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ON ENUMERATION AND CLASSIFICATION OF EL^2-SEMIHYPERGROUPS AND EL^2-H_v-SEMIHYPERGROUPS WITH 2 ELEMENTS

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Abstract. EL-hypergroups were defined by Chvalina 1995. Till now, no exact statistics of EL-hypergroups have been done. Moreover, there is no classification of EL-hypergroups and EL^2-hypergroups even over small sets. In this paper, we classify all EL-(semi)hypergroups over sets with two elements obtained from quasi ordered semigroups. Also, we characterize all quasi ordered H_v-group and then we enumerate the number of EL^2-H_v-hypergroups and EL^2-hypergroups of order 2.

1. Introduction

Hypergroups were first introduced by Marty. A hypergroup is a generalization of a group. Also, Vougioklis introduced the H_v-groups as a generalization of hypergroups[26]. The first book on algebraic hyperstructures was written by Corsini[3]. Moreover, Vougioklis wrote a book on the Hv-group [26]. After that, Corsini, Leoreanu, Davvaz published some books on the applications of hyperstructures and other branches of hyperstructures.[3, 5, 7, 6].

The connection between semihypergroups and partial ordering has been started in 1960s. This relations has been introduced by Nieminen, Corsini, Rosenberg and Novak. Ends Lemma hyperstructures(EL-hyperstructures) are hyperstructures based on po(semi)groups. These

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were first investigated by Chvalina [2] and after that introduced by Rosenberg in [23], Hoskova in [13], Rackova in [22] and Novak in [16, 17, 18, 19, 20, 21]. EL-hyperstructures were generalized and extended by Ghazavi et al. [10] and called EL^2-hyperstructures. Also, Ghazavi et al. studied EL_n-hyperstructures and EL-Γ-hyperstructures in [11, 12].

The number of hypergroups of order 3 is 23192[14]. The number of hypergroups of the same order up to isomorphism is 3999 [24]. Vougiouklis in found 8 hypergroups(up to isomorphism) of order 2 and Bayon and Lygeros computed 20 H_v-groups(up to isomorphism) of order 2 [1]. Enumerating and classification of hypergroups and related hyperstructures have many significant applications in other branches of science. Corssini, Leoreanu and Davvaz [4, 7] presented some of applications of hypergroups, H_v-groups and hyperrings. The enumeration and classification of (semi)hypergroups and H_v-(semi)groups will use to study its application in different sciences.

Recently, Ghazavi and Mirvakili classified EL-hypergroups with 2 elements[9]. In this paper, we characterize all posemihypergroups and poH_v-semigroups of order 2. Then, we concentrate on posemihypergroups and poH_v-semigroups in order to find and classify all EL^2-semihypergroups and EL^2-H_v-semigroups of order 2.

2. Preliminaries

We recall some basic notions and definitions of ordered semigroups and (semi)hypergroups[3, 6].

Let $H \neq \emptyset$ and $\mathcal{P}^*(H) = \{K \subseteq H | K \neq \emptyset\}$. A hypergroupoid (hyperstructure) is a pair $(H, \circ)$ where $\circ$ is a hyperoperation, that means $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$. Let A an B be two non-empty subsets of hypergroupoid $(H, \circ)$ and $x \in H$, then we set

$$x \circ A = \{x\} \circ A,\ A \circ x = A \circ \{x\}\ and\ A \circ B = \bigcup \{a \circ b | a \in A, b \in B\}.$$

The hyperoperation $\circ$ is called associative (weak associative) if for triple $(a, b, c) \in H^3$ we have

$$a \circ (b \circ c) = (a \circ b) \circ c (a \circ (b \circ c) \cap (a \circ b) \circ c \neq \emptyset).$$

The semihypergroup $(H, \circ)$ is called hypergroup if reproduction axiom holds. It means that for every $a \in H$

$$a \circ H = H = H \circ a$$

The hypergroupoid $(H, \circ)$ is called a(an) semihypergroup (H_v-semigroup) if $\circ$ is an(a) associative (weak associative) hyperoperation. Moreover, a(an) semihypergroup (H_v-semigroup) is called a(an) semihypergroup (H_v-semigroup if for the hyperoperation $\circ$ the reproduction axiom holds.

A relation R on non-empty set H is called *quasi order* if it is reflexive and transitive. A quasi order relation R is called partially order relation if R is antisymmetric. Let $(S, \cdot)$ is a (semi)group. A triple $(S, \cdot, R)$ is called quasi ordered (semi)group, if R is a quasi order relation on S such that monotone condition holds, i.e., for all $x, y, z \in S$,

$$xRy \;\rightarrow\; (x \cdot z)R(y \cdot z) \text{ and } (z \cdot x)R(z \cdot y).$$

If R is partially order relation then $(S, \cdot, R)$ is called partially ordered (semi)group or po(semi)group.

Let $(S, \cdot, R)$ be a po(semi)group. We set $[x)_R = \{s \in S; xRs\}$ and also $[A)_R = \bigcup_{x \in A} [x)_R$. Similarly, $(x]_R = \{s \in S; sRx\}$ and $(A]_R = \bigcup_{x \in A} (x]_R$. The EL-(semi)hypergroups are (semi)hypergroups constructed from a po(semi)groups using "Ends lemma". This concept was first introduced by Chvalina in 1995 [2]. In particular, Chvalina in Theorem 3 in [2] proved that:

Lemma 2.1. *Let $(S, \cdot, R)$ be a posemigroup. Define a hyperoperation $\circ : S \times S \longrightarrow \mathcal{P}^*(S)$ by $a \circ b = [a \cdot b)_R = \{x \in S, a \cdot bRx\}$. Then $(S, \circ)$ is a semihypergroup. Moreover, $(S, \circ)$ is commutative if and only if the semigroup $(S, \cdot)$ is commutative.*

Also, Chvalina in Theorem 1.4 in [2] showed that

Theorem 2.2. *Suppose that $(S, \cdot, R)$ is a posemigroup. Then the following conditions are equivalent:*

(I) *For every $a, b \in S$ there exist $c, d \in S$ such that $(b \cdot d)Ra$ and $(c \cdot b)Ra$.*
(II) *The hyperstructure $(S, \circ)$ is a hypergroup.*

Remark 2.3. If $(S, \cdot, R)$ is a pogroup, then the condition (II) in Theorem 2.2 is valid. Therefore, $(S, \circ, R)$ is a hypergroup.

3. Main results

Now, we try to study and count all semihypergroups and H_v-semigroups, of order 2, which has EL^2-construction. As mentioned before, EL^2-hyperstructures are a family of hyperstructures constructed on quasi ordered hyperstructures and consequently we need all quasi order relations on a set with two elements.

Theorem 3.1. *Suppose $A = \{a, b\}$. Then, there are four quasi order relations on A as follows:.*

$R_1 = \{(a, a), (b, b)\}$,
$R_2 = \{(a, a), (b, b), (a, b)\}$,
$R_3 = \{(a, a), (b, b), (b, a)\}$,
$R_4 = \{(a, a), (b, b), (a, b), (b, a))\} = A \times A$.

Definition 3.2. The triple $(H, \circ, R)$ is known as a *(partially) quasi ordered* hypergroupoid provided that $(H, \circ)$ be a hypergroupoid and "R" be a (partially) quasi order relation on H and, in addition, for all $a, b, c \in H$ with the property aRb there holds $a \circ c\overline{R}b \circ c$ and $c \circ a\overline{R}c \circ b$ (monotone condition), where if A and B are non-empty subsets of H, then we define $A\overline{R}B$ whenever for all $a \in A$, there exists $b \in B$ and for all $b \in B$ there exists $a \in A$ such that aRb.

Example 3.3. Suppose $(S, \cdot, R)$ is a (partially) quasi ordered semigroup. For $(x, y) \in S^2$, define $x \circ y = \{x^i : i \in \mathbb{N}\}$. Now, it is easy to see that the monotone condition holds and therefore the triple $(S, \circ, R)$ is a (partially) quasi ordered semihypergroup.

Example 3.4. Suppose (A, R) is a (partially) quasi ordered set. Define the hyperopration "$*$" on A as $a * b = \{a, b\}$ for all $(a, b) \in A^2$. It easy to see that $(A, *, R)$ is a (partially) quasi ordered hypergroup.

Example 3.5. Look at $(H = \{x, y, z\}, \circ, R)$ where

$$R = \{(x, x), (y, y), (z, z), (x, y), (x, z), (y, z)\}$$

and hyperoperation "$\circ$" is given by the Table 1.

TABLE 1. Ordered hypergroup with 3 elements

$\circ$	x	y	z
x	x	x, y	x, z
y	x, y	y	y, z
z	x, z	y, z	z

A simple computation shows that the triple $(H, \circ, R)$ is quasi ordered hypergroup.

Definition 3.6 ([10])**.** Let $(H, \circ, R)$ be a (partially) quasi ordered hypergroupoid. For $(a, b) \in H^2$, define the new hyperoperation $*$ on H as

$$* : H \times H \longrightarrow \mathcal{P}^*(H)$$

$$a * b = [a \circ b)_R = \bigcup_{m \in a \circ b} [m)_R.$$

Remark 3.7. This hyperopration is known as the extended version of Ends Lemma.

Remark 3.8. From now on, we name $(H, *)$ as the EL^2-hypergroupoid associated to (partially) quasi ordered hypergroupoid $(H, \circ, R)$.

Theorem 3.9 ([10]). *Suppose $(S, \cdot, R)$ is a (partially) quasi ordered H_v-semigroup. Then, its associated EL^2-hyperstructure $(H, *)$ is an H_v-semigroup i.e. "$*$" is weak associative.*

Corollary 3.10 ([10]). *If $(H, \circ, R)$ is a (partially) quasi ordered H_v-group, then $(H, *)$ is an H_v-group.*

Notice that the converse of the above corollary does not hold. Look at the following example.

Example 3.11. Look at the hypergroupoid $(H = \{a, b\}, \circ)$ in Table 2. Then the triple $(H = \{a, b\}, \circ, R)$ in which $R = H \times H$ is a quasi

TABLE 2. hypergroupoid

∘	a	b
a	a	a
b	b	a

ordered hypergroupoid. Now, because $(b \circ a) \circ b \bigcap b \circ (a \circ b) = \emptyset$, the pair $(H, \circ)$ is not an H_v-group. Setting EL^2-construction on $(H = \{a, b\}, \circ, R)$, we get its associated EL^2-hyperstructure as in the Table 3. Clearly, the hypergroupoid $(H, *)$ is an H_v-group.

TABLE 3. EL^2-hypergroup

*	a	b
a	H	H
b	H	H

Theorem 3.12. *Suppose $(H, \circ, R)$ is a quasi (partially) ordered semihypergroup i.e. "$\circ$" is an associative hyperoperation and the monotone condition holds. Then, its associated EL^2-hyperstructure $(H, *)$ is a semihypergroup.*

3.1. EL^2**-semihypergroup.** .

EL^2-semihypergroups are hyperstructures constructed on a quasi (partially) ordered semihypergroup using extended version of Ends Lemma. By 3.12, if we start with a quasi (partially) ordered semihypergroup $(S, \circ, R)$ and set the EL^2-construction on it, its associatd EL^2 hyperstructure $(S, *)$ would be a semihypergroup. So, at the begining, we should know all semihypergroups of order 2. Then,

Theorem 3.13. [8] *There exist, up tp isomorphism, 17 semihypergroups of order 2 in Table 4.*

TABLE 4. Classification of the semihypergroups of order 2

$\circ_1$	a	b
a	a	a
b	a	a

$\circ_2$	a	b
a	a	a
b	a	b

$\circ_3$	a	b
a	a	a
b	b	b

$\circ_4$	a	b
a	a	b
b	b	a

$\circ_5$	a	b
a	a	b
b	a	b

$\circ_6$	a	b
a	S	a
b	a	b

$\circ_7$	a	b
a	a	S
b	a	b

$\circ_8$	a	b
a	a	S
b	b	b

$\circ_9$	a	b
a	S	b
b	b	b

$\circ_{10}$	a	b
a	S	S
b	a	b

$\circ_{11}$	a	b
a	S	a
b	S	b

$\circ_{12}$	a	b
a	S	S
b	b	b

$\circ_{13}$	a	b
a	a	S
b	S	b

$\circ_{14}$	a	b
a	S	b
b	S	b

$\circ_{15}$	a	b
a	S	S
b	S	a

$\circ_{16}$	a	b
a	a	S
b	S	S

$\circ_{17}$	a	b
a	S	S
b	S	S

Similar to EL-construction, in EL^2-construction we have to recognize and determine all triples $(S, \circ_i, R_j)$ which have the monotone condition. (i.e. those who are quasi ordered semihypergroups). So, we can see that:

Proposition 3.14. *For all* $i \in \{1, 2, \ldots, 17\}$ *and* $j \in \{1, 4\}$, *the triple* $(S, \circ_i, R_j)$ *is a quasi ordered semihypergroup.*

Proof. Let $S = \{a, b\}$. Then,

(i) For $j = 1$, the proof is straightforward, since R_1 consists diagonal pairs $(x, x), x \in S$ and product of any element in these pairs are again diagonal.

(ii) Because $R_4 = S \times S$, all possible pairs $(x \circ_i y, x \circ_i z)$ and $(y \circ_i x, z \circ_i x)$ for all $x, y, z, \in S$, are contained in R_4 and therefore the monotone condition holds.

□

Proposition 3.15. *For all* $i \in \{1, 2, 3, 5, 7, 8, 9, 12, 13, 14, 16, 17\}$ *and* $j \in \{2, 3\}$*, triples* $(S, \circ_i, R_j)$ *are quasi ordered semihypergroups.*

Proof. We focus on all of these 12 cases in separate parts. Notice that we should ignore diagonal pairs and focus on nondiagonal pairs $(b, a) \in R_3$ and $(a, b) \in R_2$.

1) The triple $(S, \circ_1, R_j)$ has monotone condition for $j \in \{2, 3\}$ since for all $(x, y) \in S^2$ it holds $x \circ_1 y = a$ and $(a, a) \in R_j$, $j \in \{2, 3\}$.
2) Look at $(b, a) \in R_3$. As $(b \circ_i x, a \circ_i x)$ and $(x \circ_i b, x \circ_i a)$, $x \in S = \{a, b\}$, are all in R_3 we can conclude that $(S, \circ_i, R_3)$ is a quasi ordered semihypergroup for all $i \in \{2, 3, 5\}$. Also, for $(a, b) \in R_2$ we can see that $(a \circ_i x, b \circ_i x) \in R_2$ and $(x \circ_i b, x \circ_i a) \in R_2$, $x \in \{a, b\}$, which means that $(S, \circ_i, R_2)$, $i \in \{2, 3, 5\}$, is a quasi ordered semihypergroup.
3) For $(b, a) \in R_3$ ($(a, b) \in R_2$) and $x \in \{a, b\}$ it holds $(b \circ_i x, a \circ_i x) \in \overline{R}_3$ and $(x \circ_i b, x \circ_i a) \in \overline{R}_3$ ($(a \circ_i x, b \circ_i x) \in \overline{R}_2$ and $(x \circ_i b, x \circ_i a) \in \overline{R}_2$) for all $i \in \{7, 8, 9, 12, 13, 14, 16, 17\}$. So, $(S, \circ_i, R_j)$ is a quasi ordered semihypegroup for $i \in \{2, 3\}$ and $i \in \{7, 8, 9, 12, 13, 14, 16, 17\}$.

□

Proposition 3.16. *For all* $i \in \{4, 6, 10, 11, 15\}$ *and* $j \in \{2, 3\}$*, the triple* $(H, \circ_i, R_j)$ *is not a quasi ordered semihypergroup.*

Proof. Consider the following five cases:

1) Since $(b, a) \in R_3$ and $(b \circ_4 b, a \circ_4 b) \notin R_3$, $(S, \circ_4, R_3)$ is not a quasi ordered semihypergroup. Also, $(S, \circ_4, R_2)$ is not a quasi ordered semihypergroup as $(a, b) \in R_2$ but $(a \circ_4 b, b \circ_4 b) \notin R_2$.
2) $(S, \circ_6, R_3)$ is not a quasi ordered semihypergroup because $(a, b) \in R_2$ but $(b \circ_6 a, a \circ_6 a) \notin \overline{R}_3$. Similarly, $(S, \circ_6, R_2)$ is not a quasi ordered semihypergroup because $(a, b) \in R_2$ but $(a \circ_6 a, b \circ_6 a) \notin \overline{R}_2$.
3) $(S, \circ_{10}, R_3)$ is not a quasi ordered semihypergroup because $(b, a) \in R_2$ but $(b \circ_{10} a, a \circ_{10} a) \notin \overline{R}_3$. Similarly, $(S, \circ_{10}, R_2)$ is not a quasi ordered semihypergroup because $(a, b) \in R_2$ but $(a \circ_{10} a, b \circ_{10} a) \notin \overline{R}_2$.

4) $(S, \circ_{11}, R_3)$ is not a quasi ordered semihypergroup because $(b, a) \in R_3$ but $(a \circ_{11} b, a \circ_{11} a) \notin \overline{R}_3$. Similarly, $(S, \circ_{11}, R_2)$ is not a quasi ordered semihypergroup because $(a, b) \in R_2$ but $(a \circ_{11} a, a \circ_{11} b) \notin \overline{R}_2$.
5) $(S, \circ_{15}, R_3)$ is not a quasi ordered semihypergroup because $(b, a) \in R_3$ but $(b \circ_{15} b, a \circ_{15} b) \notin \overline{R}_3$. Similarly, $(S, \circ_{15}, R_2)$ is not a quasi ordered semihypergroup because $(a, b) \in R_2$ but $(a \circ_{15} b, b \circ_{15} b) \notin \overline{R}_2$.

□

Now, regarding Propositions 3.14, 3.15 and 3.16, we have:

Corollary 3.17. *There exist 58 quasi ordered semihypergroups with 2 elements.*

Definition 3.18. The semihypergroup $(S, *)$ is said to be a non-trivial semihypergroup if it is not total semihypergroup (i.e. $a * b = H$ for all $(a, b) \in H$) nor it is not associated to $(H, \circ_i, R_1)$, $i \in \{1, 2, \cdots, 17\}$ in EL^2-construction.

Theorem 3.19. *There exist only 5 non-trivial EL^2-semihypergroups of order 2. ($(S, \circ_i)$ has the EL^2-construction for $i \in \{1, 9, 12, 14, 16\}$.)*

Proof. At first look at the quasi ordered semihypergroups founded in 3.14. Clearly, $(S, \circ_i, R_1)$ tends to $(S, \circ_i)$, for all $i = 1, 2, \cdots, 17$, via EL^2-construction, which are trivial by Definition 3.18. Also, for all $i \in \{1, 2, \cdots, 17\}$, the triple $(H, \circ_i, R_4)$ leads to total hypergroup. In addition, focus on 24 non-trivial ones founded in 3.15. We have:

1) Setting EL^2-construction on $(S, \circ_1, R_3)$, we can achieve $(S, \circ_1)$.
2) Setting EL^2-construction on $(S, \circ_9, R_2)$, we can achieve $(S, \circ_9)$.
3) Setting EL^2-construction on $(S, \circ_5, R_3)$, $(S, \circ_7, R_3)$, $(S, \circ_3, R_2)$, $(S, \circ_8, R_2)$ and $(S, \circ_{12}, R_2)$ we can achieve $(S, \circ_{12})$.
4) Setting EL^2-construction on $(S, \circ_5, R_2)$ and $(S, \circ_{14}, R_2)$ we can achieve $(S, \circ_{14})$.
5) Setting EL^2-construction on $(S, \circ_{10}, R_3)$, $(S, \circ_8, R_3)$, $(S, \circ_{13}, R_3)$, $(S, \circ_2, R_2)$, $(S, \circ_{13}, R_2)$ and $(S, \circ_{16}, R_2)$ we can achieve $(S, \circ_{16})$.
6) Setting EL^2-construction on $(S, \circ_i, R_2)$, $i \in \{1, 15, 16\}$, and $(S, \circ_i, R_3)$, $i \in \{9, 12, 14\}$ we can get $(S, \circ_{17})$ which is trivial by Definition 3.18.

Notice that by Setting EL^2-construction on $(S, \circ_2, R_3)$, $(S, \circ_3, R_3)$ and $(S, \circ_5, R_3)$ we get

	a	b
a	a	S
b	a	S

which is clearly isomorphic to $(S, \circ_{12})$. □

3.2. EL^2-H_v-semigroups with 2 elements. .

In this section, we determine all EL^2-H_v-semigroups with two elements. Now, in order to find and study EL^2-H_v-semigroups of order 2, we need all H_v-semigroups with two elements. Then, we obtain the next theorem:

Theorem 3.20. [8] *There exist 36 non-isomorphic H_v-semigroups of order 2 given in Table 5. In this table the Cayley table $(abcd)$ of H_v-semigroups $(H = \{a, b\}, \circ)$ means that $a = a \circ a$, $b = a \circ b$, $c = b \circ a$ and $d = b \circ b$. Also, $H_i = (abcd)$ means that the H_v-semigroups $(H = \{a, b\}, \circ_i)$*

TABLE 5. H_v-semigroups of order 2

$H_1^* = (a, a, a, a)$	$H_{10} = (H, b, b, a)$	$H_{19} = (H, a, H, a)$	$H_{28} = (a, H, H, a)$
$H_2^* = (a, a, a, b)$	$H_{11}^* = (H, b, b, b)$	$H_{20}^* = (H, H, a, b)$	$H_{29}^* = (a, H, H, b)$
$H_3^* = (a, a, b, b)$	$H_{12} = (a, H, a, a)$	$H_{21}^* = (H, a, H, b)$	$H_{30} = (b, H, H, a)$
$H_4^* = (a, b, a, b)$	$H_{13} = (a, a, H, a)$	$H_{22} = (H, H, b, a)$	$H_{31}^* = (H, H, H, a)$
$H_5^* = (a, b, b, a)$	$H_{14}^* = (a, H, a, b)$	$H_{23} = (H, b, H, a)$	$H_{32} = (H, H, a, H)$
$H_6 = (H, a, a, a)$	$H_{15}^* = (a, H, b, b)$	$H_{24}^* = (H, H, b, b)$	$H_{33} = (H, b, a, H)$
$H_7^* = (H, a, a, b)$	$H_{16} = (a, H, b, a)$	$H_{25}^* = (H, b, H, b)$	$H_{34} = (H, a, H, H)$
$H_8 = (H, a, b, b)$	$H_{17} = (b, H, a, b)$	$H_{26} = (H, a, a, H)$	$H_{35}^* = (a, H, H, H)$
$H_9 = (H, b, a, b)$	$H_{18} = (H, H, a, a)$	$H_{27} = (H, a, b, H)$	$H_{36}^* = (H, H, H, H)$

Among these 36 H_v-semigroups there are 17 ones which are semihypergroups. We mention them by a " $*$ " sign in the related Cayley tables of Table 5.

By Theorem 3.1 there are 4 quasi order relations on a set with two elements. Hence, there are 4*36=144 triples $(H, \circ_i, R_j)$ for $i \in \{1, 2, ..., 36\}$ and $j \in \{1, 2, 3, 4\}$. To find those which has monotone condition among these 144 cases, we have:

Theorem 3.21. *For all $i \in \{1, 2, \ldots, 36\}$ and $j \in \{1, 4\}$, the triple $(H, \circ_i, R_j)$ is a quasi ordered H_v-semigroups.*

Proof. The proof is straightforward. □

Theorem 3.22. *For all $i \in \{1, 2, 3, 4, 11, 14, 15, 24, 25, 30, 35, 36\}$ and $j \in \{2, 3\}$, triples $(H, \circ_i, R_j)$ are quasi ordered H_v-semigroups.*

Proof. First of all, we mention that among these 12 H_v-semigroups, there are 11 cases which are semihypergroups (i.e. they have $*$ sign). So, by Proposition 3.15 the triple $(H, \circ_i, R_j)$ has monotone condition for $i \in \{1, 2, 3, 4, 11, 14, 15, 24, 25, 35, 36\}$ and $j \in \{2, 3\}$. To complete the proof, we should only show that $(H, \circ_{30}, R_3)$ and $(H, \circ_{30}, R_2)$ have monotone condition. To do this, Look at $(b, a) \in R_3$. Since $(b \circ_{30} a, a \circ_{30} a)$ and $(b \circ_{30} b, a \circ_{30} b)$ are both in $\overline{R}_3$. So, we can see that $(H, \circ_{30}, R_3)$ is a partially ordered H_v-semigroup. Notice that $(H, \circ_{30})$ is abelian. The same argument holds for $(H, \circ_{30}, R_2)$ □

Proposition 3.23. *For all $i \in \{5, 6, 7, 8, 9, 10, 12, 13, 16, 17, 18, 19, 20, 21, 22, 23, 26, 27, 28, 29, 31, 32, 33, 34\}$ and $j \in \{2, 3\}$, the triple $(H, \circ_i, R_j)$ is not a partially ordered H_v-semigroup.*

Proof. We prove the proposition in the following steps:

1) For $i \in \{5, 7, 20, 21, 29, 31\}$, the pair $(H, \circ_i)$ is a semihypergroup and by proposition 3.16 it does not have the monotone condition.
2) The triple $(H, \circ_i, R_3)$ does not have the monotone condition because $(b, a) \in R_3$ but $(a \circ_i b, a \circ_i a) \notin \overline{R}_3$ for all $i = 6, 8, 10, 17, 26, 27, 28, 34$. Also, Since $(b \circ_i b, a \circ_i b) \notin \overline{R}_3$, $(H, \circ_i, R_3)$ is not a partially ordered H_v-semigroup for $i = 12, 16, 18, 22, 23, 32$. In addition, $(H, \circ_9, R_3)$, $(H, \circ_{20}, R_3)$ and $(H, \circ_{33}, R_3)$ do not have monotone condition since $(b \circ_i a, a \circ_i a) \notin \overline{R}_3$, $i = 9, 20, 33$ and finally because $(b \circ_{13} b, b \circ_{13} a) \notin \overline{R}_3$, the triple $(H, \circ_{13}, R_3)$ is not a partially ordered H_v-semigroup.
3) The triple $(H, \circ_i, R_2)$ does not have the monotone condition because $(a, b) \in R_2$ but $(a \circ_i a, a \circ_i b) \notin \overline{R}_2$ for all $i = 6, 8, 17, 26, 27, 31, 32, 34$. Also, Since $(a \circ_i b, b \circ_i b) \notin \overline{R}_2$, $(H, \circ_i, R_2)$ is not a quasi ordered H_v-semigroup for $i = 10, 12, 16, 22, , 23, 28, 29, 32, 33$. In addition, $(H, \circ_9, R_2)$, $(H, \circ_{18}, R_2)$ are not quasi ordered as $(a \circ_i a, b \circ_i a) \notin \overline{R}_3$, $i = 9, 18$ and finally because $(b \circ_{13} a, b \circ_{13} b) \notin \overline{R}_2$ and $(b \circ_{19} a, b \circ_{19} b) \notin \overline{R}_2$ two triples $(H, \circ_{13}, R_2)$ and $(H, \circ_{19}, R_2)$ are not partially ordered H_v-semigroups.

□

Now, by Theorems 3.21 and 3.22, we have:

Corollary 3.24. *There exist 96 quasi ordered H_v-semigroups of order 2.*

Definition 3.25. Suppose $(H, *)$ is an H_v-semigroups. Then, $(H, *)$ is said to be a nontrivial H_v-semigroups if it is not total H_v-semigroups (

i.e. $a * b = H$ for all $(a, b) \in H$) nor it is not associated to $(H, \circ_i, R_1)$, $i \in \{1, 2, \cdots, 36\}$ in EL^2-construction.

Theorem 3.26. *There are 5 non-trivial EL^2-H_v-semigroups of order 2. (H_i has the EL^2-construction for $i \in \{1, 11, 24, 25, 35\}$.)*

Proof. In order to find non-trivial EL^2-H_v-semihypergroup, it is enough to set EL^2-construction on quasi ordered H_v-semigroup founded in 3.22. But all of them except one, $(H, \circ_{30})$, are semigroup and we study them in 3.19. Now, by setting EL^2-construction on $(H, \circ_{30}, R_3)$ we get $(H, \circ_{35})$ and by setting EL^2-construction on $(H, \circ_{30}, R_2)$ we achieve

	a	b
a	a	S
b	a	S

which is isomorphic to $(S, \circ_{35})$. At the end, it should be mentioned that $(S, \circ_1) = (H, \circ_1)$, $(S, \circ_9) = (H, \circ_{11})$, $(S, \circ_{12}) = (H, \circ_{24})$, $(S, \circ_{14}) = (H, \circ_{25})$ and $(S, \circ_{16}) = (H, \circ_{35})$. □

Definition 3.27. The H_v-semigroups $(H, *)$ is said to be a proper H_v-semigroups if it is not a H_v-groups. (i.e. the hyperoperation $*$ is not reproductive.)

Theorem 3.28. *There are 4 proper EL^2-H_v-semigroups created by H_v-semigroups.($H_1, H_{11}, H_{24}, H_{25}$ are proper EL^2-H_v-semigroups).*

Proof. The proof is straightforward. □

Corollary 3.29. *There is only one non-trivial H_v-group with EL^2-construction which is H_{35}.*

4. Conclusion

In this contribution, we enumerated all EL^2-semihypergroups and EL^2-H_v-groups of order 2. As we showed, there are only five non-trivial EL^2-H_v-hypergroup, with two elements, which are all EL^2-semihypergroups.

The approach and method used in this paper can be use to enumerate the larger EL (EL^2)-hyperstructures. For future work, we can count and classify all EL-hypergroups of order 3 or other algebraic EL (EL^2)-hyperstructures of small order.

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