WNSC.

1. Introduction

Semicommutative rings are an interesting class of rings as they properly include reversible rings which were introduced by P.M. Cohn in [3]. A ring is called *semicommutative* if for $a, b \in R, aRb = 0$ whenever $ab = 0$ [6]. Shin [14] first mentioned this class of rings. Many generalizations of semicommutative rings have been investigated during the last three decades. We record few of them which are of interest in our study. A ring R is *weakly semicommutative* [10] if for any $a, b \in R, ab = 0$ implies $aRb \subseteq Nil(R)$, where $Nil(R)$ is the set of all nilpotent elements of R. A ring R is *nil-semicommutative-II* [2] if for any $a, b \in R, ab \in Nil(R)$, then $aRb \subseteq Nil(R)$. Another generalization of semicommutative ring is the class of *nil-semicommutative-I* rings [11] where for any $a, b \in Nil(R)$ with $ab = 0, aRb = 0$. In this paper, we introduce a new class of rings called the *weakly nil-semicommutative rings* which are related to the class of rings that we have mentioned.

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We define a ring R as a *weakly nil-semicommutative* ring when for any $a, b \in Nil(R)$, if $ab = 0$, then $aRb \subseteq Nil(R)$. It follows naturally that subrings and direct products of weakly nil-semicommutative rings are weakly nil-semicommutative. Each of these classes of rings contains the class of reduced rings.

The following figure shows the relations between the newly defined class of rings and the rings mentioned above.

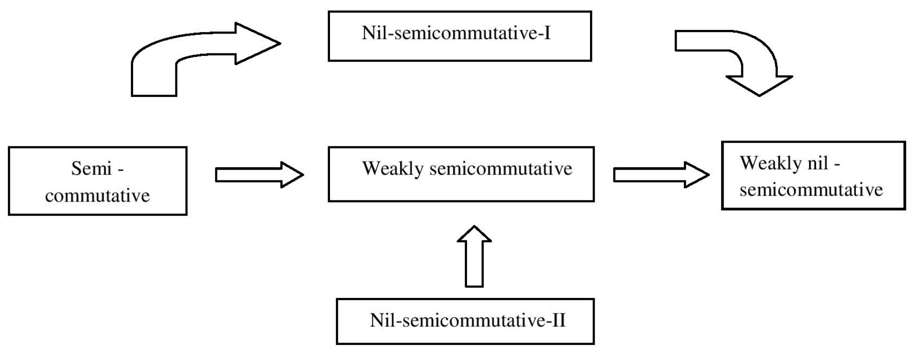

Figure 1

The reverse implications of these relations are not true. For example, in Section 2, we will see that $R = M_2(\mathbb{Z}_2)$, the 2×2 matrix ring over $\mathbb{Z}_2$ is weakly nil-semicommutative. However, it is not nil semicommutative-I as the set $Nil(R)$ is not an ideal of R [Theorem 2.5, [11]].

2. Basic Results

In this section, we collect a few basic properties of the class of weakly nil-semicommutative rings introduced in Section 1. Specifically, in Proposition 2.5, we put a condition on the ring R which establishes the equivalence of semi-commutativity and its four generalisations mentioned in the introduction. In this section, we also discuss weakly nil-semicommutativity in various ring extensions, namely the trivial extensions, the Nagata extensions and the Dorroh extensions.

Proposition 2.1. *For a ring R, if $Nil(R)$ is an ideal of R, then R is weakly nil-semicommutative.*

Proof. Suppose $a, b \in Nil(R)$. Hence as $Nil(R)$ is an ideal of R, $Rb \subseteq Nil(R)$ implying that $aRb \subseteq Nil(R)$. Hence, R is weakly nil-semicommutative. $\square$

Notice that in the proof of Proposition 2.1, it is enough to assume either $a \in Nil(R)$ or $b \in Nil(R)$.

Let us now recall that a ring R is *reduced* if zero is its only nilpotent element. Similarly an ideal of a ring is called reduced if it contains no non-zero nilpotent element.

A common observation is that we can find cases when a factor ring R/I is weakly nil-semicommutative but the ring R is not weakly nil-semicommutative. For example, the ring $R = M_3(\mathbb{Z}_2) \times \mathbb{Z}_2$ factored by the ideal $I = M_3(\mathbb{Z}_2) \times \{0\}$ is weakly nil-semicommutative, but R is not (Proposition 2.8). However, if I is either a reduced ideal or a nil-ideal, then the result holds.

Proposition 2.2. *If R is a ring and I is a reduced ideal of R such that R/I is weakly nil-semicommutative, then R is weakly nil- semicommutative.*

Proof. Let $a, b \in Nil(R)$ such that $ab = 0$. Hence there exist $k_1, k_2 \in \mathbb{N}_{>0}$ such that $a^{k_1} = 0 = b^{k_2}$. We define $\bar{R} = R/I$, $\bar{a} = a + I$, $\bar{b} = b + I$ and $\bar{r} = r + I$, for any $r \in R$. Then $\bar{a}^{k_1} = \bar{0} = \bar{b}^{k_2}$ in $\bar{R}$. Therefore $\bar{a}, \bar{b} \in Nil(\bar{R})$ such that $\bar{a}\bar{b} = \bar{0}$ in $\bar{R}$. By weak nil-semicommutativity of $\bar{R}$, $\bar{a}\bar{r}\bar{b} \in Nil(\bar{R})$, $\forall\ \bar{r} \in \bar{R}$. This implies that for each $r \in R$, there exists $k_r \in \mathbb{N}_{>0}$ such that $(arb)^{k_r} \in I$. Now $(bIa)^2 = 0$ and $bIa \subseteq I$. As I is a reduced ideal of R, we have $bIa = 0$ and $(aRbI)^2 = 0$. Also $aRbI \subseteq I$, yielding $aRbI = 0$. Now $(arb)^{k_r+1} \in arbI = 0$, i.e., $(arb)^{k_r+1} = 0$, hence $arb \in Nil(R)$, $\forall\ r \in R$. Thus R is weakly nil-semicommutative. □

An ideal J is *nil* if each of its elements is nilpotent.

Proposition 2.3. *For a nil ideal J in R, if R/J is weakly nil- semicommutative, then R is weakly nil-semicommutative.*

Proof. We define $\bar{R} = R/J, \bar{a} = a + J$ and $\bar{b} = b + J$. Let $a, b \in Nil(R)$ such that $ab = 0$. Then $\bar{a}, \bar{b} \in Nil(\bar{R})$ such that $\bar{a}\bar{b} = \bar{0}$. As $\bar{R}$ is weakly nil-semicommutative, $\bar{a}\bar{R}\bar{b} \subseteq Nil(\bar{R})$, i.e., there exists $k \in \mathbb{N}_{>0}$ such that $(aRb)^k \subseteq J \subseteq Nil(R)$. This implies $aRb \subseteq Nil(R)$. Hence R is weakly nil-semicommutative. □

By recalling that the *prime radical* of a ring R, denoted by $P(R)$ is the intersection of all the prime ideals of R, we get the following corollary.

Corollary 2.4. *If $R/P(R)$ is weakly nil-semicommutative, then R is weakly nil-semicommutative*

As observed in Figure 1, the rings that we mentioned there are weakly nil-semicommutative. A natural question is, for which class of rings will the converse also hold? As an answer to that, we have the following proposition.

Let us denote the set of all zero divisors of R as $D(R)$. Then

Proposition 2.5. *For a ring R with the property that aRb is a reduced subring of R, for all $a, b \in D(R)$, the following statements are equivalent:*
(1) R is semicommutative.
(2) R is weakly semicommutative.
(3) R is nil semicommutative-I.
(4) R is nil semicommutative-II.
(5) R is weakly nil-semicommutative.

Proof. It follows from [9, 11] and the definitions of the classes of rings involved here that

$$(1) \implies (5),\ (3) \implies (4) \implies (2).$$

Hence, we only need to show $(5) \implies (3)$ and $(2) \implies (1)$ to complete the proof.

$(5) \implies (3)$: Let $a, b \in Nil(R) \subseteq D(R)$ such that $ab = 0$. Since R is weakly nil-semicommutative, $aRb \subseteq Nil(R)$. As aRb is a reduced subring, $aRb = 0$. Thus R is nil semicommutative-I.

$(2) \implies (1)$: Let $a, b \in R$ such that $ab = 0$. Hence, $a, b \in D(R)$. Now since R is weakly semicommutative, $aRb \subseteq Nil(R)$. Since aRb is reduced for $a, b \in D(R)$, $aRb = 0$. Thus R is semicommutative. □

Proposition 2.6. *$M_2(\mathbb{Z}_2)$ is weakly nil-semicommutative.*

Proof. The set of all the nilpotent elements of $M_2(\mathbb{Z}_2)$ is

$$Nil(M_2(\mathbb{Z}_2)) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

Now it is easy to see that if $a, b \in Nil(M_2(\mathbb{Z}_2))$ such that $ab = 0$, then $a = b$. Take $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in M_2(\mathbb{Z}_2)$. Then

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c' \\ 0 & 0 \end{pmatrix} \in Nil(M_2(\mathbb{Z}_2)),$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b' & 0 \end{pmatrix} \in Nil(M_2(\mathbb{Z}_2))$$

Also,

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix} \mathit{or} \begin{pmatrix}1&1\\1&1\end{pmatrix}\in Nil(M_2(\mathbb{Z}_2))$$

Hence $M_2(\mathbb{Z}_2)$ is weakly nil-semicommutative. □

Remark 2.7. The nilradical of a weakly nil-semicommutative ring does not necessarily form an ideal, in general. In fact, it may not even form a subring. For instance in $M_2(\mathbb{Z}_2)$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\in Nil(M_2(\mathbb{Z}_2)$ but $\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}\notin Nil(M_2(\mathbb{Z}_2))$.

Proposition 2.8. *For a ring R with a non zero-divisor, $M_n(R)$ is not weakly nil-semicommutative for all $n\geq 3$.*

Proof. Let R be a ring with identity. For $1\leq i,j\leq n$, let E_{ij} denote the $n\times n$ matrix with ij-entry 1 and other entries 0. Then $E_{n2}, E_{1n}\in Nil(M_n(R))$ and $E_{n2}E_{1n}=0$. But for $E_{21}\in M_n(R)$, $E_{n2}E_{21}E_{1n}=E_{nn}\notin Nil(M_n(R))$. The proof for a ring without identity is similar where we can replace the identity element in E_{ij} with a non zero-divisor of the ring R. □

While we still cannot yet establish whether polynomial rings over weakly nil- semicommutative rings are again weakly nil- semicommutative or not, [Example 2, [8]] indicates that there are some weakly nil-semicommutative rings where this statement hold.

Let us define Ω to be a multiplicatively closed subset of a ring R consisting of central regular elements of R and $Z(R)$ be the center of R. Then we have the following result:

Proposition 2.9. *R is a weakly nil-semicommutative ring if and only if $\Omega^{-1}R$ is a weakly nil-semicommutative ring.*

Proof. The proof for the sufficient condition is straightforward as subrings of weakly nil- semicommutative rings are again weakly nil- semicommutative. Hence it is enough for us to prove the necessary condition. Let $\alpha_1=u_1^{-1}a_1, \alpha_2=u_2^{-1}a_2\in Nil(\Omega^{-1}R)$ such that $\alpha_1\alpha_2=0$. Since $\Omega\subseteq Z(R)$, we have $u_1,u_2\in\Omega$ and $a_1,a_2\in Nil(R)$. Now $0=\alpha_1\alpha_2=u_1^{-1}a_1u_2^{-1}a_2=u_1^{-1}u_2^{-1}a_1a_2=(u_1u_2)^{-1}a_1a_2$. Therefore $a_1a_2=0$. Since R is weakly nil-semicommutative, $a_1Ra_2\subseteq Nil(R)$. This implies, for every $r\in R$, there exists $k_r\in\mathbb{N}_{>0}$ such that $(a_1ra_2)^{k_r}=0$. Now for each $\beta=v^{-1}r\in\Omega^{-1}R$, where $v\in\Omega$ and $r\in R$, we have $(\alpha_1\beta\alpha_2)^{k_r}=((u_1vu_2)^{-1}a_1ra_2)^{k_r}=(u_1vu_2)^{-k_r}(a_1ra_2)^{k_r}=0$. Thus $\alpha_1\beta\alpha_2\in Nil(\Omega^{-1}R)$ for each $\beta\in\Omega^{-1}R$ and hence $\Omega^{-1}R$ is weakly nil-semicommutative. □

Corollary 2.10. *For a ring R, $R[x]$ is weakly nil-semicommutative if and only if $R[x, x^{-1}]$ is weakly nil-semicommutative.*

Proof. Let us assume $\Omega = \{1, x, x^2, ...\}$, then it is easy to see that Ω is a multiplicatively closed subset of $R[x]$ and $R[x, x^{-1}] = \Omega^{-1}R[x]$. Thus from Proposition 2.9, it follows that $R[x]$ is weakly nil-semicommutative if and only if $R[x, x^{-1}]$ is weakly nil-semicommutative. □

We now construct new examples of weakly nil -semicommutative rings with respect to Nagata and Dorroh extensions of weakly nil -semicommutative rings.

Let R be a commutative ring, M be an R-module and ρ be an endomorphism of R. The abelian group $R \oplus M$ has a ring structure with multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \rho(r_1)m_2 + r_2 m_1),$$

where $r_i \in R, m_i \in M$. This extension is called the *Nagata extension* of R by M and ρ, and is denoted by $N(R, M, \rho)$ [12].

Lemma 2.11. *For $N(R, M, \rho)$, $Nil(R \oplus M) = Nil(R) \oplus M$.*

Proof. Let $r \in R$ and $m \in M$ such that $(r, m) \in Nil(R \oplus M)$. Then, there exists $k \in \mathbb{N}_{>0}$ such that $(0, 0) = (r, m)^k = (r^k, m')$, for some $m' \in M$. This implies that $r^k = 0$, that is $r \in Nil(R)$. Thus we get $Nil(R \oplus M) \subseteq Nil(R) \oplus M$.

Conversely, let $(r, m) \in Nil(R) \oplus M$ such that $r^{k'} = 0$, where $k' \in \mathbb{N}_{>0}$. Then $(r, m)^{k'} = (0, m'')$, for some $m'' \in M$. Now

$(r, m)^{2k'} = (0, m'')(0, m'') = (0, \rho(0).m'' + 0.m'') = (0, 0)$, yielding $(r, m) \in Nil(R \oplus M)$ and therefore $Nil(R) \oplus M \subseteq Nil(R \oplus M)$.

Thus we get $Nil(R) \oplus M = Nil(R \oplus M)$. □

Remark 2.12. It may be noted that in the first part of the proof of Lemma 2.11, $m' = (\sum_{n=0}^{k-1} \rho(r)^{(k-1)-n} r^n)m = 0$. If $m \neq 0$ then we get that the homogeneous bivariate polynomial $\sum_{n=0}^{k-1} x^{(k-1)-n} y^n$ evaluated at $(\rho(r), r)$ annihilates m.

Theorem 2.13. *For a commutative ring R, $N(R, M, \rho)$ is a weakly nil-semicommutative ring.*

Proof. Let $(r_1, m_1), (r_2, m_2) \in Nil(R \oplus M)$ such that $(r_1, m_1)(r_2, m_2) = (r_1 r_2, \rho(r_1)m_2 + r_2 m_1) = (0, 0)$. Then by Lemma 2.11 we have, $r_1, r_2 \in Nil(R)$ such that $r_1 r_2 = 0$. Since R is commutative, R is trivially weakly nil-semicommutative. Hence for all $r \in R, r_1rr_2 \in Nil(R)$. Now let (r, m) be any arbitrary element of $(R \oplus M)$. Then $(r_1, m_1)(r, m)(r_2, m_2) = (r_1rr_2, m') \in Nil(R) \oplus M$, for some $m' \in M$. Hence by Lemma 2.11, $(r_1, m_1)(r, m)(r_2, m_2) \in Nil(R \oplus M)$. Thus $N(R, M, \rho)$ is weakly nil-semicommutative. ☐

Let R be a ring and ${}_RN_R$ be a bimodule which itself is a general ring (not necessarily with unity) such that $(mn)r = m(nr), (mr)n = m(rn)$ and $(rm)n = r(mn)$ holds for all $m, n \in N$ and $r \in R$. Then the abelian group $R \oplus N$ has the structure of a ring with multiplication given by:

$$(r, m)(s, n) = (rs, rn + ms + mn)$$

where $r, s \in R$, and $m, n \in N$. This extension is called the *Dorroh extension* $D(R; N)$ of R by N [4, 5].

Proposition 2.14. *For $D(R; N)$, N is a nil ring if and only if*

$$Nil(D(R; N)) = Nil(R) \oplus N.$$

Proof. Assume that N is a nil ring. Let $(r, n) \in Nil(D(R; N))$. Then there exists $k \in \mathbb{N}_{>0}$ such that $(0, 0) = (r, n)^k = (r^k, n')$, for some $n' \in N$. This yields that $r \in Nil(R)$ and so $(r, n) \in Nil(R) \oplus N$. Hence $Nil(D(R; N)) \subseteq Nil(R) \oplus N$.

To prove the other inclusion, let $(r, n) \in Nil(R) \oplus N$ such that $r^{k'} = 0$, where $k' \in \mathbb{N}_{>0}$. Then $(r, n)^{k'} = (r^{k'}, n') = (0, n')$, for some $n' \in N$. Since N is a nil ring, there exists some $s \in \mathbb{N}$ such that $n'^s = 0$. Now taking the s^{th} power of $(0, n')$ we have, $(0, n')^s = (0, n'^s) = (0, 0)$. Thus we obtain that $(r, n)^{k's} = (0, 0)$, that is, $(r, n) \in Nil(D(R; N))$ an hence finally, $Nil(R) \oplus N \subseteq Nil(D(R; N))$.

For the converse, let us assume that $Nil(D(R; N)) = Nil(R) \oplus N$. Let n be an arbitrary element of N. Then $(0, n) \in Nil(R) \oplus N = Nil(D(R; N))$. Hence, there exists some $s \in \mathbb{N}_{>0}$ such that $(0, n)^s = (0, n^s) = (0, 0)$. Therefore, $n^s = 0$ implying that n is a nilpotent element of N. Since n was chosen arbitrarily from N, thus N is a nil ring. ☐

Proposition 2.15. *For a weakly nil-semicommutative ring R and a nil ring N, the Dorroh extension $D(R; N)$ is weakly nil-semicommutative.*

Proof. The proof of this proposition is similar to the proof of Theorem 2.13. ☐

3. Main results

In Section 2, we discussed in detail the weakly nil- semicommutative condition in various ring extensions. Moving ahead, we explore further in this section the weakly nil-semicommutativity in Morita contexts and we prove that the ring of Morita context with zero pairings is weakly nil-semicommutative if and only if each of the associated rings is weakly nil-semicommutative.

A *Morita context* denoted by (R, S, M, N, ξ, χ) consists of rings R, S, bimodules ${}_RN_S$, ${}_SM_R$ and a pair of bimodule homomorphisms (called pairings) $\xi : N \bigotimes_S M \to R$ and $\chi : M \bigotimes_R N \to S$ which satisfy the following associativity conditions:

$$\xi(n \bigotimes m)n' = n\chi(m \bigotimes n') \; ; \; \chi(m \bigotimes n)m' = m\xi(n \bigotimes m').$$

These conditions ensure that the set T of generalized matrices

$$\text{T=} \left\{ \begin{pmatrix} r & n \\ m & s \end{pmatrix} \mid r \in R,\ s \in S,\ m \in M,\ n \in N \right\}$$

forms a ring, called the ring of the Morita context (R, S, M, N, ξ, χ) [1].

Lemma 3.1. *Let T be the ring of a Morita context (R, S, M, N, ξ, χ) with zero pairings. Then*

$$Nil(T) = \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}$$

Proof. Let $\begin{pmatrix} r & n \\ m & s \end{pmatrix} \in Nil(T)$. This implies

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & n \\ m & s \end{pmatrix}^2 = \begin{pmatrix} r^2 & rn + ns \\ mr + sm & s^2 \end{pmatrix}$$

So, $r^2 = 0 = s^2$ i.e., $r \in Nil(R), s \in Nil(S)$. Hence $\begin{pmatrix} r & n \\ m & s \end{pmatrix} \in \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}$.

Conversely, let $\begin{pmatrix} r & n \\ m & s \end{pmatrix} \in \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}$. Then $r^{k_1} = 0$ and $s^{k_2} = 0$, for some $k_1, k_2 \in \mathbb{N}_{>0}$. Write $k = max\{k_1, k_2\}$. Then $\begin{pmatrix} r & n \\ m & s \end{pmatrix}^k = \begin{pmatrix} 0 & n' \\ m' & 0 \end{pmatrix}$, for some $n' \in N$ and $m' \in M$. Now squaring both sides we have,

$$\begin{pmatrix} r & n \\ m & s \end{pmatrix}^{2k} = \begin{pmatrix} 0 & n' \\ m' & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $\begin{pmatrix} r & n \\ m & s \end{pmatrix} \in Nil(T)$.

□

Theorem 3.2. *Let T be the ring of a Morita context (R, S, M, N, ξ, χ) with zero pairings. Then T is weakly nil-semicommutative if and only if so are R and S.*

Proof. If T is weakly nil-semicommutative then R and S are trivially weakly nil-semicommutative.

Conversely, let R and S be weakly nil-semicommutative. Consider $\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix}, \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} \in Nil(T)$ such that

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix} \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\text{i.e., } \begin{pmatrix} r_1 r_2 & r_1 n_2 + n_1 s_2 \\ m_1 r_2 + s_1 m_2 & s_1 s_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

By Lemma 3.1 and from the above equation we have, $r_1, r_2 \in Nil(R)$, $r_1 r_2 = 0$ and $s_1, s_2 \in Nil(S)$, $s_1 s_2 = 0$. Now the weak nil- semicommutativity of R and S ensure that $r_1 R r_2 \subseteq Nil(R)$ and $s_1 S s_2 \subseteq Nil(S)$. Consider any element $\begin{pmatrix} r & n \\ m & s \end{pmatrix} \in T$. Then

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix} \begin{pmatrix} r & n \\ m & s \end{pmatrix} \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix}$$

$$= \begin{pmatrix} r_1 r r_2 & r_1 r n_2 + r_1 n s_2 + n_1 s s_2 \\ m_1 r r_2 + s_1 m r_2 + s_1 s m_2 & s_1 s s_2 \end{pmatrix}$$

$$\in \begin{pmatrix} Nil(R) & N \\ M & Nil(S) \end{pmatrix}$$

Therefore by Lemma 3.1,

$$\begin{pmatrix} r_1 & n_1 \\ m_1 & s_1 \end{pmatrix} \begin{pmatrix} r & n \\ m & s \end{pmatrix} \begin{pmatrix} r_2 & n_2 \\ m_2 & s_2 \end{pmatrix} \in Nil(T).$$

Thus T is weakly nil-semicommutative. □

A *Formal triangular matrix ring* is a ring of the form

$$\begin{pmatrix} R & 0 \\ M & S \end{pmatrix} = \left\{ \begin{pmatrix} r & 0 \\ m & s \end{pmatrix} \mid r \in R, s \in S \text{ and } m \in M \right\}$$

under the usual matrix operations, where R, S are rings and ${}_S M_R$ is a bimodule [13].

Corollary 3.3. *The formal triangular matrix ring* $\begin{pmatrix} R & 0 \\ M & S \end{pmatrix}$ *is weakly nil-semicommutative if and only if so are* R *and* S.

Given a ring R and an ${}_RM_R$-bimodule, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and multiplication defined as:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$$

where $r_1, r_2 \in R, m_1, m_2 \in M$ [7].

This is isomorphic to the ring of matrices of the form $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r \in R, m \in M$ and usual matrix operations are used.

Corollary 3.4. *A ring* R *is weakly nil-semicommutative if and only if the trivial extension* $T(R, M)$ *is weakly nil-semicommutative.*

Remark 3.5. An analogue of Theorem 3.2 may or may not hold for a ring of a Morita context with non zero pairings. For instance, taking $R = S = M = N = \mathbb{Z}_2$ in T we can see from Proposition 2.6 that $M_2(\mathbb{Z}_2)$ is a weakly nil-semicommutative ring. But taking $R = S = M = N = M_2(\mathbb{Z}_2)$ in T we have

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in Nil(T)$$

$$\text{such that } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{But } \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin Nil(T)$$

Thus T is not a weakly nil-semicommutative ring in this case.

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