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DOKDO SUB-HOOPS AND DOKDO FILTERS OF HOOPS

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ABSTRACT. The concepts of Dokdo sub-hoops and Dokdo filters are introduced, and their properties are investigated. The relationship between Dokdo sub-hoops and Dokdo filters is discussed, and characterizations of Dokdo sub-hoops and Dokdo filters are established.

1. Introduction

As a nice algebraic structure to research the many-valued logical system whose propositional value is given in a lattice, the concept of hoop is proposed by Bosbach [6, 7]. Since then, various contents of the hoop have been studied (see [1, 2, 3, 4, 5, 8, 11, 15, 16]). The operations in EQ-algebras have a similar interpretation to those in hoops. In such EQ-algebras, several types of filters have been studied by Paad and Jafari (see [17]). Jun [10] introduced Dokdo structure, a type of hybrid structure, using three concepts, namely bipolar fuzzy set, soft set, and interval-valued fuzzy set, and applied it to BCK/BCI-algebras. He first introduced the concepts of (null, full) Dokdo structure, (positive, negative) internel Dokdo structure, (positive, negative) externel Dokdo structure, and found appropriate examples. He introduce the notions of (closed) Dokdo subalgebra and Dokdo ideal, and investigated their properties. He considered conditions for Dokdo subalgebra to be closed, and provided conditions for Dokdo structure to be Dokdo subalgebra. He discussed relationship between Dokdo subalgebra and Dokdo ideal,

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and provided conditions for Dokdo structure to be Dokdo ideal in BCK-Dokdo universe. He explored the conditions under which Dokdo ideal can be Dokdo subalgebra in BCI-Dokdo universe.

The purpose of this paper is to study sub-hoops and filters in hoops using Dokdo structure. We introduce the concepts of Dokdo sub-hoops and Dokdo filters, and investigate their properties. We discuss the relationship between Dokdo sub-hoops and Dokdo filters. We establish characterizations of Dokdo sub-hoops and Dokdo filters. In general, characterization is the process through which an author reveals a character's personality. So it's an important task to find characterizations of a concept that appears in mathematics. This is because when using mathematical concepts in pure and applied science, using the characterization of concepts can make research easier.

2. Preliminaries

2.1. **Basic concepts about Dokdo structures.** Let X be a set. A *bipolar fuzzy set* in X (see [12]) is an object of the following type

$$\mathring{\varphi} = \{(\tilde{x}, \mathring{\varphi}^-(\tilde{x}), \mathring{\varphi}^+(\tilde{x})) \mid \tilde{x} \in X\} \tag{2.1}$$

where $\mathring{\varphi}^- : X \to [-1, 0]$ and $\mathring{\varphi}^+ : X \to [0, 1]$ are mappings. The bipolar fuzzy set which is described in (2.1) is simply denoted by $\mathring{\varphi} := (X; \mathring{\varphi}^-, \mathring{\varphi}^+)$.

A bipolar fuzzy set can be reinterpreted as a function:

$$\mathring{\varphi} : X \to [-1, 0] \times [0, 1],\ x \mapsto (\mathring{\varphi}^-(x), \mathring{\varphi}^+(x)).$$

Let U be an initial universe set and X be a set of parameters. For any subset A of X, a pair (φ^s, A) is called a *soft set* over U (see [13, 14]), where φ^s is a mapping described as follows:

$$\varphi^s : A \to 2^U$$

where 2^U is the power set of U. If $A = X$, the soft set (φ^s, A) over U is simply denoted by φ^s only.

A mapping $\tilde{\varphi} : X \to [[0, 1]]$ is called an *interval-valued fuzzy set* (briefly, an IVF set) in X (see [9, 18]) where $[[0, 1]]$ is the set of all closed subintervals of $[0, 1]$, and members of $[[0, 1]]$ are called *interval numbers* and are denoted by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, etc., where $\tilde{a} = [a_L, a_R]$ with $0 \leq a_L \leq a_R \leq 1$.

For every two interval numbers $\tilde{a}$ and $\tilde{b}$, we define

$$\tilde{a} \precsim \tilde{b} \text{ (or } \tilde{b} \succsim \tilde{a}) \Leftrightarrow a_L \leq b_L,\ a_R \leq b_R, \tag{2.2}$$

$$\tilde{a} = \tilde{b} \Leftrightarrow \tilde{a} \precsim \tilde{b},\ \tilde{b} \precsim \tilde{a}, \tag{2.3}$$

$$\mathrm{rmin}\{\tilde{a}, \tilde{b}\} = [\min\{a_L, b_L\}, \min\{a_R, b_R\}]. \tag{2.4}$$

Let U be an initial universe set and X a set of parameters. A triple $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is called a *Dokdo*[1] *structure* (see [10]) in (U, X) if $\mathring{\varphi} : X \to [-1, 0] \times [0, 1]$ is a bipolar fuzzy set in X, $\varphi^s : X \to 2^U$ is a soft set over U and $\tilde{\varphi} : X \to [[0, 1]]$ is an interval-valued fuzzy set in X.

The Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in (U, X) can be represented as follows:

$$\begin{aligned} &Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi}) : X \to ([-1, 0] \times [0, 1]) \times 2^U \times [[0, 1]], \\ &x \mapsto (\mathring{\varphi}(x), \varphi^s(x), \tilde{\varphi}(x)) \end{aligned} \tag{2.5}$$

where $\mathring{\varphi}(x) = (\mathring{\varphi}^-(x), \mathring{\varphi}^+(x))$ and $\tilde{\varphi}(x) = [\tilde{\varphi}_L(x), \tilde{\varphi}_R(x)]$.

Given a Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in a Dokdo universe (U, X), we consider the following sets:

$$\mathring{\varphi}(\max, \min) := \left\{ \tfrac{x}{(y,z)} \in \tfrac{X}{X \times X} \,\middle|\, \begin{array}{l} \mathring{\varphi}^-(x) \leq \max\{\mathring{\varphi}^-(y), \mathring{\varphi}^-(z)\} \\ \mathring{\varphi}^+(x) \geq \min\{\mathring{\varphi}^+(y), \mathring{\varphi}^+(z)\} \end{array} \right\}.$$

$$\mathring{\varphi}(s, N) := \{x \in X \mid \mathring{\varphi}^-(x) \leq s\},$$

$$\mathring{\varphi}(t, P) := \{x \in X \mid \mathring{\varphi}^+(x) \geq t\},$$

$$\mathring{\varphi}(s, t) := \mathring{\varphi}(s, N) \cap \mathring{\varphi}(t, P),$$

$$\varphi^s_\alpha := \{x \in X \mid \varphi^s(x) \supseteq \alpha\},$$

$$\tilde{\varphi}_{\tilde{a}} := \{x \in X \mid \tilde{\varphi}(x) \succsim \tilde{a}\},$$

where $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.

2.2. Basic concepts about hoops.

Definition 2.1 ([2])**.** *An algebra $(H, \circledcirc, \rightsquigarrow, 1)$ is called a hoop (or hoop) if the following assertions hold.*

(H1) $(H, \circledcirc, 1)$ *is a commutative monoid,*
(H2) $x \rightsquigarrow x = 1$,
(H3) $x \circledcirc (x \rightsquigarrow y) = y \circledcirc (y \rightsquigarrow x)$,
(H4) $x \rightsquigarrow (y \rightsquigarrow z) = (x \circledcirc y) \rightsquigarrow z$

for all $x, y, z \in H$.

[1]It is the name of Korea's most beautiful island.

We define a relation "$\leq$" on a hoop H by

$$(\forall x, y \in H)(x \leq y \;\Leftrightarrow\; x \rightsquigarrow y = 1). \tag{2.6}$$

It is easy to see that $(H, \leq)$ is a poset.

Definition 2.2 ([8]). *A nonempty subset S of a hoop H is called a sub-hoop of H if it is closed under two operations $\circledcirc$ and $\rightsquigarrow$.*

Note that every sub-hoop contains the element 1.

Proposition 2.3 ([8]). *Let $(H, \circledcirc, \rightsquigarrow, 1)$ be a hoop. For any $x, y, z \in H$, we have the following conditions:*

$$(H, \leq) \text{ is a meet-semilattice with } x \wedge y = x \circledcirc (x \rightsquigarrow y) \tag{2.7}$$

$$x \circledcirc y \leq z \text{ if and anly if } x \leq y \rightsquigarrow z, \tag{2.8}$$

$$x \circledcirc y \leq x, y \text{ and } x^n \leq x \text{ for any } n \in \mathbb{N}, \tag{2.9}$$

$$x \leq y \rightsquigarrow x, \tag{2.10}$$

$$1 \rightsquigarrow x = x \text{ and } x \rightsquigarrow 1 = 1, \tag{2.11}$$

$$x \circledcirc (x \rightsquigarrow y) \leq y,\ x \circledcirc y \leq x \wedge y \leq x \rightsquigarrow y, \tag{2.12}$$

$$x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightsquigarrow (x \rightsquigarrow z), \tag{2.13}$$

$$x \leq y \text{ implies } x \circledcirc z \leq y \circledcirc z,\ z \rightsquigarrow x \leq z \rightsquigarrow y \text{ and } y \rightsquigarrow z \leq x \rightsquigarrow z, \tag{2.14}$$

$$x \rightsquigarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightsquigarrow z), \tag{2.15}$$

$$(x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z) \leq x \rightsquigarrow z, \tag{2.16}$$

$$x \leq (x \rightsquigarrow y) \rightsquigarrow y. \tag{2.17}$$

Definition 2.4 ([8]). *A nonempty subset F of a hoop H is called a filter of H if it is closed under the operation $\circledcirc$ and is upward closed, that is, it satisfies:*

$$(\forall x, y \in H)(x, y \in F \;\Rightarrow x \circledcirc y \in F), \tag{2.18}$$

$$(\forall x, y \in H)(x \in F,\ x \leq y \;\Rightarrow\; y \in F). \tag{2.19}$$

Lemma 2.5 ([8]). *A subset F of a hoop H is a filter of H if and only if it satisfies:*

$$1 \in F, \tag{2.20}$$

$$(\forall x, y \in H)\,(x \in F,\ x \rightsquigarrow y \in F \;\Rightarrow\; y \in F)\,. \tag{2.21}$$

3. Dokdo sub-hoops and Dokdo filters

Let U be an initial universe set and H a set of parameters. We say that the pair (U, H) is a *hoop universe* if $(H, \circledcirc, \rightsquigarrow, 1)$ is a hoop. In what follows, a hoop $(H, \circledcirc, \rightsquigarrow, 1)$ will be simply denoted by H unless otherwise specified.

Definition 3.1. *A Dokdo structure* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *in a hoop universe* (U, H) *is called a Dokdo sub-hoop of* H *if the sets* $\mathring{\varphi}(s,t)$, φ^s_α *and* $\tilde{\varphi}_{\tilde{a}}$ *are closed under the operations* $\circledcirc$ *and* $\rightsquigarrow$ *for all* $(s,t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ *and* $\tilde{a} = [a_L, a_R]$.

A Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in a hoop universe (U, H) is a Dokdo sub-hoop of H if and only if the following assertions are valid.

$$(\forall x, y \in H) \left(\begin{array}{l} \frac{x \circledcirc y}{(x,y)} \in \mathring{\varphi}(\max, \min), \\ \varphi^s(x \circledcirc y) \supseteq \varphi^s(x) \cap \varphi^s(y), \\ \tilde{\varphi}(x \circledcirc y) \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \end{array} \right), \tag{3.1}$$

$$(\forall x, y \in H) \left(\begin{array}{l} \frac{x \rightsquigarrow y}{(x,y)} \in \mathring{\varphi}(\max, \min), \\ \varphi^s(x \rightsquigarrow y) \supseteq \varphi^s(x) \cap \varphi^s(y), \\ \tilde{\varphi}(x \rightsquigarrow y) \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \end{array} \right). \tag{3.2}$$

Example 3.2. *Let us consider a set* $H := \{0, 1, 2, 3\}$ *and binary operations "*$\circledcirc$*" and "*$\rightsquigarrow$*" in* H *which are given by Table 1.*

Table 1. Cayley tables for the binary operations "$\circledcirc$" and "$\rightsquigarrow$"

$\circledcirc$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	1	2	2
3	0	1	2	3

$\rightsquigarrow$	0	1	2	3
0	3	3	3	3
1	1	3	3	3
2	0	1	3	3
3	0	1	2	3

Then $(H,\ \circledcirc,\ \rightsquigarrow,\ 3)$ *is a hoop. Let* $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *be a Dokdo structure in a hoop universe* $(U = \mathbb{N}, H)$ *which is defined by Table 2.*

Table 2. Tabular representation of $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$

H	$\mathring{\varphi}(x)$	$\varphi^s(x)$	$\tilde{\varphi}(x)$
0	$(-0.69, 0.57)$	$2\mathbb{N}$	$[0.37, 0.64]$
1	$(-0.58, 0.46)$	$4\mathbb{N}$	$[0.33, 0.61]$
2	$(-0.37, 0.34)$	$2\mathbb{N}$	$[0.27, 0.58]$
3	$(-0.77, 0.65)$	$\mathbb{N}$	$[0.41, 0.74]$

It is routine to verify that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ *is a Dokdo sub-hoop of* $(H,\ \circledcirc,\ \rightsquigarrow,\ 3)$.

Proposition 3.3. *Every Dokdo sub-hoop $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ of H satisfies:*

$$(\forall x \in H)((1, x) \in D(\varphi)) \tag{3.3}$$

where

$D(\varphi) := \{(x, y) \mid \mathring{\varphi}^-(x) \leq \mathring{\varphi}^-(y), \mathring{\varphi}^+(x) \geq \mathring{\varphi}^+(y), \varphi^s(x) \supseteq \varphi^s(y), \tilde{\varphi}(x) \succsim \tilde{\varphi}(y)\}$.

Proof. This is immediately induced by the combination of (H2) and (3.2). □

Theorem 3.4. *Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H). Then $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo sub-hoop of H if and only if the nonempty sets $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are sub-hoops of H for all $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.*

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo sub-hoop of H and let $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$ be such that $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are nonempty. Let $x, y \in \mathring{\varphi}(s, t) \cap \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{a}}$. Then $\mathring{\varphi}^-(x) \leq s$, $\mathring{\varphi}^-(y) \leq s$, $\mathring{\varphi}^+(x) \geq t$, $\mathring{\varphi}^+(y) \geq t$, $\varphi^s(x) \supseteq \alpha$, $\varphi^s(y) \supseteq \alpha$, $\tilde{\varphi}(x) \succsim \tilde{a}$ and $\tilde{\varphi}(y) \succsim \tilde{a}$. Hence

$$\begin{aligned}
&\mathring{\varphi}^-(x \circledcirc y) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(y)\} \leq s,\\
&\mathring{\varphi}^-(x \rightsquigarrow y) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(y)\} \leq s,\\
&\mathring{\varphi}^+(x \circledcirc y) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(y)\} \geq t,\\
&\mathring{\varphi}^+(x \rightsquigarrow y) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(y)\} \geq t,
\end{aligned}$$

that is, $x \circledcirc y \in \mathring{\varphi}(s, t)$ and $x \rightsquigarrow y \in \mathring{\varphi}(s, t)$. Also, we have

$$\varphi^s(x \circledcirc y) \supseteq \varphi^s(x) \cap \varphi^s(y) \supseteq \alpha,\ \varphi^s(x \rightsquigarrow y) \supseteq \varphi^s(x) \cap \varphi^s(y) \supseteq \alpha,$$
$$\tilde{\varphi}(x \circledcirc y) \succsim \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \succsim \tilde{a},\ \tilde{\varphi}(x \rightsquigarrow y) \succsim \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \succsim \tilde{a},$$

that is, $x \circledcirc y \in \varphi^s_\alpha$, $x \rightsquigarrow y \in \varphi^s_\alpha$, $x \circledcirc y \in \tilde{\varphi}_{\tilde{a}}$ and $x \rightsquigarrow y \in \tilde{\varphi}_{\tilde{a}}$. Therefore $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are sub-hoops of H.

Conversely, suppose that $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are nonempty sub-hoops of H for all $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. Suppose that there exist $a, b \in H$ such that $a \circledcirc b \notin \mathring{\varphi}(s, t)$ or $a \rightsquigarrow b \notin \mathring{\varphi}(s, t)$. If $a \circledcirc b \notin \mathring{\varphi}(s, t)$, then

$$\mathring{\varphi}^-(a \circledcirc b) > \max\{\mathring{\varphi}^-(a), \mathring{\varphi}^-(b)\} \text{ or } \mathring{\varphi}^+(a \circledcirc b) < \min\{\mathring{\varphi}^+(a), \mathring{\varphi}^+(b)\}$$

It follows that $a, b \in \mathring{\varphi}(s, t)$ but $a \circledcirc b \notin \mathring{\varphi}(s, t)$ for $s := \max\{\mathring{\varphi}^-(a), \mathring{\varphi}^-(b)\}$ and $t := \min\{\mathring{\varphi}^+(a), \mathring{\varphi}^+(b)\}$. This is a contradiction, and so $\frac{x \circledcirc y}{(x,y)} \in \mathring{\varphi}(\max, \min)$ for all $x, y \in H$. In the same way, the case $a \rightsquigarrow b \notin \mathring{\varphi}(s, t)$ leads to contradiction. Thus $\frac{x \rightsquigarrow y}{(x,y)} \in \mathring{\varphi}(\max, \min)$ for all $x, y \in H$. For every $x, y \in H$, let $\varphi^s(x) = \alpha_x$, $\varphi^s(y) = \alpha_y$, $\tilde{\varphi}(x) = \tilde{a}$ and $\tilde{\varphi}(y) = \tilde{b}$. If we take $\alpha := \alpha_x \cap \alpha_y$ and $\tilde{c} := \mathrm{rmin}\{\tilde{a}, \tilde{b}\}$, then $x, y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$ and so $x \circledcirc y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$ and $x \rightsquigarrow y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$. Hence

$$\varphi^s(x \circledcirc y) \supseteq \alpha = \alpha_x \cap \alpha_y = \varphi^s(x) \cap \varphi^s(y),$$
$$\varphi^s(x \rightsquigarrow y) \supseteq \alpha = \alpha_x \cap \alpha_y = \varphi^s(x) \cap \varphi^s(y)$$

and

$$\tilde{\varphi}(x \circledcirc y) \succsim \tilde{c} = \mathrm{rmin}\{\tilde{a}, \tilde{b}\} = \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\},$$
$$\tilde{\varphi}(x \rightsquigarrow y) \succsim \tilde{c} = \mathrm{rmin}\{\tilde{a}, \tilde{b}\} = \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}.$$

Therefore $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo sub-hoop of H. □

Definition 3.5. *A Dokdo structure $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ in a hoop universe (U, H) is called a Dokdo filter of H if it satisfies (3.1) and*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow (y, x) \in D(\varphi))\,. \tag{3.4}$$

Example 3.6. *Consider the hoop $(H, \circledcirc, \rightsquigarrow, 3)$ which is given in Example 3.2. Let $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ be a Dokdo structure in a hoop universe $(U = \mathbb{Z}, H)$ which is defined by Table 3.*

TABLE 3. Tabular representation of $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$

H	$\mathring{\varphi}(x)$	$\varphi^s(x)$	$\tilde{\varphi}(x)$
0	$(-0.36, 0.48)$	$4\mathbb{N}$	$[0.33, 0.61]$
1	$(-0.53, 0.48)$	$4\mathbb{N}$	$[0.33, 0.61]$
2	$(-0.67, 0.64)$	$4\mathbb{Z}$	$[0.39, 0.68]$
3	$(-0.87, 0.75)$	$2\mathbb{Z}$	$[0.41, 0.74]$

It is routine to verify that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo filter of $(H, \circledcirc, \rightsquigarrow, 3)$.

It is clear that every Dokdo filter is a Dokdo sub-hoop. But the converse is not true in general as seen in the following example.

Example 3.7. *Let $H := \{0, 1, 2, 3\}$ be a set with binary operations "$\circledcirc$" and "$\rightsquigarrow$" which are given by Table 4.*
Then $(H, \circledcirc, \rightsquigarrow, 3)$ is a hoop. Let $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ be a Dokdo structure in a hoop universe $(U = \mathbb{Z}, H)$ which is defined by Table 5.
It is routine to verify that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo sub-hoop of $(H, \circledcirc, \rightsquigarrow, 3)$. Note that $2 \leq 3$ and $(2, 3) \notin D(\varphi)$. Hence $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is not a Dokdo filter of $(H, \circledcirc, \rightsquigarrow, 3)$.

We discuss characterizations of a Dokdo filter.

Table 4. Cayley tables for the binary operations "$\circledcirc$" and "$\rightsquigarrow$"

$\circledcirc$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	2
3	0	1	2	3

$\rightsquigarrow$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	2	3	3
3	0	1	2	3

Table 5. Tabular representation of $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$

H	$\mathring{\varphi}(x)$	$\varphi^s(x)$	$\tilde{\varphi}(x)$
0	$(-0.70, 0.67)$	$8\mathbb{N}$	$[0.43, 0.71]$
1	$(-0.75, 0.48)$	$4\mathbb{Z}$	$[0.36, 0.62]$
2	$(-0.55, 0.25)$	$4\mathbb{N}$	$[0.29, 0.58]$
3	$(-0.85, 0.74)$	$2\mathbb{Z}$	$[0.56, 0.84]$

Theorem 3.8. *A Dokdo structure $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ in a hoop universe (U, H) is a Dokdo filter of H if and only if it satisfies (3.3) and*

$$(\forall x, y \in H)\left(\begin{array}{l} \frac{y}{(x, x\rightsquigarrow y)} \in \mathring{\varphi}(\max, \min), \\ \varphi^s(y) \supseteq \varphi^s(x) \cap \varphi^s(x \rightsquigarrow y), \\ \tilde{\varphi}(y) \succsim \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\} \end{array}\right). \tag{3.5}$$

Proof. Assume that $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo filter of H. Since $x \leq 1$ for all $x \in H$, (3.3) is induced from (3.4). Note from (2.12) that $x \circledcirc (x \rightsquigarrow y) \leq y$ for all $x, y \in H$, and so $(y, x \circledcirc (x \rightsquigarrow y)) \in D(\varphi)$ by (3.4). It follows from (3.1) that $\mathring{\varphi}^-(y) \leq \mathring{\varphi}^-(x \circledcirc (x \rightsquigarrow y)) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(x \rightsquigarrow y)\}$ and

$$\mathring{\varphi}^+(y) \geq \mathring{\varphi}^+(x \circledcirc (x \rightsquigarrow y)) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(x \rightsquigarrow y)\},$$

that is, $\frac{y}{(x, x\rightsquigarrow y)} \in \mathring{\varphi}(\max, \min)$; and

$$\varphi^s(y) \supseteq \varphi^s(x \circledcirc (x \rightsquigarrow y)) \supseteq \varphi^s(x) \cap \varphi^s(x \rightsquigarrow y),$$

$$\tilde{\varphi}(y) \succsim \tilde{\varphi}(x \circledcirc (x \rightsquigarrow y)) \succsim \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\}.$$

Hence (3.5) is valid.

Conversely, suppose that $Dok_{\varphi} := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ satisfies (3.3) and (3.5). The combination of (H2) and (H4) induces $x \rightsquigarrow (y \rightsquigarrow (x \circledcirc y)) = 1$ for all $x, y \in H$. It follows from (3.3) and (3.5) that

$$\begin{aligned}\mathring{\varphi}^{-}(x \circledcirc y) &\leq \max\{\mathring{\varphi}^{-}(y), \mathring{\varphi}^{-}(y \rightsquigarrow (x \circledcirc y))\} \\ &\leq \max\{\mathring{\varphi}^{-}(y), \max\{\mathring{\varphi}^{-}(x), \mathring{\varphi}^{-}(x \rightsquigarrow (y \rightsquigarrow (x \circledcirc y)))\}\} \\ &= \max\{\mathring{\varphi}^{-}(y), \max\{\mathring{\varphi}^{-}(x), \mathring{\varphi}^{-}(1)\}\} \\ &= \max\{\mathring{\varphi}^{-}(y), \mathring{\varphi}^{-}(x)\},\end{aligned}$$

$$\begin{aligned}\mathring{\varphi}^{+}(x \circledcirc y) &\geq \min\{\mathring{\varphi}^{+}(y), \mathring{\varphi}^{+}(y \rightsquigarrow (x \circledcirc y))\} \\ &\geq \min\{\mathring{\varphi}^{+}(y), \min\{\mathring{\varphi}^{+}(x), \mathring{\varphi}^{+}(x \rightsquigarrow (y \rightsquigarrow (x \circledcirc y)))\}\} \\ &= \min\{\mathring{\varphi}^{+}(y), \min\{\mathring{\varphi}^{+}(x), \mathring{\varphi}^{+}(1)\}\} \\ &= \min\{\mathring{\varphi}^{+}(y), \mathring{\varphi}^{+}(x)\},\end{aligned}$$

that is, $\frac{x \circledcirc y}{(x,y)} \in \mathring{\varphi}(\max, \min)$, and

$$\begin{aligned}\varphi^{s}(x \circledcirc y) &\supseteq \varphi^{s}(y) \cap \varphi^{s}(y \rightsquigarrow (x \circledcirc y)) \\ &\supseteq \varphi^{s}(y) \cap (\varphi^{s}(x) \cap \varphi^{s}(x \rightsquigarrow (y \rightsquigarrow (x \circledcirc y)))) \\ &= \varphi^{s}(y) \cap (\varphi^{s}(x) \cap \varphi^{s}(1)) \\ &= \varphi^{s}(y) \cap \varphi^{s}(x),\end{aligned}$$

$$\begin{aligned}\tilde{\varphi}(x \circledcirc y) &\gtrsim \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y \rightsquigarrow (x \circledcirc y))\} \\ &\gtrsim \operatorname{rmin}\{\tilde{\varphi}(y), \operatorname{rmin}\{\tilde{\varphi}(x), v(x \rightsquigarrow (y \rightsquigarrow (x \circledcirc y)))\}\} \\ &= \operatorname{rmin}\{\tilde{\varphi}(y), \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(1)\}\} \\ &= \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(x)\}.\end{aligned}$$

Let $x, y \in H$ be such that $x \leq y$. Then $x \rightsquigarrow y = 1$, and so

$$\begin{aligned}&\mathring{\varphi}^{-}(y) \leq \max\{\mathring{\varphi}^{-}(x), \mathring{\varphi}^{-}(x \rightsquigarrow y)\} = \max\{\mathring{\varphi}^{-}(x), \mathring{\varphi}^{-}(1)\} = \mathring{\varphi}^{-}(x), \\ &\mathring{\varphi}^{+}(y) \geq \min\{\mathring{\varphi}^{+}(x), \mathring{\varphi}^{+}(x \rightsquigarrow y)\} = \min\{\mathring{\varphi}^{+}(x), \mathring{\varphi}^{+}(1)\} = \mathring{\varphi}^{+}(x), \\ &\varphi^{s}(y) \supseteq \varphi^{s}(x) \cap \varphi^{s}(x \rightsquigarrow y) = \varphi^{s}(x) \cap \varphi^{s}(1) = \varphi^{s}(x), \\ &\tilde{\varphi}(y) \gtrsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\} = \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(1)\} = \tilde{\varphi}(x),\end{aligned}$$

that is, $(y, x) \in D(\varphi)$. Therefore $Dok_{\varphi} := (\mathring{\varphi}, \varphi^{s}, \tilde{\varphi})$ is a Dokdo filter of H. □

Proposition 3.9. *Every Dokdo filter $Dok_{\varphi} := (\mathring{\varphi}, \varphi^{s}, \tilde{\varphi})$ of H satisfies:*

$$(\forall x, y \in H) \left(\tfrac{x \circledcirc y}{(x,y)} \in \Omega(\varphi) \right), \tag{3.6}$$

where $\Omega(\varphi) := \left\{ \frac{x}{(y,z)} \;\middle|\; \begin{array}{l} \mathring{\varphi}^{-}(x) = \max\{\mathring{\varphi}^{-}(y), \mathring{\varphi}^{-}(z)\} \\ \mathring{\varphi}^{+}(x) = \min\{\mathring{\varphi}^{+}(y), \mathring{\varphi}^{+}(z)\} \\ \varphi^{s}(x) = \varphi^{s}(y) \cap \varphi^{s}(z) \\ \tilde{\varphi}(x) = \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(z)\} \end{array} \right\}.$

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H. Since $x \circledcirc y \leq x$ and $x \circledcirc y \leq y$ for all $x, y \in H$, we have $(x, x \circledcirc y) \in D(\varphi)$ and $(y, x \circledcirc y) \in D(\varphi)$ by (3.4). Hence $\mathring{\varphi}^-(x \circledcirc y) \geq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(y)\}$, $\mathring{\varphi}^+(x \circledcirc y) \leq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(y)\}$, $\varphi^s(x \circledcirc y) \subseteq \varphi^s(x) \cap \varphi^s(y)$ and $\tilde{\varphi}(x \circledcirc y) \precsim \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}$. The combination of these and (3.1) induces (3.6). □

Theorem 3.10. *Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H). Then it is a Dokdo filter of H if and only if it satisfies* (3.4) *and* (3.6)*.*

Proof. The necessity is from Definition 3.5 and Proposition 3.9. Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H) that satisfies (3.4) and (3.6). Since $x \circledcirc (x \rightsquigarrow y) \leq y$ for all $x, y \in H$, it follows from (3.4) and (3.6) that

$$\begin{aligned}
\mathring{\varphi}^-(y) &\leq \mathring{\varphi}^-(x \circledcirc (x \rightsquigarrow y)) = \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(x \rightsquigarrow y)\},\\
\mathring{\varphi}^+(y) &\geq \mathring{\varphi}^+(x \circledcirc (x \rightsquigarrow y)) = \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(x \rightsquigarrow y)\},
\end{aligned}$$

that is, $\frac{y}{(x, x \rightsquigarrow y)} \in \mathring{\varphi}(\max, \min)$; and

$$\begin{aligned}
\varphi^s(y) &\supseteq \varphi^s(x \circledcirc (x \rightsquigarrow y)) = \varphi^s(x) \cap \varphi^s(x \rightsquigarrow y),\\
\tilde{\varphi}(y) &\succsim \tilde{\varphi}(x \circledcirc (x \rightsquigarrow y)) = \mathrm{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\}.
\end{aligned}$$

Since $x \leq 1$ for all $x \in H$, (3.3) is induced from (3.4). Therefore $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

Theorem 3.11. *Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H). Then it is a Dokdo filter of H if and only if it satisfies* (3.4) *and*

$$(\forall x, y, z \in H)\left(\begin{array}{l}
\frac{x \rightsquigarrow z}{(x \rightsquigarrow y,\, y \rightsquigarrow z)} \in \mathring{\varphi}(\max, \min),\\
\varphi^s(x \rightsquigarrow z) \supseteq \varphi^s(x \rightsquigarrow y) \cap \varphi^s(y \rightsquigarrow z),\\
\tilde{\varphi}(x \rightsquigarrow z) \succsim \mathrm{rmin}\{\tilde{\varphi}(x \rightsquigarrow y), \tilde{\varphi}(y \rightsquigarrow z)\}
\end{array}\right). \tag{3.7}$$

Proof. Suppose that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H and let $x, y, z \in H$. Since $(x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z) \leq x \rightsquigarrow z$ by (2.16), it follows from (3.4) and (3.6) that

$$\begin{aligned}
\mathring{\varphi}^-(x \rightsquigarrow z) &\leq \mathring{\varphi}^-((x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z)) = \max\{\mathring{\varphi}^-(x \rightsquigarrow y), \mathring{\varphi}^-(y \rightsquigarrow z)\},\\
\mathring{\varphi}^+(x \rightsquigarrow z) &\geq \mathring{\varphi}^+((x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z)) = \min\{\mathring{\varphi}^+(x \rightsquigarrow y), \mathring{\varphi}^+(y \rightsquigarrow z)\},
\end{aligned}$$

that is, $\frac{x \rightsquigarrow z}{(x \rightsquigarrow y, y \rightsquigarrow z)} \in \mathring{\varphi}(\max, \min)$; and

$$\begin{aligned}
\varphi^s(x \rightsquigarrow z) &\supseteq \varphi^s((x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z)) = \varphi^s(x \rightsquigarrow y) \cap \varphi^s(y \rightsquigarrow z),\\
\tilde{\varphi}(x \rightsquigarrow z) &\succsim \tilde{\varphi}((x \rightsquigarrow y) \circledcirc (y \rightsquigarrow z)) = \mathrm{rmin}\{\tilde{\varphi}(x \rightsquigarrow y), \tilde{\varphi}(y \rightsquigarrow z)\}.
\end{aligned}$$

This proves (3.7).

Conversely, let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U,H) that satisfies (3.4) and (3.7). Since $x \leq 1$ for all $x \in H$, (3.3) is induced from (3.4). If we take $x := 1$ in (3.7) and use (2.11), then we have (3.5). Therefore $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

Theorem 3.12. *Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U,H). Then it is a Dokdo filter of H if and only if it satisfies (3.4) and*

$$(\forall x,y,z \in H)\left(\begin{array}{l} \frac{z\circledcirc y}{(z\circledcirc x,\, x\rightsquigarrow y)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(z\circledcirc y) \supseteq \varphi^s(z\circledcirc x) \cap \varphi^s(x \rightsquigarrow y), \\ \tilde{\varphi}(z\circledcirc y) \succsim \operatorname{rmin}\{\tilde{\varphi}(z\circledcirc x), \tilde{\varphi}(x\rightsquigarrow y)\} \end{array}\right). \tag{3.8}$$

Proof. Suppose that $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H and let $x,y,z \in H$. Note that $(z\circledcirc x)\circledcirc(x\rightsquigarrow y) = z\circledcirc(x\circledcirc(x\rightsquigarrow y)) \leq z\circledcirc y$ by (H1), (2.12) and (2.14). it follows from (3.4) and (3.6) that

$$\begin{aligned} &\mathring{\varphi}^-(z\circledcirc y) \leq \mathring{\varphi}^-((z\circledcirc x)\circledcirc(x\rightsquigarrow y)) = \max\{\mathring{\varphi}^-(z\circledcirc x), \mathring{\varphi}^-(x\rightsquigarrow y)\}, \\ &\mathring{\varphi}^+(z\circledcirc y) \geq \mathring{\varphi}^+((z\circledcirc x)\circledcirc(x\rightsquigarrow y)) = \min\{\mathring{\varphi}^+(z\circledcirc x), \mathring{\varphi}^+(x\rightsquigarrow y)\}, \end{aligned}$$

that is, $\frac{z\circledcirc y}{(z\circledcirc x,\, x\rightsquigarrow y)} \in \mathring{\varphi}(\max,\min)$; and

$$\begin{aligned} &\varphi^s(z\circledcirc y) \supseteq \varphi^s((z\circledcirc x)\circledcirc(x\rightsquigarrow y)) = \varphi^s(z\circledcirc x)\cap\varphi^s(x\rightsquigarrow y), \\ &\tilde{\varphi}(z\circledcirc y) \succsim \tilde{\varphi}((z\circledcirc x)\circledcirc(x\rightsquigarrow y)) = \operatorname{rmin}\{\tilde{\varphi}(z\circledcirc x), \tilde{\varphi}(x\rightsquigarrow y)\}. \end{aligned}$$

This proves (3.8).

Conversely, assume that a Dokdo structure $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ in a hoop universe (U,H) satisfies (3.4) and (3.8). Since $x \leq 1$ for all $x \in H$, (3.3) is induced from (3.4). The assertion (3.5) is induced by taking $z := 1$ in (3.8). Hence $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

Theorem 3.13. *Let $Dok_\varphi := (\mathring{\varphi},\ \varphi^s,\ \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U,H). Then it is a Dokdo filter of H if and only if it satisfies:*

$$(\forall x,y,z \in H)\left(x \leq y \rightsquigarrow z \;\Rightarrow \left\{\begin{array}{l} \frac{z}{(x,y)} \in \mathring{\varphi}(\max,\min), \\ \varphi^s(z) \supseteq \varphi^s(x)\cap\varphi^s(y), \\ \tilde{\varphi}(z) \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\} \end{array}\right.\right) \tag{3.9}$$

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo filter of H and let $x, y, z \in H$ be such that $x \leq y \rightsquigarrow z$. Then

$$\begin{aligned}
&\mathring{\varphi}^-(y \rightsquigarrow z) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(x \rightsquigarrow (y \rightsquigarrow z))\} = \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(1)\} = \mathring{\varphi}^-(x),\\
&\mathring{\varphi}^+(y \rightsquigarrow z) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(x \rightsquigarrow (y \rightsquigarrow z))\} = \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(1)\} = \mathring{\varphi}^+(x),\\
&\varphi^s(y \rightsquigarrow z) \supseteq \varphi^s(x) \cap \varphi^s(x \rightsquigarrow (y \rightsquigarrow z)) = \varphi^s(x) \cap \varphi^s(1) = \varphi^s(x),\\
&\tilde{\varphi}(y \rightsquigarrow z) \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow (y \rightsquigarrow z))\} = \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(1)\} = \tilde{\varphi}(x).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathring{\varphi}^-(z) \leq \max\{\mathring{\varphi}^-(y), \mathring{\varphi}^-(y \rightsquigarrow z)\} \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(y)\},\\
&\mathring{\varphi}^+(z) \geq \min\{\mathring{\varphi}^+(y), \mathring{\varphi}^+(y \rightsquigarrow z)\} = \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(y)\},
\end{aligned}$$

that is, $\frac{z}{(x,\,y)} \in \mathring{\varphi}(\max, \min)$; and

$$\begin{aligned}
&\varphi^s(z) \supseteq \varphi^s(y) \cap \varphi^s(y \rightsquigarrow z) \supseteq \varphi^s(x) \cap \varphi^s(y),\\
&\tilde{\varphi}(z) \succsim \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y \rightsquigarrow z)\} \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(y)\}.
\end{aligned}$$

This proves (3.9).

Conversely, let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H) that satisfies (3.9). Since $x \leq x \rightsquigarrow 1$ for all $x \in H$, it is clear that $(1, x) \in D(\varphi)$ for all $x \in H$ by (3.9). The combination of (2.17) and (3.9) induces (3.5). Therefore $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

Theorem 3.14. *Let $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H). Then it is a Dokdo filter of H if and only if it satisfies (3.4) and*

$$(\forall x, y, z \in H)\left(\begin{array}{l} \frac{x \rightsquigarrow z}{((x \rightsquigarrow y) \rightsquigarrow z,\, y)} \in \mathring{\varphi}(\max, \min),\\ \varphi^s(x \rightsquigarrow z) \supseteq \varphi^s((x \rightsquigarrow y) \rightsquigarrow z) \cap \varphi^s(y),\\ \tilde{\varphi}(x \rightsquigarrow z) \succsim \operatorname{rmin}\{\tilde{\varphi}((x \rightsquigarrow y) \rightsquigarrow z), \tilde{\varphi}(y)\} \end{array}\right). \tag{3.10}$$

Proof. Assume that $Dok_\varphi := (\mathring{\varphi},\, \varphi^s,\, \tilde{\varphi})$ is a Dokdo filter of H and let $x, y, z \in H$. Using (2.10), (2.12) and (2.14), we have $(x \rightsquigarrow y) \rightsquigarrow z \leq y \rightsquigarrow z$ and

$$y \circledcirc ((x \rightsquigarrow y) \rightsquigarrow z) \leq y \circledcirc (y \rightsquigarrow z) \leq z \leq x \rightsquigarrow z.$$

which imply from (3.4) and Proposition 3.9 that

$$\begin{aligned}
\mathring{\varphi}^-(x \rightsquigarrow z) &\leq \mathring{\varphi}^-(z) \leq \mathring{\varphi}^-(y \circledcirc (y \rightsquigarrow z))\\
&= \max\{\mathring{\varphi}^-(y), \mathring{\varphi}^-(y \rightsquigarrow z)\}\\
&\leq \max\{\mathring{\varphi}^-(y), \mathring{\varphi}^-((x \rightsquigarrow y) \rightsquigarrow z)\},
\end{aligned}$$

$$\begin{aligned}\mathring{\varphi}^+(x \rightsquigarrow z) &\geq \mathring{\varphi}^+(z) \geq \mathring{\varphi}^+(y \odot (y \rightsquigarrow z)) \\ &= \min\{\mathring{\varphi}^+(y), \mathring{\varphi}^+(y \rightsquigarrow z)\} \\ &\geq \min\{\mathring{\varphi}^+(y), \mathring{\varphi}^+((x \rightsquigarrow y) \rightsquigarrow z)\},\end{aligned}$$

that is, $\frac{x \rightsquigarrow z}{((x \rightsquigarrow y) \rightsquigarrow z,\, y)} \in \mathring{\varphi}(\max, \min)$; and

$$\begin{aligned}\varphi^s(x \rightsquigarrow z) &\supseteq \varphi^s(z) \supseteq \varphi^s(y \odot (y \rightsquigarrow z)) \\ &= \varphi^s(y) \cap \varphi^s(y \rightsquigarrow z)\} \\ &\supseteq \varphi^s(y) \cap \varphi^s((x \rightsquigarrow y) \rightsquigarrow z)\},\end{aligned}$$

and

$$\begin{aligned}\tilde{\varphi}(x \rightsquigarrow z) &\succsim \tilde{\varphi}(z) \succsim \tilde{\varphi}(y \odot (y \rightsquigarrow z)) \\ &= \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}(y \rightsquigarrow z)\} \\ &\succsim \operatorname{rmin}\{\tilde{\varphi}(y), \tilde{\varphi}((x \rightsquigarrow y) \rightsquigarrow z)\}.\end{aligned}$$

This proves (3.10).

Conversely, suppose that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ satisfies (3.4) and (3.10). The condition (3.3) is induced from (2.11) and (3.4). If you put $x := 1$ in (3.10) and use (2.11), we can obtain (3.5). Therefore $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

Theorem 3.15. *Let $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ be a Dokdo structure in a hoop universe (U, H). Then $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo filter of H if and only if the nonempty sets $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are filters of H for all $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$.*

Proof. Assume that $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo filter of H. Then it is a Dokdo sub-hoop of H, and so $(1, x) \in D(\varphi)$ for all $x \in H$ by Proposition 3.3. Let $(s, t) \in [-1, 0] \times [0, 1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$ be such that $\mathring{\varphi}(s, t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are nonempty. Then there exist $x \in \mathring{\varphi}(s, t)$, $y \in \varphi^s_\alpha$ and $z \in \tilde{\varphi}_{\tilde{a}}$ which imply that $\mathring{\varphi}^-(1) \leq \mathring{\varphi}^-(x) \leq s$, $\mathring{\varphi}^+(1) \geq \mathring{\varphi}^+(x) \geq t$, $\varphi^s(1) \supseteq \varphi^s(y) \supseteq \alpha$ and $\tilde{\varphi}(1) \succsim \tilde{\varphi}(z) \succsim \tilde{a}$. Hence $1 \in \mathring{\varphi}(s, t) \cap \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{a}}$. Let $x, y \in H$ be such that $x \in \mathring{\varphi}(s, t) \cap \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{a}}$ and $x \rightsquigarrow y \in \mathring{\varphi}(s, t) \cap \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{a}}$. Then $\mathring{\varphi}^-(x) \leq s$, $\mathring{\varphi}^-(x \rightsquigarrow y) \leq s$, $\mathring{\varphi}^+(x) \geq t$, $\mathring{\varphi}^+(x \rightsquigarrow y) \geq t$, $\varphi^s(x) \supseteq \alpha$, $\varphi^s(x \rightsquigarrow y) \supseteq \alpha$, $\tilde{\varphi}(x) \succsim \tilde{a}$ and $\tilde{\varphi}(x \rightsquigarrow y) \succsim \tilde{a}$. It follows that

$$\begin{aligned}&\mathring{\varphi}^-(y) \leq \max\{\mathring{\varphi}^-(x), \mathring{\varphi}^-(x \rightsquigarrow y)\} \leq s, \\ &\mathring{\varphi}^+(y) \geq \min\{\mathring{\varphi}^+(x), \mathring{\varphi}^+(x \rightsquigarrow y)\} \geq t, \\ &\varphi^s(y) \supseteq \varphi^s(x) \cap \varphi^s(x \rightsquigarrow y) \supseteq \alpha, \\ &\tilde{\varphi}(y) \succsim \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\} \succsim \tilde{a},\end{aligned}$$

that is, $y \in \mathring{\varphi}(s,t) \cap \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{a}}$. Therefore $\mathring{\varphi}(s,t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are filters of H by Lemma 2.5.

Conversely, suppose that $\mathring{\varphi}(s,t)$, φ^s_α and $\tilde{\varphi}_{\tilde{a}}$ are nonempty filters of H for all $(s,t) \in [-1,0] \times [0,1]$, $\alpha \in 2^U$ and $\tilde{a} = [a_L, a_R]$. Assume that $(1,a) \notin D(\varphi)$ for some $a \in H$. Then $\frac{1}{(a,a)} \notin \mathring{\varphi}(\max,\min)$, $\varphi^s(1) \not\supseteq \varphi^s(a)$ or $\tilde{\varphi}(1) \not\succsim \tilde{\varphi}(a)$. If $\frac{1}{(a,a)} \notin \mathring{\varphi}(\max,\min)$, then $\mathring{\varphi}^-(1) > \mathring{\varphi}^-(a)$ or $\mathring{\varphi}^+(1) < \mathring{\varphi}^+(a)$, and so $1 \notin \mathring{\varphi}(s,t)$ for $s := \mathring{\varphi}^-(a)$ and $t := \mathring{\varphi}^+(a)$. The case $\varphi^s(1) \not\supseteq \varphi^s(a)$ induces $1 \notin \varphi^s_\alpha$ for $\alpha := \varphi^s(a)$. If $\tilde{\varphi}(1) \not\succsim \tilde{\varphi}(a)$, then $1 \notin \tilde{\varphi}_{\tilde{a}}$ for $\tilde{a} := \tilde{\varphi}(a)$. This is a contradiction, and thus $(1,x) \in D(\varphi)$ for all $x \in H$. Now suppose that $\frac{b}{(a,\, a \rightsquigarrow b)} \notin \mathring{\varphi}(\max,\min)$ for some $a, b \in H$. Then $\mathring{\varphi}^-(b) > \max\{\mathring{\varphi}^-(a), \mathring{\varphi}^-(a \rightsquigarrow b)\}$ or $\mathring{\varphi}^+(b) < \min\{\mathring{\varphi}^+(a), \mathring{\varphi}^+(a \rightsquigarrow b)\}$. Hence $a \in \mathring{\varphi}(s,t)$ and $a \rightsquigarrow b \in \mathring{\varphi}(s,t)$ but $b \notin \mathring{\varphi}(s,t)$ for $s := \max\{\mathring{\varphi}^-(a), \mathring{\varphi}^-(a \rightsquigarrow b)\}$ and $t := \min\{\mathring{\varphi}^+(a), \mathring{\varphi}^+(a \rightsquigarrow b)\}$. This is a contradiction, and so $\frac{y}{(x,\, x \rightsquigarrow y)} \in \mathring{\varphi}(\max,\min)$ for all $x, y \in H$. For every $x, y \in H$, let $\varphi^s(x) = \alpha_x$, $\varphi^s(x \rightsquigarrow y) = \alpha_y$, $\tilde{\varphi}(x) = \tilde{a}$ and $\tilde{\varphi}(x \rightsquigarrow y) = \tilde{b}$. If we take $\alpha := \alpha_x \cap \alpha_y$ and $\tilde{c} := \operatorname{rmin}\{\tilde{a}, \tilde{b}\}$, then $x \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$ and $x \rightsquigarrow y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$. Hence $y \in \varphi^s_\alpha \cap \tilde{\varphi}_{\tilde{c}}$ which implies that

$$\varphi^s(y) \supseteq \alpha = \alpha_x \cap \alpha_y = \varphi^s(x) \cap \varphi^s(x \rightsquigarrow y)$$

and

$$\tilde{\varphi}(y) \succsim \tilde{c} = \operatorname{rmin}\{\tilde{a}, \tilde{b}\} = \operatorname{rmin}\{\tilde{\varphi}(x), \tilde{\varphi}(x \rightsquigarrow y)\}.$$

Therefore $Dok_\varphi := (\mathring{\varphi}, \varphi^s, \tilde{\varphi})$ is a Dokdo filter of H by Theorem 3.8. □

4. Conclusion

A bipolar fuzzy set is proposed as an extension of a classical fuzzy set to solve a particular class of decision-making problems. Soft set theory is a generalization of fuzzy set theory proposed by Molodtsov in 1999 to deal with uncertainty in a parametric way. An interval value fuzzy set is proposed to express the conjugation concept based on the normal format in which the variable is assumed to be fuzzy as well as the language connection. Bipolar fuzzy set, soft set and interval value fuzzy set are all useful tools for dealing with uncertainty. Uncertainty may contain a number of factors. To deal with uncertainty involving more than one factor, we feel the need for a hybrid structure. Based on this, Jun introduced the so-called Dokdo structure which is one of hybrid structure. A hoop is a a partially ordered commutative residuated integral monoid, and it is a particular class of algebraic structures. For the purpose of using Dokdo structure to study sub-hoops and filters in hoops, we introduced the concepts of Dokdo sub-hoops and Dokdo filters and examined various related properties. We discussed the relationship between Dokdo sub-hoops and Dokdo filters. In general, characterization is the process through which an author reveals a character's personality. So it's an important task to find characterizations of a concept that appears in mathematics. This is because when using mathematical concepts in pure and applied science, using the characterization of concepts can make research easier. So we established characterizations of Dokdo sub-hoops and Dokdo filters. Against the backdrop of the ideas and results obtained in this paper, we will conduct other substructures in hoops, such as (positive) implicative filter, fantastic filter etc., and further use them for related algebraic systems, for example, MV-algebras, EQ-algebras, equality algebras, BL-algebras, etc.

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