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ON CP-FRAMES AND THE ARTIN-REES PROPERTY

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ABSTRACT. The set $\mathcal{C}_c(L) = \Big\{\alpha \in \mathcal{R}L : \big|\{r \in \mathbb{R} : \operatorname{coz}(\alpha - \mathbf{r}) \neq 1\}\big| \leq \aleph_0\Big\}$ is a sub-f-ring of $\mathcal{R}L$, that is, the ring of all continuous real-valued functions on a completely regular frame L. The main purpose of this paper is to continue our investigation begun in [3] of extending ring-theoretic properties in $\mathcal{R}L$ to the context of completely regular frames by replacing the ring $\mathcal{R}L$ with the ring $\mathcal{C}_c(L)$ to the context of zero-dimensional frames. We show that a frame L is a CP-frame if and only if $\mathcal{C}_c(L)$ is a regular ring if and only if every ideal of $\mathcal{C}_c(L)$ is pure if and only if $\mathcal{C}_c(L)$ is an Artin-Rees ring if and only if every ideal of $\mathcal{C}_c(L)$ with the Artin-Rees property is an Artin-Rees ideal if and only if the factor ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an Artin-Rees ring for any $\alpha \in \mathcal{C}_c(L)$ if and only if every minimal prime ideal of $\mathcal{C}_c(L)$ is an Artin-Rees ideal.

1. Introduction

Throughout, A is a commutative ring with an identity element $1 \neq 0$. For a Noetherian ring A, every ideal I of A satisfies the following condition:

(**AR**) For each ideal J of A there is $n \in \mathbb{N}$, depending on J, for which
$$I^n \cap J \subseteq IJ.$$

E. Artin (unpublished) and D. Rees [20] proved this famous fact, the *Artin-Rees property*, more than sixty years ago, independently.

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Similarly to the above, it is said that an ideal I of a ring A has the *Artin-Rees property* if for each ideal J of A there is $n \in \mathbb{N}$ for which $I^n \cap J \subseteq IJ$. A ring A is called an *Artin-Rees ring* (*AR-ring*) if every ideal of A has the Artin-Rees property. An ideal I of a ring A is said to be an *Artin-Rees ideal* (*AR-ideal*) if for two sub-ideals E and F of I there is $n \in \mathbb{N}$ for which $E^n \cap F \subseteq EF$.

As in [13], for any $\alpha \in \mathcal{R}L$, we put

$$R_\alpha = \{r \in \mathbb{R} : \operatorname{coz}(\alpha - \mathbf{r}) \neq 1\}.$$

The set

$$\mathcal{C}_c(L) = \{\alpha \in \mathcal{R}L : R_\alpha \text{ is countable}\}$$

has been studied as a sub-f-ring of $\mathcal{R}L$ (see [12, 13, 14, 15, 21, 22]).

A *P-frame* is a frame L in which $c \vee c^* = 1$ for any cozero element $c \in \operatorname{Coz}[L]$. Motivated by this definition and following Estaji and et al. [14, 21], we define what we call a CP-frame as follow:

A frame L is a CP-frame in case $s \vee s^* = 1$ for any cozero element $s \in \operatorname{Coz}_c[L]$.

To begin with, in Lemma 3.3, we show that the density limitation of cozero elements of $\operatorname{Coz}_c[L]$ can really be relaxed. Two of the characterizations are that a frame L is a P-frame if and only if $\mathcal{R}L$ is regular (that is, for each $\alpha \in \mathcal{R}L$ there exists $\beta \in \mathcal{R}L$ with $\alpha = \alpha\beta\alpha$) if and only if every ideal of $\mathcal{R}L$ is pure (see [1, 2, 9] for other ring-theoretic characterizations of P-frames). By extending these characterizations, we show that a frame L is a CP-frame if and only if $\mathcal{C}_c(L)$ is regular if and only if every ideal of $\mathcal{C}_c(L)$ is pure (see Proposition 3.6 and Theorem 3.9).

In Section 4, we are going to characterize CP-frames in terms of the Artin-Rees property in $\mathcal{C}_c(L)$ and in the factor rings of $\mathcal{C}_c(L)$. In Theorem 4.3, we prove that an ideal of $\mathcal{C}_c(L)$ is pure if and only if it is a z_c-ideal with the Artin-Rees property. This theorem enables us to characterize CP-frame as precisely those L for which every ideal of $\mathcal{C}_c(L)$ with the Artin-Rees property is an Artin-Rees ideal (Theorem 4.6).

Given an element $\alpha \in \mathcal{C}_c(L)$, in Proposition 4.10, we show that $\operatorname{coz}(\alpha)$ is a complemented element of L and the closed quotient $\uparrow\operatorname{coz}(\alpha)$ is a CP-frame if and only if the factor ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an Artin-Rees ring if and only if the ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is regular.

To conclude Section 4, our aim is to characterize CP-frames in terms of the Artin-Rees property in some ideals of $\mathcal{C}_c(L)$ and in some factor rings of $\mathcal{C}_c(L)$ (Theorem 4.12).

2. Preliminaries

2.1. **Frames.** For a general theory of frames and locales, we refer to [19]. A *frame* or *locale* L is a complete lattice in which finite meets distribute over arbitrary joins. We use 0 and 1 for the bottom element and the top element of L, respectively. The *pseudocomplement* of an element $a \in L$, denoted by a^*, is the element

$$a^* = \bigvee \{x \in L : a \wedge x = 0\}.$$

An element $a \in L$ is called *complemented* if $a \vee a^* = 1$. We write

$$B(L) = \{a \in L : a \vee a^* = 1\}$$

for the set of all complemented elements of L and, clearly, it is a sublattice of L.

As usual, the *rather below* and the *completely below* relations are denoted by $\prec$ and $\prec\!\!\prec$, respectively. Recall that $a \prec b$ in case there is an element $c \in L$ such that $a \wedge c = 0$ and $c \vee b = 1$.

A *frame homomorphism* is a map between frames that preserves finite meets and all joins. By a *quotient* of a frame L, we mean a surjective frame homomorphism. In particular, for any $a \in L$, the *closed quotient* of a is the frame $\uparrow a = \{b \in L : a \leq b\}$ with quotient map $L \xrightarrow{-\vee a} \uparrow a$ defined by $b \mapsto a \vee b$, and the *open quotient* of a is the frame $\downarrow a = \{b \in L : b \leq a\}$ with quotient map $L \xrightarrow{-\wedge a} \downarrow a$ defined by $b \mapsto a \wedge b$.

2.2. **The ring $\mathcal{R}L$ and the cozero part of L.** Recall from [7] that the frame $\mathcal{L}(\mathbb{R})$ of reals is the frame generated by the ordered pairs (p, q) of rational numbers subject to the following relations.

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$.
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$.
(R3) $(p, q) = \bigvee \{(r, s) : p < r < s < q\}$.
(R4) $\bigvee \{(p, q) : p, q \in \mathbb{Q}\} = 1$.

For any complete regular frame, the ring $\mathcal{R}L$ is the set of all frame homomorphisms $\mathcal{L}(\mathbb{R}) \to L$. It is a reduced f-ring (see [5] and [7] for details). We use the notation of [7] and we write $\mathbf{0}$ and $\mathbf{1}$ for the zero element and the identity element of $\mathcal{R}L$, respectively. For any $r \in \mathbb{R}$, the constant frame map $\mathbf{r} \in \mathcal{R}L$ is defined by

$$\mathbf{r}(p, q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

To recall, for any $\alpha \in \mathcal{R}L$,

$$\operatorname{coz}(\alpha) = \alpha(-,0) \vee \alpha(0,-) = \alpha\big(\bigvee\{(r,0) : r < 0\}\big) \vee \alpha\big(\bigvee\{(0,s) : 0 < s\}\big).$$

An element $c \in L$ is a *cozero element* if $c = \operatorname{coz}(\alpha)$ for some $\alpha \in \mathcal{R}L$. The *cozero part* of L, that is, $\operatorname{Coz}[L] = \{\operatorname{coz}(\alpha) : \alpha \in \mathcal{R}L\}$, is a sub-$\sigma$-frame of L. It is known that L is completely regular if and only if $\operatorname{Coz}[L]$ generates L.

2.3. **The ring $\mathcal{C}_c(L)$.** As stated in the introduction, the ring $\mathcal{C}_c(L) = \{\alpha \in \mathcal{R}L : R_\alpha \text{ is countable}\}$, where $R_\alpha = \{r \in \mathbb{R} : \operatorname{coz}(\alpha - \mathbf{r}) \neq 1\}$, has been studied as a sub-f-ring of $\mathcal{R}L$, and we denote its bounded part by $\mathcal{C}_c^*(L)$ (see [12, 13, 14, 15, 21, 22] for details).

A frame L is called *zero-dimensional* if each of its elements is a join complemented elements. Let us recall that every zero-dimensional frame is complete regular. In [22], the authors show that, for any frame L, there is a zero-dimensional frame M which is a continuous image of L and $\mathcal{C}_c(L) \cong \mathcal{C}_c(M)$. Thus, all frames considered in this paper are zero-dimensional. In [12, 21, 22], it is shown that:

(1) Every element of $\mathcal{C}_c(L)$ has an n^{th} root for any odd $n \in \mathbb{N}$.
(2) Every positive element of $\mathcal{C}_c(L)$ has an n^{th} root for any $n \in \mathbb{N}$.
(3) Every prime ideal P in $\mathcal{C}_c(L)$ is contained in a unique maximal ideal.
(4) For any unit element $\alpha \in \mathcal{R}L$), we have $\alpha^{-1} \in \mathcal{C}_c(L)$ when $\alpha \in \mathcal{C}_c(L)$. Also, any $\alpha \geq \mathbf{1}$ in $\mathcal{C}_c(L)$ has an inverse.

Recall that a σ-frame is a bounded distributive lattice in which every countable subset has a join and binary meet distributes over these joins, and regularity (complete regularity) of a σ-frame is the countable counterparts of regularity (complete regularity) of frames, that is, $a = \bigvee_{a_n \prec a} a_n$ ($a = \bigvee_{a_n \prec\!\prec a} a_n$) for each element a.

In [15], it is shown that $\operatorname{Coz}_c[L] = \{\operatorname{coz}(\alpha) : \alpha \in \mathcal{C}_c(L)\}$ is a sub-σ-frame of L such that

$$s \in \operatorname{Coz}_c[L] \Leftrightarrow s = \bigvee_{n=1}^{\infty} s_n, \text{ where } s_n \in B(L).$$

This is to say that $\operatorname{Coz}_c[L]$ is a regular sub-σ-frame of L and hence, by [6], we deduce that it is normal (that is, given a and b with $a \vee b = 1$, we can find c and d such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$). So, in $\operatorname{Coz}_c[L]$, we have $\prec = \prec\!\prec$.

3. CP-frames

Recall that a frame is said to be a P-frame if $c \vee c^* = 1$ for any $c \in \mathrm{Coz}[L]$. Motivated by this definition and following Estaji et al. [14, 21], the following definition is formulated.

Definition 3.1. A frame L is a CP-frame in case $s \vee s^* = 1$ for any $s \in \mathrm{Coz}_c[L]$.

An element a in a frame L is called *dense* if $a^* = 0$. We are going to show that, in the above definition, the density limitation of cozero elements can really be relaxed. Before proceeding, we need some background. Let L be a frame. For $a, b \in L$, the *relative pseudo-complement* of a with respect to b, denoted by $a \to b$, is the element $a \to b = \bigvee \{x \in L : a \wedge x \leq b\}$. Note that $a^* = a \to 0$. A subset S of a frame L is a *sublocale* if it is closed under all meets, and $a \to s \in S$ for every $a \in L$ and $s \in S$. Corresponding to any $a \in L$, the closed and the open sublocales are defined, respectively, by the formulas

$$\mathfrak{c}(a) = \{x \in L : a \leq x\} = \uparrow a$$

and

$$\mathfrak{o}(a) = \{x : x = a \to x\} = \{a \to x : x \in L\},$$

We use the symbols $\uparrow a$, $\mathfrak{c}(a)$ interchangeably. Recall that

(1) $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ if and only if $a \vee b = 1$.
(2) $a \in B(L)$ if and only if $\mathfrak{c}(a) = \mathfrak{o}(a^*)$.

Let $a \in B(L)$. Now, for each $p, q \in \mathbb{Q}$, define

$$\eta_a(p,q) = \begin{cases} 0 & \text{if } p < q \leq 0 \text{ or } 1 \leq p < q \\ a^* & \text{if } p < 0 < q \leq 1 \\ a & \text{if } 0 \leq p < 1 < q \\ 1 & \text{if } p < 0 < 1 < q. \end{cases}$$

Then, by [5, 8.4], $\eta_a \in \mathcal{R}L$ such that $\mathrm{coz}(\eta_a) = a$, $\mathrm{coz}(\eta_a - \mathbf{1}) = a^*$, and ${\eta_a}^2 = \eta_a$. Since $\mathrm{coz}(\eta_a - \mathbf{r}) = 1$ for every $0, 1 \neq r \in \mathbb{R}$, we have $R_{\eta_a} = \{0, 1\}$, which implies that $\eta_a \in \mathcal{C}_c(L)$. As an immediate consequence, we have the following.

Proposition 3.2. *The following hold for any $a \in L$.*

(1) *If $a \in B(L)$, then $a, a^* \in \mathrm{Coz}_c[L]$.*
(2) *$a \in B(L)$ if and only if $a \prec a$ in $\mathrm{Coz}_c[L]$.*

Lemma 3.3. *A frame L is a CP-frame if and only if every non-dense cozero element of $\mathrm{Coz}_c[L]$ is complemented in $\mathrm{Coz}_c[L]$.*

Proof. To prove the nontrivial part, let $s \in \mathrm{Coz}_c[L]$ be a dense cozero element of L. Therefore $s \neq 0$, and so, by zero-dimensionality, there exists a nonzero $b \in \mathrm{Coz}_c[L]$ with $b \prec\!\prec s$. If $b^* = 0$, then we are done. So, by the hypothesis, we can infer that $b \vee b^* = 1$, which implies $b^* \in \mathrm{Coz}_c[L]$, showing that $s \wedge b^* \in \mathrm{Coz}_c[L]$. If $(s \wedge b^*)^* = 0$, then we have

$$(s \wedge b^*)^{**} = 1 \Rightarrow s^{**} \wedge b^{***} = 1 \Rightarrow b^* = b^{***} = 1 \Rightarrow b = 0,$$

which is of course false. So, we can assume that $(s \wedge b^*)^* \neq 0$, and hence, by the hypothesis, we can conclude that $s \wedge b^* \in B(L)$. Now, consider any $x \in \mathfrak{c}(s)$, and hence, $x \in \mathfrak{c}(s \wedge b^*)$. Since $\mathfrak{c}(s \wedge b^*) = \mathfrak{o}\big((s \wedge b^*)^*\big)$, we obtain

$$x = \big((s \wedge b^*)^* \to x\big) = \big((s \wedge b^*) \to 0\big) \to x.$$

By [19, Ch III. Proposition 3.1.1, (H7)], we would have

$$\begin{aligned} x &= \big(b^* \to (s \to 0)\big) \to x, \\ &= (b^* \to s^*) \to x \\ &= (b^* \to 0) \to x, && \text{since} \quad s^* = 0 \\ &= b^{**} \to x \\ &= b \to x, && \text{since} \quad b \in B(L), \end{aligned}$$

which would imply $x \in \mathfrak{o}(b)$. Thus, $\mathfrak{c}(s) \subseteq \mathfrak{o}(b)$, that is, $s \vee b = 1$, in consequence, $s = s \vee b = 1$, and the proof is complete. □

Now, we intend to give two ring-theoretic characterizations of CP-frames. For these, we shall need a series of results which will also be used when discussing the Artin-Rees property in the ring $\mathcal{C}_c(L)$.

Let us remind the reader that the proof of the following lemma is a $\mathcal{C}_c(L)$ version of $\mathcal{R}L$ proved in [10, Lemma 2.1].

Lemma 3.4. *Suppose* $\mathrm{coz}(\varphi) \prec \mathrm{coz}(\delta)$ *in* $\mathrm{Coz}_c[L]$ *for some* $\varphi, \delta \in \mathcal{C}_c(L)$. *Then there exists an invertible* $\rho \in \mathcal{C}_c(L)$ *such that* $\varphi = \varphi\rho\delta^2$.

Proof. Since $\mathrm{coz}(\varphi) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c\, L$, we can choose $\alpha \in \mathcal{C}_c(L)$ such that $\mathrm{coz}(\varphi) \wedge \mathrm{coz}(\alpha) = 0$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta) = 1$. The latter implies that

$$1 = \mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta) = \mathrm{coz}(\alpha^2) \vee \mathrm{coz}(\delta^2) = \mathrm{coz}(\alpha^2 + \delta^2),$$

this means that $\alpha^2 + \delta^2$ is invertible. By the former case, we have $\mathrm{coz}(\varphi\alpha) = 0$, that is, $\varphi\alpha = \mathbf{0}$. Putting $\rho = \frac{\mathbf{1}}{\alpha^2+\delta^2}$, we then have

$$\varphi = \varphi\frac{\alpha^2 + \delta^2}{\alpha^2 + \delta^2} = \frac{\varphi\delta^2}{\alpha^2 + \delta^2} = \varphi\rho\delta^2.$$

□

An element a of a ring A is said to be a *regular element* if there is an element $b \in A$ such that $a = aba$. A ring A is called *regular* if all its elements are regular. An ideal I of A is said to be a *regular ideal* if any element of I is a regular element of A.

Lemma 3.5. *Suppose $s = \mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{C}_c(L)$. Then $s \in B(L)$ if and only if α is a regular element in $\mathcal{C}_c(L)$ if and only if there is $\delta \in \mathcal{C}_c(L)$ such that $\alpha = \delta\alpha$ with $\mathrm{coz}(\delta) \leq \mathrm{coz}(\alpha)$*

Proof. Suppose that $s = \mathrm{coz}(\alpha) \in B(L)$. Then $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\alpha)$ in $\mathrm{Coz}_c[L]$. Now, lemma 3.4 implies that there is $\rho \in \mathcal{C}_c(L)$ such that $\alpha = \rho\alpha^2$, which means that α is a regular element in $\mathcal{C}_c(L)$.

Let α be a regular element in $\mathcal{C}_c(L)$. Then there exists $\beta \in \mathcal{C}_c(L)$ such that $\alpha = \beta\alpha^2$. Now, if $\delta = \beta\alpha$, then $\delta \in \mathcal{C}_c(L)$, $\alpha = \delta\alpha$, and $\mathrm{coz}(\delta) = \mathrm{coz}(\beta) \wedge \mathrm{coz}(\alpha) \leq \mathrm{coz}(\alpha)$.

Suppose $\delta \in \mathcal{C}_c(L)$ and $\alpha = \delta\alpha$ with $\mathrm{coz}(\delta) \leq \mathrm{coz}(\alpha)$. Then $\alpha(\mathbf{1} - \delta) = \mathbf{0}$, this implies that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\mathbf{1} - \delta) = 0$. Now, since

$$1 = \mathrm{coz}(\mathbf{1} - \delta + \delta) \leq \mathrm{coz}(\mathbf{1} - \delta) \vee \mathrm{coz}(\delta) \leq \mathrm{coz}(\mathbf{1} - \delta) \vee \mathrm{coz}(\alpha),$$

we can conclude that $s = \mathrm{coz}(\alpha) \in B(L)$. Therefore the proof is complete. □

An immediate corollary to the previous lemma is the next corollary.

Corollary 3.6. *A frame L is a CP-frame if and only if $\mathcal{C}_c(L)$ is a regular ring.*

The pure part of an ideal I of a ring A is the ideal

$$mI = \{a \in A : a = ax \text{ for some } x \in I\}.$$

An ideal I is called pure whenever $I = mI$, or equivalently, for every ideal J of A, the equality $I \cap J = IJ$ holds.

Let us remind the reader that the following lemma and corollary are the $\mathcal{C}_c(L)$ versions of $\mathcal{R}L$ proved in [11, Lemma 3.2 and Corollary 3.3].

Lemma 3.7. *An ideal Q of $\mathcal{C}_c(L)$ is pure if and only if for every $\alpha \in Q$, there exists $\delta \in Q$ such that $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$.*

Proof. ($\Rightarrow$): Let Q be pure and $\alpha \in Q$. Now, pick $\delta \in Q$ such that $\alpha = \alpha\delta$. It follows that $\alpha(\mathbf{1} - \delta) = \mathbf{0}$, implying that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\mathbf{1} - \delta) = 0$. On the other hand, we have $1 = \mathrm{coz}(\delta + (\mathbf{1} - \delta)) \leq \mathrm{coz}(\delta) \vee \mathrm{coz}(\mathbf{1} - \delta)$. Thus, $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$ for some $\delta \in Q$.

($\Leftarrow$): Assume $\alpha \in Q$. Then the present hypothesis implies that $\mathrm{coz}(\alpha)$ is rather below $\mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$ for some $\delta \in Q$. Then, by Lemma 3.4, there exists an invertible $\rho \in \mathcal{C}_c(L)$ such that $\alpha = \alpha\rho\delta^2$. Putting $\beta = \rho\delta^2$, we then have $\alpha = \alpha\beta$ and $\beta \in Q$. This is to say that Q is a pure ideal. □

Using the above lemma, we give the following frame-theoretic characterization of the pure part of each ideal Q of $\mathcal{C}_c(L)$.

Corollary 3.8. *For any ideal Q of $\mathcal{C}_c(L)$, we have*

$$mQ = \{\alpha \in \mathcal{C}_c(L) : \mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta) \text{ in } \mathrm{Coz}_c[L] \text{ for some } \delta \in Q\}.$$

We close this section by the next ring-theoretic characterization of CP-frames. For an element a of a ring A, we write $\langle a\rangle$ for the principal ideal is generated by a.

Theorem 3.9. *L is a CP-frame if and only if every ideal of $\mathcal{C}_c(L)$ is pure.*

Proof. ($\Rightarrow$): Let Q be an ideal of $\mathcal{C}_c(L)$ and $\alpha \in Q$. Then, in view of L being a CP-frame, $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\alpha)$ in $\mathrm{Coz}_c[L]$. This shows that Q is pure by Lemma 3.7. Conseqently, every ideal of $\mathcal{C}_c(L)$ is pure.

($\Leftarrow$): Let $s \in \mathrm{Coz}_c[L]$ such that $s = \mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{C}_c(L)$. We must show that $s \vee s^* = 1$. The present hypothesis implies that the principal ideal $\langle\alpha\rangle = \alpha\mathcal{C}_c(L)$ is a pure ideal of $\mathcal{C}_c(L)$. Since $\alpha \in \langle\alpha\rangle$, by Lemma 3.7, there exists $\delta \in \langle\alpha\rangle$ such that $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$. This shows that $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta) \leq \mathrm{coz}(\alpha)$, showing $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\alpha)$ in $\mathrm{Coz}_c[L]$. Now, by Proposition 3.2, we would have $s = \mathrm{coz}(\alpha) \in B(L)$, meaning that $s \vee s^* = 1$. □

4. Artin-Rees property in the ring $\mathcal{C}_c(L)$

It is routine to show that every pure ideal of a ring A has the AR property. We intend to show that, as pure ideals in $\mathcal{R}L$ can be characterized in terms of the AR property, pure ideals in $\mathcal{C}_c(L)$ can also be characterized in terms of the AR property. We begin with the following propositions. Let us recall from [12] that an ideal Q in $\mathcal{C}_c(L)$ is called a z_c-ideal if, for any $\alpha \in \mathcal{C}_c(L)$ and $\beta \in Q$, $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ implies $\alpha \in Q$. By some simple changes in the proof of Proposition 3.27 in [12], we can deduce the next proposition. Recall from [18] that an ideal I of a ring A is said to be a *z-ideal* if $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $a \in I$ imply $b \in I$, where the intersection of all maximal ideal containing $x \in A$ is denoted by $\mathfrak{M}(x)$.

Proposition 4.1. *An ideal Q in $\mathcal{C}_c(L)$ is a z_c-ideal if and only if it is a z-ideal.*

Proposition 4.2. *Every pure ideal in $\mathcal{C}_c(L)$ is a z_c-ideal.*

Proof. Let Q be a pure ideal in $\mathcal{C}_c(L)$. Now, suppose $\mathrm{coz}(\alpha) = \mathrm{coz}(\beta)$ with $\beta \in Q$. Then, by Lemma 3.7, $\mathrm{coz}(\beta) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$ for some $\delta \in Q$. It follows that $\mathrm{coz}(\alpha) \prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c[L]$, which implies, by Lemma 3.4, that α is a multiple of δ, and hence $\alpha \in Q$. So the proposition holds. ☐

In the paper [3], we show that every z-ideal of a ring A with the AR property is pure. Hence, combining the two previous propositions we obtain the following theorem.

Theorem 4.3. *An ideal of $\mathcal{C}_c(L)$ is a z_c-ideal with the Artin-Rees property if and only if it is pure.*

In the next theorem, we aim to show that CP-frames can be characterized in terms of the AR property. Let us make the following lemma about the pure ideals.

Lemma 4.4. *Every pure ideal of a regular ring is an Artin-Rees ideal.*

Proof. Suppose I is a pure ideal of a regular ring A with two sub-ideals J and K. Let $x \in J \cap K$. Then, by regularity, there exists $y \in A$ such that $x = xyx$. Since I is a pure ideal, we can conclude that $x = xyxz$ for some $z \in I$. Putting $t = zy$, we then have $tx \in K$ and $x = xtx \in JK$. So, $J \cap K = JK$, which means that I is an AR-ideal. ☐

Corresponding to any $a \in L$, two ideals $\mathbf{M}_a$ and $\mathbf{O}_a$ of $\mathcal{R}L$ are defined by the formulas

$$\mathbf{M}_a = \{\alpha \in \mathcal{R}L : \operatorname{coz}(\alpha) \leq a\} \qquad \text{and} \qquad \mathbf{O}_a = \{\alpha \in \mathcal{R}L : \operatorname{coz}(\alpha) \prec\!\!\prec a\}.$$

Now, we put

$$\mathbf{M}_a^c = \mathbf{M}_a \cap \mathcal{C}_c(L) = \{\alpha \in \mathcal{C}_c(L) : \operatorname{coz}(\alpha) \leq a\}$$

and

$$\mathbf{O}_a^c = \mathbf{O}_a \cap \mathcal{C}_c(L) = \{\alpha \in \mathcal{C}_c(L) : \operatorname{coz}(\alpha) \prec\!\!\prec a\}.$$

Clearly, the ideals $\mathbf{M}_a^c$ and $\mathbf{O}_a^c$ are z_c-ideals. To recall, a ring A is regular if and only if all maximal ideals of A are pure (see [4]). This fact is used in part of the proof below.

Theorem 4.5. *The following are equivalent for a frame L.*

1. *L is a CP-frame.*
2. *The ring $\mathcal{C}_c(L)$ is an Artin-Rees ring.*
3. *Every z_c-ideal of $\mathcal{C}_c(L)$ has the Artin-Rees property.*
4. *Every maximal ideal of $\mathcal{C}_c(L)$ has the Artin-Rees property.*
5. *Every ideal of $\mathcal{C}_c(L)$ with the Artin-Rees property is an Artin-Rees ideal.*

Proof. (1) ⇒ (2). By Theorem 3.9, L is a CP-frame if and only if every ideal of $\mathcal{C}_c(L)$ is pure. So, in view of Theorem 4.3, (1) clearly implies (2).

(2) ⇒ (3) ⇒ (4). Obvious.

(4) ⇒ (1). Since every maximal ideal of $\mathcal{C}_c(L)$ is a z_c-ideal, Theorem 4.3 and the present hypothesis imply that all maximal ideals of $\mathcal{C}_c(L)$ are pure. So, $\mathcal{C}_c(L)$ is a regular ring, that is, L is a CP-frame.

To complete of the poof, it remains to show that (1) and (5) are equivalent. Assume L is a CP-frame. Then, the combination of Theorem 3.9 and Lemma 4.4 implies that every ideal of $\mathcal{C}_c(L)$ is an AR-ideal. This shows that (5) holds.

Now, assume (5), and to prove (1) suppose $s \in \mathrm{Coz}_c[L]$ such that $s = \mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{C}_c(L)$. Lemma 3.3 tells us that we can assume $s^* \neq 0$. Then, by zero-dimensionality, there is $t \in \mathrm{Coz}_c[L]$ such that $t \prec\!\!\prec s^*$, and hence $s = \mathrm{coz}(\alpha) \leq s^{**} \prec\!\!\prec t^*$, which implies $\alpha \in \mathbf{O}^c_{t^*}$. By Lemma 3.7, it is easy to see that $\mathbf{O}^c_{t^*}$ is a pure ideal, and so, it has the AR property. Now, the hypothesis implies that $\mathbf{O}^c_{t^*}$ is an AR-ideal. Since $\mathbf{M}^c_s, \langle\alpha\rangle \subseteq \mathbf{O}^c_{t^*}$ and $\left(\mathbf{M}^c_s\right)^n = \mathbf{M}^c_s$, we can conclude that $\mathbf{M}^c_s \cap \langle\alpha\rangle \subseteq \langle\alpha\rangle\mathbf{M}^c_s$. Next, $\alpha \in \mathbf{M}^c_s \cap \langle\alpha\rangle$ implies $\alpha \in \langle\alpha\rangle\mathbf{M}^c_s$, it follows that there exist $\delta \in \mathbf{M}^c_s$ and $\gamma \in \mathcal{C}_c(L)$ so that $\alpha = \alpha\gamma\delta$. Since $\mathrm{coz}(\delta\gamma) \leq \mathrm{coz}\,\delta \leq \mathrm{coz}(\alpha)$, Lemma 3.5 implies that $s = \mathrm{coz}(\alpha) \in B(L)$. □

Let $C(X)$ be the ring of all real-valued continuous functions on a completely regular Hausdorff space X, and let $C_c(X)$ be the subring of $C(X)$ whose elements have countable image. A space X is called a CP-space if $C_c(X)$ is regular (see [16]) The frame of open subsets of a topological space X is denoted by $\mathfrak{O}X$. In [22], it is shown that $C_c(\mathfrak{O}X) \cong C_c(X)$. A direct consequence of the above is the following result.

Corollary 4.6. *The following are equivalent for a space X.*

1. *X is a CP-space.*
2. *The ring $\mathcal{C}_c(X)$ is an Artin-Rees ring.*
3. *Every z-ideal of $\mathcal{C}_c(X)$ has the Artin-Rees property.*
4. *Every maximal ideal of $\mathcal{C}_c(X)$ has the Artin-Rees property.*
5. *Every ideal of $\mathcal{C}_c(X)$ with the Artin-Rees property is an Artin-Rees ideal.*

Let $\mathcal{R}^*L$ be the bounded part of $\mathcal{R}L$. Recall from [5] that a quotient $f : L \to M$ is a *C-quotient* (resp. a *C^*-quotient*) if, for any $\alpha \in \mathcal{R}M$ ($\alpha \in \mathcal{R}^*M$), there is $\bar{\alpha} \in \mathcal{R}L$ ($\bar{\alpha} \in \mathcal{R}^*L$) such that $f\bar{\alpha} = \alpha$. This motivates the following definition.

Definition 4.7. A quotient $f : L \to M$ is a *C_c-quotient* (resp. a *C_c^*-quotient*) if, for any $\alpha \in \mathcal{C}_c(M)$ ($\alpha \in \mathcal{C}_c^*(M)$), there is $\bar{\alpha} \in \mathcal{C}_c(L)$ ($\bar{\alpha} \in \mathcal{C}_c^*(L)$) such that $f\bar{\alpha} = \alpha$.

Let $a \in L$ and $\alpha \in \mathcal{R}L$. We write $\alpha_{|a}$ and $\alpha_{|\mathfrak{c}(a)}$ for the elements of $\mathcal{R}(\downarrow a)$ and $\mathcal{R}(\uparrow a)$ defined by the composites $\mathcal{L}(\mathbb{R}) \xrightarrow{\alpha} L \xrightarrow{-\wedge a} \downarrow a$ and $\mathcal{L}(\mathbb{R}) \xrightarrow{\alpha} L \xrightarrow{-\vee a} \uparrow a$, respectively. It is easy to see $\alpha_{|a} \in \mathcal{C}_c(\downarrow a)$ and $\alpha_{|\mathfrak{c}(a)} \in \mathcal{C}_c(\uparrow a)$ whenever $\alpha \in \mathcal{C}_c(L)$.

Lemma 4.8. *The following statements hold for any $a \in B(L)$.*

(1) *Let $f \in \mathcal{C}_c(\uparrow a)$. For each $p, q \in \mathbb{Q}$, define*

$$\alpha(p,q) = \begin{cases} f(p,q) & \text{if} \quad p < 0 < q \\ a^* \wedge f(p,q) & \text{if} \quad p < q \leq 0 \text{ or } 0 \leq p < q. \end{cases}$$

Then $\alpha \in \mathcal{C}_c(L)$ such that $\alpha_{|\mathfrak{c}(a)} = f$.

(2) *Let $g \in \mathcal{C}_c(\downarrow a)$. For each $p, q \in \mathbb{Q}$, define*

$$\beta(p,q) = \begin{cases} a^* \vee g(p,q) & \text{if} \quad p < 0 < q \\ g(p,q) & \text{if} \quad p < q \leq 0 \text{ or } 0 \leq p < q. \end{cases}$$

Then $\beta \in \mathcal{C}_c(L)$ such that $\beta_{|a} = f$.

(3) *The closed quotient $\uparrow a$ and the open quotient $\downarrow a$ are C_c-quotients.*

Proof. (1). Lemma 6 in [3] implies that $\alpha \in \mathcal{R}L$ and $\alpha_{|\mathfrak{c}(a)} = f$. It remains to show that $\alpha \in \mathcal{C}_c(L)$. Let $r \in \mathbb{R}$. Then

$$\begin{aligned} \operatorname{coz}(\alpha - \mathbf{r}) &= \alpha(-,r) \vee \alpha(r,-) \\ &= \begin{cases} \Big(a^* \wedge f(-,r)\Big) \vee f(r,-) & \text{if} \quad r < 0 \\ a^* \wedge \operatorname{coz}(f) & \text{if} \quad r = 0 \\ f(-,r) \vee \Big(a^* \wedge f(r,-)\Big) & \text{if} \quad r > 0 \end{cases} \\ &= \begin{cases} \operatorname{coz}(f - \mathbf{r}) \wedge \Big(f(r,-) \vee a^*\Big) & \text{if} \quad r < 0 \\ a^* \wedge \operatorname{coz}(f) & \text{if} \quad r = 0 \\ \operatorname{coz}(f - \mathbf{r}) \wedge \Big(f(-,r) \vee a^*\Big) & \text{if} \quad r > 0 \end{cases} \\ &= \begin{cases} \operatorname{coz}(f - \mathbf{r}) & \text{if} \quad r \neq 0 \\ a^* \wedge \operatorname{coz}(f) & \text{if} \quad r = 0, \end{cases} \end{aligned}$$

since $f(r,-) \vee a^* = 1 = f(-,r) \vee a^*$. Therefore $R_\alpha \subseteq R_f \cup \{0\}$. Now since R_f is countable, we can conclude that R_α is countable, meaning that $\alpha \in \mathcal{C}_c(L)$.

(2). We show first that the map β satisfies the relations (R1) – (R4).

(R1). We must show that $\beta(p,q) \wedge \beta(r,s) = \beta(p \vee r, q \wedge s)$ for every $p,q,r,s \in \mathbb{Q}$. Consider the following four cases:

Case 1: $p < q \leq 0$ and $r < s \leq 0$.

$$\begin{aligned} \beta(p,q) \wedge \beta(r,s) &= g(p,q) \wedge g(r,s) \\ &= g(p \vee r, q \wedge s) \\ &= \beta(p \vee r, q \wedge s). \end{aligned}$$

Case 2: $0 \leq p < q$ and $0 \leq r < s$. Similar to Case 1.

Case 3: $p < q \leq 0$ and $r < 0 < s$. Now, if $r < p < q \leq 0 < s$, then

$$\begin{aligned} \beta(p,q) \wedge \beta(r,s) &= g(p,q) \wedge \big(a^* \vee g(r,s)\big) \\ &= \big(g(p,q) \wedge a^*\big) \vee \big(g(p,q) \wedge g(r,s)\big) \\ &= g(p,q) \wedge g(r,s) \qquad \text{since } g(p,q) \leq a \\ &= g(p \vee r, q \wedge s) \\ &= g(p \vee r, q) \\ &= \beta(p \vee r, q) \\ &= \beta(p \vee r, q \wedge s). \end{aligned}$$

Case 4: $p < 0 < q$ and $0 \leq r < s$. Similar to Case 3.

(R2). We must show that $\beta(p,q) \vee \beta(r,s) = \beta(p,s)$ whenever $p \leq r < q \leq s$. Consider the following five cases:

Case 1: $p \leq r < q \leq s < 0$.

$$\begin{aligned} \beta(p,q) \vee \beta(r,s) &= g(p,q) \vee g(r,s) \\ &= g(p,s) \\ &= \beta(p,s). \end{aligned}$$

Case 2: $0 \leq p \leq r < q \leq s$. Similar to Case 1.

Case 3: $p \leq r < 0 < q \leq s$.

$$\begin{aligned} \beta(p,q) \vee \beta(r,s) &= \big(a* \vee g(p,q)\big) \vee \big(a^* \vee g(r,s)\big) \\ &= a^* \vee \big(g(p,q) \vee g(r,s)\big) \\ &= a^* \vee g(p,s) \\ &= \beta(p,s). \end{aligned}$$

Case 4: $p \leq r < q \leq 0 < s$.

$$\begin{aligned} \beta(p,q) \vee \beta(r,s) &= g(p,q) \vee \big(a^* \vee g(r,s)\big) \\ &= a^* \vee \big(g(p,q) \vee g(r,s)\big) \\ &= a^* \vee g(p,s) \\ &= \beta(p,s). \end{aligned}$$

Case 5: $p \leq 0 < r < q \leq s$. Similar to Case 4.

(R3). We must show that $\beta(p,q) = \bigvee\{\beta(r,s) : p < r < s < q\}$ for every $p,q \in \mathbb{Q}$. Consider the following three cases:

Case 1: $p < q \leq 0$.

$$\begin{aligned} \beta(p,q) &= g(p,q) \\ &= \bigvee\{g(r,s) : p < r < s < q\} \\ &= \bigvee\{\beta(r,s) : p < r < s < q\}. \end{aligned}$$

Case 2: $0 \leq p < q$. Similar to Case 1.
Case 3: $p < 0 < q$. Since

$$\begin{aligned} \bigvee\{\beta(r,s) : p < r < 0 < s < q\} &= \bigvee\{a^* \vee g(r,s) : p < r < 0 < s < q\} \\ &= a^* \vee \bigvee\{g(r,s) : p < r < 0 < s < q\} \\ &= a^* \vee g(p,q) \\ &= \beta(p,q) \end{aligned}$$

and $\{(r,s) : p < r < 0 < s < q\} \subseteq \{(r,s) : p < r < s < q\}$, so

$$g(p,q) \leq \bigvee\{g(r,s) : p < r < s < q\}.$$

By R1 or R2, g preserves order. Therefore

$$g(p,q) = \bigvee\{g(r,s) : p < r < s < q\}.$$

(R4). We must show that $\bigvee\{\beta(p,q) : p,q \in \mathbb{Q}\} = \top$.

$$\begin{aligned} \bigvee\{\beta(p,q) : p,q \in \mathbb{Q}\} &= \bigvee\{\beta(p,q) : p < 0 < q\} \\ &\quad \vee \bigvee\{\beta(p,q) : p < q \leq 0 \text{ or } 0 \leq p < q\} \\ &= \bigvee\{a^* \vee g(p,q) : p < 0 < q\} \\ &\quad \vee \bigvee\{g(p,q) : p < q \leq 0 \text{ or } 0 \leq p < q\} \\ &= a^* \vee \bigvee\{g(p,q) : p,q \in \mathbb{Q}\} \\ &= a^* \vee \top_{\downarrow a} \\ &= a^* \vee a = \top, \quad \text{since } a \in B(L). \end{aligned}$$

Hence $\beta \in \mathcal{R}L$.

Now, we show that $\beta_{|a} = f$. Let $p,q \in \mathbb{Q}$. If $p < 0 < q$, then

$$(\beta_{|a})(p,q) = a \wedge \big(a^* \vee g(p,q)\big) = \big(a \wedge a^*\big) \vee g(p,q) \geq g(p,q).$$

Otherwise,

$$(\beta_{|a})(p,q) = a \wedge g(p,q) = g(p,q).$$

In consequence, for all $p,q \in \mathbb{Q}$, $g(p,q) \leq (\beta_{|a})(p,q)$. In light of the fact that if M is a regular frame and $h, g : M \to L$ are frame maps such that $h(x) \leq g(x)$ for all $x \in M$, then $h = g$, we have $\beta_{|a} = f$ because $\mathcal{L}(\mathbb{R})$ is completely regular.

It remains to show that $\beta \in \mathcal{C}_c(L)$. Let $r \in \mathbb{R}$. Then

$$\begin{aligned}
\mathrm{coz}(\beta - \mathbf{r}) &= \beta(-,r) \vee \beta(r,-) \\
&= \begin{cases} g(-,r) \vee \Big(a^* \vee g(r,-)\Big) & \text{if} \quad r < 0 \\ \mathrm{coz}(g) & \text{if} \quad r = 0 \\ \Big(a^* \vee g(-,r)\Big) \vee g(r,-) & \text{if} \quad r > 0 \end{cases} \\
&= \begin{cases} a^* \vee \mathrm{coz}(g - \mathbf{r}) & \text{if} \quad r \neq 0 \\ \mathrm{coz}(g) & \text{if} \quad r = 0. \end{cases}
\end{aligned}$$

This implies that $R_\beta \subseteq R_g$. Since R_g is countable, we can conclude that R_β is countable, meaning that $\beta \in \mathcal{C}_c(L)$.

(3). By (1), it is clear that the closed quotient $\uparrow a$ is a C_c-quotient. Also, by (2), it is clear that the open quotient $\downarrow a$ is a C_c-quotient. □

To end, we intend to show that CP-frames can be characterized in terms of the Artin-Rees property in some ideals of $\mathcal{C}_c(L)$ and in some factor rings of $\mathcal{C}_c(L)$. Let us pause to make the next lemma and proposition. We write $()^{\circledast}$ for pseudocomplementation in $\uparrow a$. Recall from [8, Lemma 4.5] that if $a = a^{**}$, then $b^{\circledast} = (b \wedge a^*)^*$ for any $b \in \uparrow a$. The following lemma is an immediate corollary from [3, Lemma 7]. Here we give a directly proof.

Lemma 4.9. *Let $a \in B(L)$ and $b \in \uparrow a$. Then $b \in B(L)$ if and only if $b \in B(\uparrow a)$.*

Proof. Since $b \vee b^* \leq b \vee \big(b \wedge a^*\big)^* = b \vee b^{\circledast}$, we can conclude that if $b \in B(L)$, then we have $b \in B(\uparrow a)$. For the converse, let $b \in B(\uparrow a)$. We must show hat $b \vee b^* = 1$. First, we claim that $\big(b \wedge a^*\big)^* = a \vee b^*$. Clearly, the inequality $a \vee b^* \leq \big(b \wedge a^*\big)^*$ is true. To show the inequality $\big(b \wedge a^*\big)^* \leq a \vee b^*$, let $x \wedge b \wedge a^* = 0$. Putting $y_x = x \wedge a^*$, we then have

$$a \vee y_x = a \vee (x \wedge a^*) = (a \vee x) \wedge (a \vee a^*) = (a \vee x) \wedge 1 = a \vee x \geq x.$$

But $y_x \wedge b = x \wedge b \wedge a^* = 0$ implies that $x \leq \bigvee\{a \vee y : y \wedge b = 0\} = a \vee b^*$. Thus

$$(b \wedge a^*)^* = \bigvee\{x : x \wedge b \wedge a^* = 0\} \leq a \vee b^*,$$

this concludes the proof of the claim. Now, due to the present hypothesis, we obtain

$$1 = b \vee b^{\circledast} = b \vee (b \wedge a^*)^* = b \vee a \vee b^* = b \vee b^*,$$

this means that $b \in B(L)$. □

Proposition 4.10. *For any element $\alpha \in \mathcal{C}_c(L)$, the following are equivalent.*

1. *The ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an Artin-Rees ring.*
2. *The closed quotient $\uparrow\mathrm{coz}(\alpha)$ is a CP-frame and $\mathrm{coz}(\alpha) \in B(L)$.*
3. *The ideal $\mathrm{Ann}(\alpha) = \{\beta \in \mathcal{C}_c(L) : \alpha\beta = \mathbf{0}\}$ and the element α are regular.*
4. *The ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is regular.*
5. *Any ideal $\mathbf{M}^c_{\mathrm{coz}(\beta)}$ of $\mathcal{C}_c(L)$ containing α has the Artin-Rees property.*

Proof. (1) $\Rightarrow$ (2). We first show that $s = \mathrm{coz}(\alpha) \in B(L)$. Since $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an AR-ring, there is $n \in \mathbb{N}$ such that

$$\Big(\mathbf{M}^c_s/\langle\alpha\rangle\Big)^n \cap \Big(\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle\Big) \subseteq \Big(\mathbf{M}^c_s/\langle\alpha\rangle\Big)\Big(\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle\Big).$$

Since $\Big(\mathbf{M}^c_s/\langle\alpha\rangle\Big)^n = \mathbf{M}^c_s/\langle\alpha\rangle$ and $\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle \subseteq \mathbf{M}^c_s/\langle\alpha\rangle$, we can conclude that

$$\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle \subseteq \Big(\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle\Big)\Big(\mathbf{M}^c_s/\langle\varphi\rangle\Big),$$

which shows that $\alpha^{\frac{1}{3}} + \alpha \in \Big(\langle\alpha^{\frac{1}{3}}\rangle/\langle\alpha\rangle\Big)\Big(\mathbf{M}^c_s/\langle\alpha\rangle\Big)$. So there are $\alpha_1 \cdots \alpha_m \in \mathbf{M}^c_s$ and $\beta_1 \cdots \beta_m \in \mathcal{C}_c(L)$ such that $\alpha^{\frac{1}{3}} - \sum_{i=1}^m \alpha_i\alpha^{\frac{1}{3}}\beta_i \in \langle\alpha\rangle$, implying that $\alpha^{\frac{1}{3}} - \alpha^{\frac{1}{3}}\sum_{i=1}^m \alpha_i\beta_i = \alpha\delta$ for some $\delta \in \mathcal{C}_c(L)$. Now, we set $\gamma = \sum_{i=1}^m \alpha_i\beta_i$ and consider $\varphi = \mathbf{1} - \gamma - \alpha^{\frac{2}{3}}\delta$. We then have $\alpha^{\frac{1}{3}}\varphi = \mathbf{0}$, that is, $0 = \mathrm{coz}(\alpha^{\frac{1}{3}}) \wedge \mathrm{coz}(\varphi) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\varphi)$. On the other hand,

$$\begin{aligned}
1 = \mathrm{coz}(\varphi + \gamma + \alpha^{\frac{2}{3}}\delta) &\leq \mathrm{coz}(\varphi) \vee \mathrm{coz}(\gamma + \alpha^{\frac{2}{3}}\delta)\\
&\leq \mathrm{coz}(\varphi) \vee \mathrm{coz}(\gamma) \vee \mathrm{coz}(\alpha^{\frac{2}{3}}\delta)\\
&\leq \mathrm{coz}(\varphi) \vee \mathrm{coz}(\gamma) \vee \mathrm{coz}(\alpha)\\
&= \mathrm{coz}(\varphi) \vee \mathrm{coz}(\alpha), \quad \text{since} \quad \mathrm{coz}(\gamma) \leq \mathrm{coz}(\alpha).
\end{aligned}$$

Therefore, $s = \mathrm{coz}(\alpha) \in B(L)$.

We now show that the closed quotient $\uparrow\mathrm{coz}(\alpha)$ is a CP-frame. Define

$$\Theta : \mathcal{C}_c(L) \to \mathcal{C}_c(\uparrow\mathrm{coz}(\alpha)) \quad \text{by giving} \quad \Theta(\beta) = \beta_{|\mathfrak{c}(\mathrm{coz}(\alpha))}.$$

By Lemma 4.8(2), we can conclude $\uparrow\mathrm{coz}(\alpha)$ is a C_c-quotient. Now, it is easy to see that the map Θ is an onto ring homomorphism. We claim that the rings $\mathcal{C}_c(L)/\langle\alpha\rangle$ and $\mathcal{C}_c(\uparrow\mathrm{coz}\,\varphi)$ are isomorphic. To see this, it is enough to show that

$$\ker(\Theta) = \{\beta \in \mathcal{C}_c(L) : \Theta(\beta) = \mathbf{0}_{\mathcal{C}_c(\uparrow\mathrm{coz}(\alpha))} = \mathbf{0}_{\mathfrak{c}(\mathrm{coz}(\alpha))}\} = \langle\alpha\rangle.$$

Assume $\beta \in \ker(\Theta)$. Then $\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) = \operatorname{coz}(\alpha)$, which yields $\operatorname{coz}(\beta) \leq \operatorname{coz}(\alpha)$, and so, we infer from Proposition 3.2 that $\operatorname{coz}(\beta) \prec \operatorname{coz}(\alpha)$ in $\operatorname{Coz}_c[L]$ since $\operatorname{coz}(\alpha) \in B(L)$. Now, Lemma 3.4 implies that β is multiple of α, that is, $\beta \in \langle\alpha\rangle$. Therefore, $\ker(\Theta) \subseteq \langle\alpha\rangle$. Now let $\beta \in \langle\alpha\rangle$. This implies that there exist $\delta \in \mathcal{C}_c(L)$ such that $\beta = \delta\alpha$, showing that $\operatorname{coz}(\beta) \leq \operatorname{coz}(\alpha)$. In consequence,

$$\operatorname{coz}(\Theta(\beta)) = \operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) = \operatorname{coz}(\alpha) = 0_{\uparrow \operatorname{coz}(\alpha)},$$

and hence $\Theta(\beta) = \mathbf{0}_{\mathfrak{c}(\operatorname{coz}(\alpha))}$, that is, $\beta \in \ker(\Theta)$. So the reverse inclusion also holds. Thus, the rings $\mathcal{C}_c(L)/\langle\alpha\rangle$ and $\mathcal{C}_c(\uparrow\operatorname{coz}\varphi)$ are isomorphic.

Now, due to the present hypothesis, we have $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an AR-ring. Consequently, $\mathcal{C}_c(\uparrow\operatorname{coz}(\alpha))$ is an AR-ring, and hence, Theorem 4.6 shows that $\uparrow\operatorname{coz}(\alpha)$ is a CP-frame.

(2) $\Rightarrow$ (3). Since $\operatorname{coz}(\alpha) \in B(L)$, Lemma 3.5 shows that α is a regular element. Now, it remains to show that $\operatorname{Ann}(\alpha)$ is a regular ideal. To see this, we must prove that every element of $\operatorname{Ann}(\alpha)$ is a regular element of $\mathcal{C}_c(L)$. Let $\beta \in \operatorname{Ann}(\alpha)$. Again by Lemma 3.5, we intend to prove that $\operatorname{coz}(\beta) \in B(L)$. Since $\beta \in \operatorname{Ann}(\alpha)$, we have $\operatorname{coz}(\beta) \wedge \operatorname{coz}(\alpha) = 0$, and hence $\operatorname{coz}(\beta) \leq (\operatorname{coz}(\alpha))^*$. This shows that

$$\begin{aligned}
\operatorname{coz}(\beta) &= (\operatorname{coz}(\alpha))^* \wedge \operatorname{coz}(\beta) \\
&= ((\operatorname{coz}(\alpha))^* \wedge \operatorname{coz}(\beta)) \vee 0 \\
&= \big((\operatorname{coz}(\alpha))^* \wedge \operatorname{coz}(\beta)\big) \vee \big((\operatorname{coz}(\alpha))^* \wedge \operatorname{coz}(\alpha)\big) \\
&= (\operatorname{coz}(\alpha))^* \wedge (\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha)).
\end{aligned}$$

In consequence, we need only show $\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) \in B(L)$ since $(\operatorname{coz}(\alpha))^* \in B(L)$. But $\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) \in B(\uparrow\operatorname{coz}(\alpha))$ since $\uparrow\operatorname{coz}(\alpha)$ is a CP-frame and $\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) = \operatorname{coz}\left(\beta_{|\mathfrak{c}(\operatorname{coz}(\alpha))}\right) \in \operatorname{Coz}_c[\uparrow\operatorname{coz}(\alpha)]$. So, by Lemma 4.9, we can conclude that $\operatorname{coz}(\beta) \vee \operatorname{coz}(\alpha) \in B(L)$ because $\operatorname{coz}(\alpha) \in B(L)$.

(3) $\Rightarrow$ (4). By Lemma 3.5, $\operatorname{coz}(\alpha) \in B(L)$ since α is a regular element. So, as proved in the proof of the implication (1) $\Rightarrow$ (2), we have $\mathcal{C}_c(L)/\langle\alpha\rangle \cong \mathcal{C}_c(\uparrow\operatorname{coz}(\alpha))$. We now intend to show that the ring $\mathcal{C}_c(\uparrow\operatorname{coz}(\alpha))$ is regular, or equivalently, it suffices to prove that $\uparrow\operatorname{coz}(\alpha)$ is a CP-frame. To see this, we must prove $t \vee t^{\circledast} = 1$ for any $t \in \operatorname{Coz}_c[\uparrow \operatorname{coz}(\alpha)]$. Let $t = \operatorname{coz}(f) \in \operatorname{Coz}_c[\uparrow\operatorname{coz}(\alpha)]$ with $f \in \mathcal{C}_c(\uparrow\operatorname{coz}(\alpha))$. Then define

$$\delta(p,q) = \begin{cases} f(p,q) & \text{if} \quad p < 0 < q \\ (\operatorname{coz}(\alpha))^* \wedge f(p,q) & \text{if} \quad p < q \leq 0 \text{ or } 0 \leq p < q. \end{cases}$$

since $\mathrm{coz}(\alpha) \in B(L)$, By Lemma 4.8, we can conclude that $\delta \in \mathcal{C}_c(L)$. Clearly, $\mathrm{coz}(\delta) = (\mathrm{coz}(\alpha))^* \wedge t$, and so $\mathrm{coz}(\delta) \wedge \mathrm{coz}(\alpha) = \mathrm{coz}(\alpha) \wedge (\mathrm{coz}(\alpha))^* \wedge t = 0$. This shows that $\delta\alpha = \mathbf{0}$, that is, $\delta \in \mathrm{Ann}(\alpha)$. Now, the hypothesis in (3) implies that δ is a regular element of $\mathcal{C}_c(L)$, and so, by Lemma 3.5, we have $\mathrm{coz}(\delta) \in B(L)$. But

$$\begin{aligned} t = t \wedge 1 = t \wedge \big(\mathrm{coz}(\alpha) \vee (\mathrm{coz}(\alpha))^*\big) &= \big(t \wedge \mathrm{coz}(\alpha)\big) \vee \big((\mathrm{coz}(\alpha))^* \wedge t\big) \\ &= \mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta) \end{aligned}$$

implies that

$$\begin{aligned} t \vee t^{\circledast} &= t \vee (\mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta))^{\circledast} \\ &= t \vee \Big(\big(\mathrm{coz}(\alpha) \vee \mathrm{coz}(\delta)\big) \wedge (\mathrm{coz}(\alpha))^*\Big)^* \\ &= t \vee \Big(\big(\mathrm{coz}(\alpha) \wedge (\mathrm{coz}(\alpha))^* \vee (\mathrm{coz}(\delta) \wedge (\mathrm{coz}(\alpha))^*\big)\Big)^* \\ &= t \vee \Big(\mathrm{coz}(\delta) \wedge (\mathrm{coz}(\alpha))^*\Big)^* \\ &= t \vee (\mathrm{coz}(\delta))^* && \text{since} \quad \mathrm{coz}(\delta) \leq (\mathrm{coz}(\alpha))^* \\ &\geq \mathrm{coz}(\delta) \vee (\mathrm{coz}(\delta))^* && \text{since} \quad t \geq \mathrm{coz}(\delta) \\ &= 1 && \text{since} \quad \mathrm{coz}(\delta) \in B(L). \end{aligned}$$

(4) $\Rightarrow$ (1). This is a consequence of the fact that any regular ring is an AR-ring.

To complete the proof of the proposition, we aim to show that (2) and (5) are equivalent. Assume (2), and to prove (5) consider $\beta \in \mathcal{C}_c(L)$ such that $\alpha \in \mathbf{M}^c_{\mathrm{coz}(\beta)}$. We must prove that the ideal $\mathbf{M}^c_{\mathrm{coz}(\beta)}$ has the Artin-Rees property. Let Q be an ideal of $\mathcal{C}_c(L)$. We claim that $Q \cap \mathbf{M}^c_{\mathrm{coz}(\beta)} \subseteq Q\mathbf{M}^c_{\mathrm{coz}(\beta)}$. We first show that $\beta = \eta_{coz(\beta)}\beta$. Since $\alpha \in \mathbf{M}^c_{\mathrm{coz}(\beta)}$, that is, $\mathrm{coz}(\alpha) \leq \mathrm{coz}(\beta)$, we obtain

$$\mathrm{coz}(\beta) = \mathrm{coz}(\beta) \vee \mathrm{coz}(\alpha) = \mathrm{coz}(\beta|_{\mathfrak{c}(\mathrm{coz}(\alpha))}) \in \mathrm{Coz}_c[{\uparrow}\mathrm{coz}(\alpha)].$$

This implies $\mathrm{coz}(\beta) \in B({\uparrow}\mathrm{coz}(\alpha))$ since ${\uparrow}\mathrm{coz}(\alpha)$ is a CP-frame. But one of the hypotheses says $\mathrm{coz}(\alpha) \in B(L)$, and so we get $\mathrm{coz}(\beta) \in B(L)$ by Lemma 4.9. This shows that $\mathrm{coz}(\eta_{coz(\beta)}) = \mathrm{coz}(\beta)$ and $\mathrm{coz}(\mathbf{1} - \eta_{coz(\beta)}) = (\mathrm{coz}(\beta))^*$. In sequence, $\mathrm{coz}(\beta) \wedge \mathrm{coz}(\mathbf{1} - \eta_{coz(\beta)}) = 0$, that is, $\beta(\mathbf{1} - \eta_{coz(\beta)}) = \mathbf{0}$, so that $\beta = \eta_{coz(\beta)}\beta$.

Now to the proof of the claim, consider any $\varphi \in Q \cap \mathbf{M}^c_{\mathrm{coz}(\beta)}$. Then $\mathrm{coz}(\varphi) \leq \mathrm{coz}(\beta)$. Since $\mathrm{coz}(\beta) \in B(L)$, we infer from Proposition 3.2 that $\mathrm{coz}(\varphi) \prec \mathrm{coz}(\beta)$ in $\mathrm{Coz}_c[L]$, and so, by Lemma 3.4, there is $\delta \in \mathcal{C}_c(L)$ such that $\varphi = \beta\delta$. But $\beta = \eta_{coz(\beta)}\beta$ implies that $\varphi = (\beta\delta)\eta_{coz(\beta))}$. Now, since $\beta\delta = \varphi \in Q$ and $\eta_{coz(\beta))} \in \mathbf{M}^c_{\mathrm{coz}(\beta)}$, we would have $\varphi = (\beta\delta)\eta_{coz(\beta))} \in Q\mathbf{M}^c_{\mathrm{coz}(\beta)}$, which completes the proof of the claim.

Now assume (5) holds and, to establish (2), we begin by showing that $\mathrm{coz}(\alpha) \in B(L)$. By the hypothesis, $\alpha \in \mathbf{M}^c_{\mathrm{coz}(\alpha)}$ implies that $\mathbf{M}^c_{\mathrm{coz}(\alpha)}$ has the AR property. Hence there is $n \in \mathbb{N}$ such that $\left(\mathbf{M}^c_{\mathrm{coz}(\alpha)}\right)^n \cap \langle\alpha\rangle \subseteq \mathbf{M}^c_{\mathrm{coz}(\alpha)}\langle\alpha\rangle$, this shows that $\mathbf{M}_{\mathrm{coz}(\alpha)} \cap \langle\alpha\rangle \subseteq \mathbf{M}^c_{\mathrm{coz}(\alpha)}\langle\alpha\rangle$ because $\left(\mathbf{M}^c_{\mathrm{coz}\,\alpha}\right)^n = \mathbf{M}^c_{\mathrm{coz}\,\alpha}$. But $\alpha \in \mathbf{M}^c_{\mathrm{coz}(\alpha)} \cap \langle\alpha\rangle \subseteq \mathbf{M}^c_{\mathrm{coz}(\alpha)}\langle\alpha\rangle$ implies $\alpha = \delta\alpha$ for some $\delta \in \mathbf{M}_{\mathrm{coz}(\alpha)}$. Thus, by Lemma 3.5, we obtain $\mathrm{coz}\,\alpha \in B(L)$.

To complete the proof of (2) from (5), it remains to prove that the closed quotient $\uparrow\mathrm{coz}(\alpha)$ is a CP-frame. Let $t \in \mathrm{Coz}_c[\uparrow\mathrm{coz}(\alpha)]$ with $t = \mathrm{coz}\,f$ for some $f \in \mathcal{C}_c(\uparrow\mathrm{coz}(\alpha))$. We intend to prove $t \vee t^{\circledast} = 1$. Define

$$\varphi(p,q) = \begin{cases} f(p,q) & \text{if} \quad p < 1 < q \\ (\mathrm{coz}(\alpha))^* \wedge f(p,q) & \text{if} \quad p < q \leq 1 \text{ or } 1 \leq p < q. \end{cases}$$

Similar to Lemma 6 in [3], we can show that $\varphi \in \mathcal{R}L$. Since $\mathrm{coz}\,\alpha \in B(L)$, Lemma 4.8 implies that $\varphi \in \mathcal{C}_c(L)$. And because

$$\begin{aligned} \mathrm{coz}(\varphi) &= \varphi(-,0) \vee \varphi(0,-) = \big((\mathrm{coz}(\alpha))^* \wedge f(-,0)\big) \vee f(0,-) \\ &= \big((\mathrm{coz}(\alpha))^* \vee f(0,-)\big) \wedge \big(f(-,0) \vee f(0,-)\big) \\ &= \mathrm{coz}(f), \qquad \text{since} \qquad \big((\mathrm{coz}(\alpha))^* \vee f(0,-) = 1, \end{aligned}$$

we would have $\mathrm{coz}(\alpha) \leq \mathrm{coz}\,f = \mathrm{coz}(\varphi)$, that is, $\alpha \in \mathbf{M}^c_{\mathrm{coz}(\varphi)}$. So the hypothesis in (5) implies that the ideal $\mathbf{M}^c_{\mathrm{coz}\,\varphi}$ has the AR property. Now, as already shown above, we can conclude that $\mathrm{coz}(\varphi) \in B(L)$, and so $t = \mathrm{coz}(f) = \mathrm{coz}(\varphi) \in B(L)$. In consequence,

$$t \vee t^{\circledast} = t \vee (t \wedge (\mathrm{coz}(\alpha))^*)^* \geq t \vee t^* = 1.$$

This completes the proof of the proposition. □

We can now state the characterizations of CP-frames in terms of the Artin-Rees property in some ideals of $\mathcal{C}_c(L)$ and in some factor rings of $\mathcal{C}_c(L)$.

Theorem 4.11. *The following are equivalent for any frame L.*

1. *L is a CP-frame.*
2. *The ring $\mathcal{C}_c(L)/\langle\alpha\rangle$ is an Artin-Rees ring for any $\alpha \in \mathcal{C}_c(L)$.*
3. *The ring $\mathcal{C}_c(L)/P$ is an Artin-Rees ring for any prime ideal P of $\mathcal{C}_c(L)$.*
4. *The ring $\mathcal{C}_c(L)$ contains an Artin-Rees maximal ideal.*
5. *The ring $\mathcal{C}_c(L)$ contains a regular maximal ideal.*

(6) *There exists an element $\gamma \in \mathcal{C}_c(L)$ such that $\langle\gamma\rangle$ is a regular ideal and $\mathcal{C}_c(L)/\langle\gamma\rangle$ is an Artin-Rees ring.*
(7) *Every minimal prime ideal of $\mathcal{C}_c(L)$ is a regular ideal.*
(8) *Every minimal prime ideal of $\mathcal{C}_c(L)$ is an Artin-Rees ideal.*

Proof. The equivalence of (1) and (2) is a immediate from Proposition 4.10. Let L is a CP-frame. Then every prime ideal of $\mathcal{C}_c(L)$ is a maximal ideal, and hence $\mathcal{C}_c(L)/P$ is a feild for any prime ideal P of $\mathcal{C}_c(L)$. In consequence, the implication from (1) to (3) is immediate. Next, since every prime ideal of $\mathcal{C}_c(L)$ is a regular ideal when L is a CP-frame, we can conclude that (1) implies (4), (5), (6), (7) and (8). To complete the proof of the theorem, it remains to prove that (3), (4), (5), (6), (7) and (8) imply (1).

To show (3) implies (1), we intend to prove every prime ideal of $\mathcal{C}_c(L)$ is a maximal ideal. Suppose P is a prime ideal of $\mathcal{C}_c(L)$ and let M be a unique maximal ideal containing P. If $P = M$, then we are done. If $P \neq M$, then we arrive at a contradiction as follows. Take $\alpha \in M \setminus P$. Then the present hypothesis implies that there is $n \in \mathbb{N}$ for which

$$\frac{\langle\alpha\rangle + P}{P} \bigcap \Big(\frac{\mathbf{M}^c_{\operatorname{coz}(\alpha)} + P}{P}\Big)^n \subseteq \frac{\langle\alpha\rangle + P}{P}\frac{\mathbf{M}^c_{\operatorname{coz}(\alpha)} + P}{P}.$$

We claim that

$$\frac{\mathbf{M}_{\operatorname{coz}(\alpha)} + P}{P} = \Big(\frac{\mathbf{M}_{\operatorname{coz}(\alpha)} + P}{P}\Big)^n.$$

Since $\mathbf{M}^c_{\operatorname{coz}(\alpha)}$ is a z_c-ideal, the above equality holds when $P \subseteq \mathbf{M}^c_{\operatorname{coz}(\alpha)}$. Otherwise, since $\mathbf{M}^c_{\operatorname{coz}(\alpha)} \nsubseteq P$, the above equality follows from the fact that the sum of a z-ideals and a prime ideal in a ring which are not in a chain is a prime z-ideal (see [18]). Thus the claimed equality holds. But $\alpha \in \mathbf{M}^c_{\operatorname{coz}\varphi}$ implies

$$\frac{\langle\alpha\rangle + P}{P} \subseteq \frac{\mathbf{M}^c_{\operatorname{coz}(\alpha)} + P}{P}.$$

In consequence, we can conclude

$$\frac{\langle\alpha\rangle + P}{P} \subseteq \frac{\langle\alpha\rangle + P}{P}\frac{\mathbf{M}^c_{\operatorname{coz}(\alpha)} + P}{P}.$$

This shows that there is $\beta \in \mathbf{M}^c_{\operatorname{coz}(\alpha)}$ such that $\alpha(\mathbf{1} - \beta) = \alpha - \alpha\beta \in P$. Now $\alpha \notin P$ implies $\mathbf{1} - \beta \in P \subseteq M$. So, we obtain

$$1 = \operatorname{coz}(\mathbf{1} - \beta + \beta) \leq \operatorname{coz}(\mathbf{1} - \beta) \vee \operatorname{coz}(\beta) \leq \operatorname{coz}(\mathbf{1} - \beta) \vee \operatorname{coz}(\alpha),$$

which shows that $1 \in \operatorname{Coz}_c[M] = \{\operatorname{coz}\varphi : \varphi \in M\}$. This is to say that $\mathbf{1} \in M$, which is of course false.

Assume (4), and to show (1), suppose M is a maximal AR-ideal of $\mathcal{C}_c(L)$. To prove (1), consider $\operatorname{coz}(\alpha) \in \operatorname{Coz}_c[L]$ with $\alpha \in \mathcal{C}_c(L)$. If $\alpha \in M$, then $\mathbf{M}^c_{\operatorname{coz}(\alpha)} \subseteq M$, and so, by the current hypothesis, there exists $n \in \mathbb{N}$ such that $\langle\alpha\rangle \cap \big(\mathbf{M}^c_{\operatorname{coz}(\alpha)}\big)^n \subseteq \langle\alpha\rangle\mathbf{M}^c_{\operatorname{coz}(\alpha)}$. Since $\big(\mathbf{M}^c_{\operatorname{coz}(\alpha)}\big)^n = \mathbf{M}^c_{\operatorname{coz}(\alpha)}$, we would have

$$\alpha \in \langle\alpha\rangle \cap \mathbf{M}_{\operatorname{coz}(\alpha)} = \langle\alpha\rangle \cap \big(\mathbf{M}^c_{\operatorname{coz}(\alpha)}\big)^n \subseteq \langle\alpha\rangle\mathbf{M}_{\operatorname{coz}(\alpha)}.$$

This shows that $\alpha = \alpha\delta\beta$ where $\delta \in \mathcal{C}_c(L)$ and $\beta \in \mathbf{M}^c_{\operatorname{coz}(\alpha)}$. Since $\operatorname{coz}(\delta\beta) \leq \operatorname{coz}(\beta) \leq \operatorname{coz}(\alpha)$, Lemma 3.5 implies that $\operatorname{coz}(\alpha) \in B(L)$.

Now, let $\alpha \notin M$. Then $M + \langle\alpha\rangle = \mathcal{C}_c(L)$. So there are $\beta \in M$ and $\delta \in \mathcal{C}_c(L)$ such that $\mathbf{1} = \beta + \delta\alpha$, which implies $\operatorname{coz}\alpha \vee \operatorname{coz}(\beta) = 1$. Since $\operatorname{coz}(\beta) \wedge \operatorname{coz}(\alpha) = \operatorname{coz}(\alpha\beta) = 0_{\uparrow(\operatorname{coz}(\alpha\beta))}$ and $\operatorname{coz}(\beta) \in \uparrow(\operatorname{coz}(\alpha\beta))$, we can conclude that $\operatorname{coz}(\alpha) \in B(\uparrow(\operatorname{coz}(\alpha\beta)))$. On the other hand, as shown above, $\operatorname{coz}(\alpha\beta) \in B(L)$ because $\alpha\beta \in M$. Now, by Lemma 4.9, we can deduce that $\operatorname{coz}(\alpha) \in B(L)$.

The implication from (5) to (1) is similar to the foregoing implication.

Suppose (6) holds and, to establish (1), consider $\operatorname{coz}(\alpha) \in \operatorname{Coz}_c[L]$ for some $\alpha \in \mathcal{C}_c(L)$. We must show that $\operatorname{coz}(\alpha) \in B(L)$. The current hypothesis tells us that $\mathcal{C}_c(L)/\langle\gamma\rangle$ is an AR-ring, and hence, by Proposition 4.10, we obtain $\operatorname{coz}(\gamma) \in B(L)$ and $\uparrow\operatorname{coz}(\gamma)$ is a CP-frame. Since $\operatorname{coz}(\alpha) \vee \operatorname{coz}(\gamma) = \operatorname{coz}(\alpha|_{\mathfrak{c}(\operatorname{coz}(\gamma))})$, we get that $\operatorname{coz}(\alpha) \vee \operatorname{coz}(\gamma) \in B(\uparrow \operatorname{coz}(\gamma))$, and hence $\operatorname{coz}(\alpha) \vee \operatorname{coz}(\gamma) \in B(L)$ by lemma 4.9. And because

$$\operatorname{coz}(\alpha) = \operatorname{coz}(\alpha) \wedge 1 = \big(\operatorname{coz}(\alpha) \vee \operatorname{coz}(\gamma)\big) \wedge \big(\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\gamma))^*\big),$$

it is enough to show $\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\gamma))^* \in B(L)$. But

$$\operatorname{coz}(\gamma) \vee \big(\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\gamma))^*\big) = 1$$

and

$$\begin{aligned}
\operatorname{coz}(\alpha\gamma) &= \operatorname{coz}(\alpha) \wedge \operatorname{coz}(\gamma) \\
&= (\operatorname{coz}(\alpha) \wedge \operatorname{coz}(\gamma)) \vee (\operatorname{coz}(\gamma) \wedge (\operatorname{coz}(\gamma))^*) \\
&= \operatorname{coz}(\gamma) \wedge \big(\operatorname{coz}(\alpha) \vee (\operatorname{coz}(\gamma))^*\big)
\end{aligned}$$

imply that $\operatorname{coz}\delta \vee (\operatorname{coz}(\gamma))^* \in B\big(\uparrow(\operatorname{coz}(\delta\gamma))\big)$. On the other hand, since $\langle\gamma\rangle$ is a regular ideal and $\alpha\gamma \in \langle\gamma\rangle$, by Lemma 3.5, we can deduce that $\operatorname{coz}(\delta\gamma) \in B(L)$. Therefore, Lemma 4.9 implies that $\operatorname{coz}\delta \vee (\operatorname{coz}(\gamma))^* \in B(L)$.

To show (7) implies (1), choose $\operatorname{coz}(\alpha) \in \operatorname{Coz}_c[L]$ with $\alpha \in \mathcal{C}_c(L)$. Lemma 3.3 tells us that we can assume $(\operatorname{coz}(\alpha))^* \neq 0$. Then, by zero-dimensionality, there is $s \in \operatorname{Coz}_c[L]$ such that $s \prec\!\!\prec (\operatorname{coz}\alpha)^*$, this shows that $\operatorname{coz}(\alpha) \leq (\operatorname{coz}(\alpha))^{**} \prec\!\!\prec s^*$, which implies $\alpha \in \mathbf{O}^c_{s^*}$. Consequently, in light of the fact that every z-ideal is the interesction of the minimal prime ideals containing it (see [18, Lemma 1.0]), we can infer that $\alpha \in P$ for some minimal prime ideal P of $\mathcal{C}_c(L)$ since $\mathbf{O}^c_{s^*}$ is a z_c-ideal. Thus, our hypothesis shows that α is a regular element of $\mathcal{C}_c(L)$, and hence, by Lemma 3.5, we would have $\operatorname{coz}(\alpha) \in B(L)$.

To prove (8) implies (1), take $\operatorname{coz}(\alpha) \in \operatorname{Coz}_c[L]$ for some $\alpha \in \mathcal{C}_c(L)$. As shown in the proof of (7) implies (1), we can choose a minimal prime ideal P of $\mathcal{C}_c(L)$ such that $\alpha \in P$. But, by the fact that every minimal prime ideal is a z-ideal (see [18]), we get that $\mathbf{M}^c_{\operatorname{coz}(\alpha)} \subseteq P$. Now, by the hypothesis, there exists $n \in \mathbb{N}$ for which

$$\langle \alpha \rangle \cap \left(\mathbf{M}^c_{\operatorname{coz}(\alpha)}\right)^n \subseteq \langle \alpha \rangle \mathbf{M}_{\operatorname{coz}(\alpha)}.$$

Since $\left(\mathbf{M}^c_{\operatorname{coz}(\alpha)}\right)^n = \mathbf{M}^c_{\operatorname{coz}(\alpha)}$, we have $\alpha \in \langle \alpha \rangle \cap \mathbf{M}_{\operatorname{coz}(\alpha)} \subseteq \langle \alpha \rangle \mathbf{M}_{\operatorname{coz}(\alpha)}$. This shows that $\alpha = \alpha\delta\beta$ where $\delta \in \mathcal{C}_c(L)$ and $\beta \in \mathbf{M}_{\operatorname{coz}(\alpha)}$. Now, Lemma 3.5 implies that $\operatorname{coz}(\alpha) \in B(L)$ since $\operatorname{coz}(\delta\beta) \leq \operatorname{coz}(\beta) \leq \operatorname{coz}(\alpha)$. $\square$

A direct consequence of the above is the following result.

Corollary 4.12. *The following are equivalent for any space X.*

1. *X is a CP-space.*
2. *The ring $\mathcal{C}_c(X)/\langle f \rangle$ is an Artin-Rees ring for any $f \in \mathcal{C}_c(X)$.*
3. *The ring $\mathcal{C}_c(X)/P$ is an Artin-Rees ring for any prime ideal P of $\mathcal{C}_c(X)$.*
4. *The ring $\mathcal{C}_c(X)$ contains an Artin-Rees maximal ideal.*
5. *The ring $\mathcal{C}_c(X)$ contains a regular maximal ideal.*
6. *There exists an element $g \in \mathcal{C}_c(X)$ such that $\langle g \rangle$ is a regular ideal and $\mathcal{C}_c(X)/\langle g \rangle$ is an Artin-Rees ring.*
7. *Every minimal prime ideal of $\mathcal{C}_c(X)$ is a regular ideal.*
8. *Every minimal prime ideal of $\mathcal{C}_c(X)$ is an Artin-Rees ideal.*

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