Journal of Algebra and Related Topics
Vol. 11, No 1, (2023), pp 65-80

ON PRE-TOPOLOGICAL BCK-ALGEBRAS

A. B. KHALAF * AND N. K. AHMED

Abstract. The concept of pre-open set in topological spaces was introduced by Mashhour [13] and the concept of BCK-algebras was defined by Iseki [9]. The aim of this paper is to supply the BCK-algebra by a topology that makes the operation defined on X satisfy a special type of continuity with help of pre-open sets we call it p-topological BCK-algebra. This concept is an extension of the concept of topological BCK-algebra which was defined in [12]. By using properties of pre-open sets and axioms of BCK-algebras, several topological properties on BCK-algebras are found.

1. Introduction

The study of algebraic structures has an important role in pure mathematics. Several mathematicians studied various types of algebraic structures from different points of view. The structure of BCK-algebras was initiated by Imai and Iseki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since they gave an algebraic formulation for the BCK-propositional calculus system, a lot of mathematical papers have been written for investigating the algebraic properties of the BCK-algebras. On the other hand so many types of nearly open sets were defined in topological spaces and many papers were witten for investigating topological concepts by replacing these sets instead of elements of a given topology. Among these

MSC(2010): Primary: 03G25; Secondary: 54A05
Keywords: BCK-algebra, PBCK-algebras, preopen set, Pre-T_i spaces, T_i spaces ($i = 0, 1, 2$).
Received: 26 September 2022, Accepted: 6 March 2023.
∗Corresponding author .

sets Mashhour in 1983 defined the concept of pre-open sets in topological spaces. After that many topological concepts were introduced using pre-open sets such as continuity, compactness, separation axioms and so on. [2], studied the topological aspects of the BCK-structure. They studied the various topologies in a manner analogous to the study of lattices. However, no attempts have been made to study the topological structures making the star operation of BCK-algebra continuous. Theories of topological structures such as groups, rings and modules are well known and still investigated by many mathematicians. Even topological algebraic structures have been studied by some authors. [12], defined a topology on a BCK-algebra which makes the operation continuous and they obtain many topological properties. In [16], the first author and Yousef introduced the concept of semi-topological UP-algebras and gave some characterizations on this concept. In this paper, we initiate the study of a topology on BCK-algebras and we investigate a type of continuity by the aid of pre-open sets. By a space we mean a BCK-algebra with a topology τ. A subset A of X is open if it is a member of τ and a set is closed if its complement is open, the closure of A denoted by $\overline{A}$ is the intersection of all closed sets containing A while the interior of A denoted by $int(A)$ is the union of all open sets contained in A. A set A is called pre-open [13] if $A \subseteq int(\overline{A})$ by using pre-open sets of the topology we introduce the concept of PBCK-algebra as a generalization of the concept of TBCK-algebra introduced by Lee and Ryu.

2. Definitions and Background

In this section, we give necessary definitions and results related to topological concepts as well as some axioms and necessary properties of BCK-algebras which are used in the next section. More details on BCK-algebras can be found in [14].

For a subset A of a topological space (X, τ), we say that A is regular open if $A = Int(\overline{A})$ and it is pre-open if $A \subseteq int(\overline{A})$. The complement of a pre-open set is called a pre-closed. The pre-closure and pre-interior of A is denoted respectively by $\overline{A}^p$ and $pInt(A)$ and they are defined analogously as the intersection of all pre-closed sets containing A and the union of pre-open sets contained in A respectively. For details of more topological concepts we refer to [7]. A point $x \in \overline{A}^p$ if $A \cap G \neq \phi$ for each pre-open subset G of X.

Lemma 2.1. [5] *If V is pre-open in X such that $V \subseteq Y \subseteq X$, then V is also pre-open in (Y, τ_Y).*

Definition 2.2. [7, 11] A topological space (X, τ) is said to be:

1. T_0 (resp., pre-T_0) If for each $x, y \in X$, there exists an open (pre-open) set containing one of them but not the other.
2. T_1 (resp., pre-T_1) If for each $x, y \in X$, there exist open (pre-open) sets G, H such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.
3. T_2 (resp., pre-T_2) If for each $x, y \in X$, there exist open (pre-open) sets G, H such that $x \in G$, $y \in H$ and $G \cap H = \phi$.

Definition 2.3. By a BCK-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms: for every $x, y, z \in X$,

1. $((x * y) * (x * z)) * (z * y) = 0$,
2. $(x * (x * y)) * y = 0$,
3. $x * x = 0$,
4. $x * y = 0$ and $y * x = 0 \Rightarrow x = y$,
5. $0 * x = 0$.

In a BCK-algebra $(X, *, 0)$, we define a partial order relation $(\leq)$ by $x \leq y$ if and only if $x * y = 0$.

From the definition of BCK-algebras we can get the following properties very easily see ([6]; Proposition 5.1.3).

Proposition 2.4. In a BCK-algebra X, the following statements are true for all $x, y, z \in X$:

1. $x * 0 = x$,
2. $x * y \leq x$,
3. $(x * y) * z = (x * z) * y$,
4. $x \leq y \Rightarrow x * z \leq y * z$ and $z * y \leq z * x$,
5. $x * (x * (x * y)) = x * y$.

Definition 2.5. [6] A nonempty subset A of a BCK-algebra $(X, *, 0)$ is called an ideal of X if the following two conditions are satisfied:

1. $0 \in A$,
2. For all $x \in X$ and for all $y \in A$, if $x * y \in A$, then $x \in A$.

If there is an element 1 of X satisfying $x \leq 1$, for all $x \in X$, then the element 1 is called unit of X. A BCK-algebra with unit is called a bounded BCK-algebra [6].

Definition 2.6. [12] A BCK-algebra X equipped with a topology τ is called a topological BCK-algebra (for short TBCK-algebra) if $f : X \times X \to X$ defined by $f(x, y) = x * y$ is continuous for all $(x, y) \in X \times X$ where $X \times X$ has the product topology.
Equivalently, if for each open set O containing $x * y$, there exist open sets U and V containing x and y respectively such that $U * V \subseteq O$.

Definition 2.7. [6] A subset Y of a BCK-algebra X is called a BCK-subalgebra if Y itself is a BCK-algebra.

Definition 2.8. [4] Let X be a BCK-algebra, and $a \in X$. A left map $L_a : X \to X$ defined by, $L_a(x) = a * x, \forall x \in X$ and a right map $R_a : X \to X$ by $R_a(x) = x * a\ \forall x \in X$.
We denote $L(X)$ to be the family of all L_a for all $a \in X$ and $R(X)$ the family of all right maps R_a for all $a \in X$.

Definition 2.9. [4] A BCK-algebra X is called a positive implicative BCK-algebra, if $(y * x) * (z * x) = (y * z) * x$ for all $x, y, z \in X$.

3. Pre-topological BCK-algebras

In this section we introduce the concept of p-topological BCK-algebras and investigate some of their properties.

Definition 3.1. A BCK-algebra X equipped with a topology τ is called a pre-topological BCK-algebra(for short PBCK-algebra) if the function $f : X \times X \to X$ defined by $f(x, y) = x * y$ has the property that for each open set O containing $x * y$, there exist an open set U containing x and a pre-open set V containing y such that $f(U, V) = U * V \subseteq O$ for all $x, y \in X$.

From the above definitions, we conclude that every TBCK-algebra is PBCK-algebra. The following example shows that there are PBCK-algebras which are not TBCK-algebras.

Example 3.2. Let $X = \{0, a, b, c\}$ and $*$ be defined as in the following Cayley table:

$\star$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Table 1. A PBCK-algebra which is not TBCK-algebra

From Table 1, we can easily check that $(X, *, 0)$ is a BCK-algebra. Now consider the topology τ on X defined as: $\tau = \{X, \phi, \{0, b, c\}\}$. Then X is not a TBCK-algebra because $a * a = 0$, and the only open set containing a is X and $X \times X \nsubseteq \{0, b, c\}$. It is not difficult to verify that the pre-open sets in (X, τ) are $P(X) \setminus \{a\}$ because every subset of X is dense in X except $\{a\}$ and by simple calculation we can show that (X, τ) is PBCK-algebra.

In general for a subset A of a PBCK-algebra X and an element $x \in X$, $\overline{A} * x$ may not equal to $\overline{A * x}$ and also $x * \overline{A}^p$ may not equal to $\overline{x * A}$. In Example 3.2, for the first case if we take $A = \{a\}$ and $x = b$ we can verify that $\overline{A * b} = X \neq \{0\} = \overline{A} * b$. In the same example if we take $A = \{0, a\}$, then $b * \overline{A}^p = \{b\}$ and $\overline{b * A} = X$, so $b * \overline{A}^p \neq \overline{b * A}$. However we can prove the following results:

Proposition 3.3. *For any subset A of a PBCK-algebra X and any element $x \in X$, the following statements are true:*

(1) $\overline{A} * x \subseteq \overline{A * x}$.
(2) *If $\overline{A} * x$ is closed, then $\overline{A} * x = \overline{A * x}$.*

Proof. (1) Let $y = a * x \in \overline{A} * x$ where $a \in \overline{A}$ and let U be any open set containing y. Since X is a PBCK-algebra, so there exists an open set V containing a and a pre-open set G containing x such that $V * G \subseteq U$. Also we have $a \in \overline{A}$ implies that $A \cap V \neq \phi$. Suppose that $b \in A \cap V$, so $b * x \in A * x$ and $b * x \in V * x \subseteq V * G \subseteq U$. Hence $b * x \in U \cap (A * x)$ implies that $y \in \overline{A * x}$. Thus, $\overline{A} * x \subseteq \overline{A * x}$.

(2) Suppose that $\overline{A} * x$ is closed and let $y \in \overline{A * x}$. If $y \notin \overline{A} * x$, then $y \in (\overline{A} * x)^c$ which is an open set. It is clear that $A * x \subseteq \overline{A} * x$, so we get $A * x \cap (\overline{A} * x)^C = \phi$ which is contradiction and hence the proof.

Proposition 3.4. *For any subset A of a PBCK-algebra X and any element $x \in X$, the following statements are true:*

(1) $x * \overline{A}^p \subseteq \overline{x * A}$.
(2) *If $x * \overline{A}^p$ is closed, then $x * \overline{A}^p = \overline{x * A}$.*

Proof. (1) Let $y \in x * \overline{A}^p$ and U be any open set containing y. So $y = x * a$ where $a \in \overline{A}^p$. Since X is a PBCK-algebra, so there exists an open set V containing x and a pre-open set G containing a such that $V * G \subseteq U$. Also we have $a \in \overline{A}^p$ implies that $A \cap G \neq \phi$. Suppose that $b \in A \cap G$, so $x * b \in x * A$ and $x * b \in x * G \subseteq V * G \subseteq U$. Hence we obtain that $y \in \overline{x * A}$.

(2) Suppose that $x * \overline{A}^p$ is closed and let $y \in \overline{x * A}$. If $y \notin x * \overline{A}^p$, then $y \in (x * \overline{A}^p)^c$ which is an open set. Since $x * A \subseteq x * \overline{A}^p$, so we get $(x * A) \cap (x * \overline{A}^p)^c = \phi$ which is contradiction and hence the proof.

Proposition 3.5. *For any subsets A and B of a PBCK-algebra X, the following statements are true:*

(1) $\overline{A} * \overline{B}^p \subseteq \overline{A * B}$.
(2) *If* $\overline{A} * \overline{B}^p$ *is closed, then* $\overline{A} * \overline{B}^p = \overline{A * B}$.

Proof. (1) Let $x \in \overline{A} * \overline{B}^p$ and U be any open set containing x. So $x = a*b$ where $a \in \overline{A}$ and $b \in \overline{B}^p$. Since X is a PBCK-algebra, so there exists an open set V containing a and a pre-open set G containing b such that $V * G \subseteq U$. Also we have $a \in \overline{A}$ and $b \in \overline{B}^p$, implies that $A \cap V \neq \phi$ and $B \cap G \neq \phi$. Suppose that $a_1 \in A \cap V$ and $b_1 \in B \cap G$, so $a_1 * b_1 \in A * B$ and $a_1 * b_1 \in V * G \subseteq U$. Hence we obtain that $x \in \overline{A * B}$.

(2) Suppose that $\overline{A} * \overline{B}^p$ is closed and let $x \in \overline{A * B}$. If $x \notin \overline{A} * \overline{B}^p$, then $x \in (\overline{A} * \overline{B}^p)^c$ which is an open set. It is clear that $A * B \subseteq \overline{A} * \overline{B}^p$, so we get $(A * B) \cap (\overline{A} * \overline{B}^p)^c = \phi$ which is contradiction and hence the proof.
From Proposition 3.3 and Proposition 3.4, we get the following result:

Corollary 3.6. *For a subset A of a PBCK-algebra X and an element $x \in X$, the following statements are true:*
(1) *If $A * x$ is closed, then $\overline{A} * x = A * x$.*
(2) *If $x * A$ is closed, then $x * \overline{A}^p = x * A$.*

Proof. (1) From Proposition 3.3, we have $\overline{A} * x \subseteq \overline{A * x} = A * x$ and $A * x \subseteq \overline{A} * x$. Hence the proof.
(2) is similar.

It is known, (Proposition 2.2, [12]), that in a TBCK-algebra if $\{0\}$ is an open set then the space is discrete and if $\{0\}$ is a closed set then the space is Hausdorff (see Proposition 2.3 [12]).

Proposition 3.7. *In a PBCK-algebra X, if $\{0\}$ is open, then every singleton set in X is pre-open.*

Proof. Suppose that $\{0\}$ is open and let x be any point in X. Since $x * x = 0$ for all $x \in X$ and X is PBCK-algebra, so there exists an open set U containing x and a pre-open set G containing x such that $U * G \subseteq \{0\}$. Hence $W = U \cap G$ is a pre-open set containing x. If W contains any other point y, then we obtain that $x * y = 0$ and $y * x = 0$ which is contradiction. Hence W is a pre-open set contains x only. Therefore, $\{x\}$ is pre-open for each $x \in X$.

Corollary 3.8. *If $\{0\}$ is open in a PBCK-algebra X, then it is pre-T_2.*

Proof. Follows from Proposition 3.7.

Proposition 3.9. *In a PBCK-algebra X, if $\{0\}$ is closed, then X is T_2.*

Proof. Suppose that $\{0\}$ is closed and let x and y be any two distinct points in X, then either $x * y \neq 0$ or $y * x \neq 0$ without loss of generality suppose that $x*y \neq 0$. Hence there exists an open set V containing x and a pre-open set G containing y such that $V * G \subseteq X \setminus \{0\}$ which implies that $V \cap G = \phi$. Hence, $G \subseteq V^c$ and so $cl(G) \subseteq V^c$ implies that $V \cap cl(G) = \phi$ and hence $V \cap int(cl(G)) = \phi$. Since G is pre-open, so $G \subseteq int(cl(G))$ and hence $y \in G \in int(cl(G))$. Therefore, V and $int(cl(G))$ are two disjoint open sets containing x, y respectively. Hence, X is T_2.

Proposition 3.10. *If a PBCK-algebra $(X, *, \tau)$ is T_0, then it is pre-T_1.*

Proof. Let $x, y \in X$ and $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $x * y \neq 0$. Since X is T_0 space, there is an open set W containing one of them but not the other.
Case 1. Suppose that W contains $x * y$ and $0 \notin W$.
Since $(X, *, \tau)$ is a PBCK-algebra, then there exists an open set U of x and a pre-open set V of y such that $U * V \subseteq W$. Then U is an open set and V is a pre-open sets containing x and y respectively. If $U \cap V \neq \phi$, then there is a point $z \in U \cap V$ implies that $0 = z * z \in U * V \subseteq W$ which is contradiction.
Case 2. If $0 \in W$ and $x * y \notin W$. Then we have, $x * x = 0 \in W$, so there exists an open set U_x of x and a pre-open set V_x of x such that $U * V \in W$, again $y * y = 0 \in W$, so there exists an open set U_y of y and a pre-open set V_y of y such that $U_y * V_y \subseteq W$.
Therefore, $G = U_x \cap V_x$ and $H = U_y \cap V_y$ are two pre-open sets the first contains x but not y and the second contains y but not x.
Hence, in both cases we showed that $(X, *, \tau)$ is a pre-T_1 space.

Proposition 3.11. *A PBCK-algebra $(X, *, \tau)$ is T_1 if and only if it is T_2.*

Proof. Let $x, y \in X$ and $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. Suppose that $x * y \neq 0$. Since X is a T_1 space, there is an open set W containing $x * y$ and $0 \notin W$.
Since $(X, *, \tau)$ is a PBCK-algebra, then there exists an open set U of x and a pre-open set V of y such that $U * V \subseteq W$. Then U is an open set and V is a pre-open set containing x and y respectively. If $U \cap V \neq \phi$, then there is a point $z \in U \cap V$ implies that $0 = z * z \in U * V \subseteq W$ which is contradiction. Hence $U \cap V = \phi$ implies that $V \subseteq U^c$ and so

$cl(V) \subseteq U^c$ implies that $U \cap cl(V) = \phi$ and hence $U \cap int(cl(V)) = \phi$. Therefore, X is T_2.
The converse part is obvious.

Proposition 3.12. *If Y is an open BCK-subalgebra of a PBCK-algebra X, then Y is also a PBCK-algebra.*

Proof. Let $x, y \in Y$ and let U be any open set in the subspace Y containing $x * y$. Since Y is open in X, so U is open in X. Since X is a PBCK-algebra, so there exists an open set W in X containing x and a pre-open set G in X containing y such that $W * G \subseteq U$. Then we have $O = W \cap Y$ is an open set in Y containing x and by Lemma 2.1, $H = G \cap Y$ is a pre-open set in Y containing y, and $(W \cap Y) * G \cap Y \subseteq (W * G) \subseteq U$. Hence the proof.

Proposition 3.13. *If I is an ideal in a PBCK-algebra X and $0 \in int(I)$, then I is open.*

Proof. Let $x \in I$. Since $0 \in int(I)$ and $x * x = 0$, so there is an open set U such that $0 \in U \subseteq I$. Since X is a PBCK-algebra, so there exists an open set V containing x such that $V * x \subseteq U$. If there is a point $y \in V \cap (X \setminus I)$, so we obtain that $y * x \in I$. Since $x \in I$ and I is an ideal, so $y \in I$ which is contradiction. Hence $x \in V \subseteq I$ implies that I is open.

Proposition 3.14. *If I is an open ideal in a PBCK-algebra X, then I is closed.*

Proof. Let $x \notin I$. Since I is an ideal so $x * x = 0 \in I$. Since X is a PBCK-algebra, so there exists an open set V of x and a pre-open set U of x such that $V * U \subseteq I$. If $W = V \cap U$, then we have W is a pre-open set containing x and $W * W \subseteq I$. If there exists $y \in (W \cap I)$ and since I is an ideal, then we obtain that $W \subseteq I$ which is contradiction. Hence $W \subseteq X \setminus I$ and therefore, I is pre-closed. Since I is open, so we obtain that $\overline{int(I)} \subseteq I$ which implies that $I = \overline{I}$. Hence I is closed.

Definition 3.15. Let $(X, *, 0)$ be a PBCK-algebra and $F \subseteq X$. Then we say that F is a filter when it satisfies the conditions:

(1) $0 \in F$,
(2) If $0 \neq x \in F$ and $x * y \in F$, then $y \in F$.

Proposition 3.16. *Let $(X, *, \tau)$ be PBCK-algebra and F be a filter on X. If 0 is an interior point of F, then F is pre-open.*

Proof. Suppose that 0 is an interior point of F. Then there exists an open set U of 0 such that $U \subseteq F$. Let $x \in F$ be an arbitrary element. Since $x * x = 0$, there exists an open set V of x and a pre-open set W of x such that $V * W \subseteq U \subseteq F$.
Now, for each y in a pre-open set W, we have $x * y \in F$. Since F is a filter and $x \in F$, we have $y \in F$. Hence $x \in W \subseteq F$ and so F is pre-open set.

Proposition 3.17. *Let $(X, *, \tau)$ be a PBCK-algebra and F be a filter of X. If F is open, then it is closed.*

Proof. Let F be a filter of X which is open in (X, τ). We show that $X \setminus F$ is open. Let $x \in X \setminus F$. Since F is open, 0 is an interior point of F. Since $x * x = 0$, there exists an open set V containing x and a pre-open set W containing x such that $V * W \subseteq F$. We claim that $V \subseteq X \setminus F$. If $V \nsubseteq X \setminus F$, then there exists an element $y \in V \cap F$. For each $z \in W$, we have $y * z \in V * W \subseteq F$, since $y \in F$ and F is a filter, $z \in F$. Hence $W \subseteq F$ and so $x \in F$ which is contradiction. Therefore, $x \in V \subseteq X \setminus F$ which implies that $X \setminus F$ is open and hence, F is closed.

From Proposition 3.14, Proposition 3.17 and the fact that a space is disconnected [7], if it contains a proper non-empty subset which is both open and closed, we get the following result.

Corollary 3.18. *If F is a non-trivial open filter in a PBCK-algebra X, then X is topologically disconnected.*

Corollary 3.19. *In a PBCK-algebra $(X, *, 0)$, if $\{0\}$ is open, then X is disconnected.*

Proof. Since $\{0\}$ is an ideal in X, by Proposition 3.14 $\{0\}$ is closed also. Hence, X is disconnected because it contains a proper non-empty set which is open and closed.

Definition 3.20. Let X be a BCK-algebra, U be a non-empty subset of X and $a \in X$. The subsets U_a and ${}_aU$ are defined as follows:
$U_a = \{x \in X : x * a \in U\}$ and ${}_aU = \{x \in X : a * x \in U\}$.
Also if $K \subseteq X$ we define

$$ {}_KU = \bigcup_{a \in K} {}_aU \qquad \& \qquad U_K = \bigcup_{a \in K} U_a $$

.

The proof of the following results obtained directly from their definitions.

Proposition 3.21. Let X be an BCK-algebra and A, B, W, K are subsets of X then:
(1) If $A \subseteq B$ then ${}_AW \subseteq {}_BW$.
(2) If $W \subseteq K$ then ${}_AW \subseteq {}_AK$.
(3) If $F \subseteq X$, then $(F_a)^c = (F^c)_a$ and $({}_aF)^c =_a (F^c)$ for each $a \in X$.

Proposition 3.22. *Let X be a PBCK-algebra, U and F be two non-empty subsets of X, the following statements are true:*

(1) *If U is an open set, then U_a is open and ${}_aU$ is a pre-open set.*
(2) *If F is closed set then F_a is closed and ${}_aF$ is a pre closed set.*

Proof. (1) Let U be an open set, $a \in X$ and let $x \in U_a$. Then $x * a \in U$. Since X is PBCK-algebra, then there exists an open set G containing x and a pre-open set A containing a such that $G * A \subseteq U$, $x * a \in G * a \subseteq U$, thus $G * a \subseteq U$. Hence, $x \in G \subseteq U_a$ implies that U_a is open.

To prove that ${}_aU$ is pre-open, let $x \in {}_aU$ implies that $a * x \in U$. Since X is PBCK-algebra, then there exist an open set A containing a and a pre-open set H containing x such that $A * H \subseteq U$, so $a * x \in {}_aH \subseteq U$, thus $a * H \subseteq U$. Hence, $x \in H \subseteq {}_aU$. Therefore, ${}_aU$ is a pre-open set.

(2) Let F be a closed set, then F^c is open. Hence, by (1), $(F^c)_a$ is open and ${}_a(F^c)$ is pre-open. By Proposition 3.21, $(F_a)^c = (F^c)_a$ and $({}_aF)^c =_a (F^c)$. Hence, $(F_a)^c$ is open and $({}_aF)^c$ is pre-open. Consequently, F_a is closed and ${}_aF$ is pre-closed.

Definition 3.23. Let X be a BCK-algebra. The binary operation $\circ$ will be defined on $L(X)$ as $(L_a \circ L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.

Theorem 3.24. *Let X be a positive implicative BCK-algebra, then $(L(X), \circ, L_0)$ is a BCK-algebra.*

Proof. Let L_a and L_b be any two elements of $L(X)$. Then, by definition $(L_a \circ L_b)(x) = L_a(x) * L_b(x) = (a * x) * (b * x)$. Since X is positive implication BCK-algebra, so $(a * x) * (b * x) = (a * b) * x$. Hence, $(L_a \circ L_b)(x) = L_{a*b}(x)$ which implies that $L_a \circ L_b = L_{a*b}$ for all $a, b \in X$. Also we have the following are true.

(1) $((L_x \circ L_y) \circ (L_x \circ L_z)) \circ (L_z \circ L_y) = (L_{x*y} \circ L_{x*z}) \circ L_{z*y}$
$= L_{((x*y)*(x*z))*(z*y)} = L_0$,

(2) $(L_x \circ (L_x \circ L_y)) \circ L_y = (L_x \circ (L_{x*y}) \circ L_y = L_{(x*(x*y)} \circ L_y$
$= L_{(x*(x*y))*y} = L_0$
(3) $L_x \circ L_x = L_{x*x} = L_0$,
(4) $L_x \circ L_y = L_0$ and $L_y \circ L_x = L_0$, then $L_{x*y} = L_0$ and $L_{y*x} = L_0$ which implies that $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ and hence, $L_x = L_y$,
(5) $L_0 \circ L_x = L_{0*x} = L_0$.

Hence, $L(X)$ is a BCK-algebra.

Definition 3.25. Let X be a BCK-algebra, we define a map $\psi : X \to L(X)$ by $\psi(x) = L_x$ for all $x \in X$ and if A is any subset of X, then $L_A = \{L_a : a \in A\}$.

Remark 3.26. If X is a positive implicative BCK-algebra, then the following statements can be easily proved.

(1) If $A \subseteq B$, then $\psi(A) \subseteq \psi(B)$.
(2) If A and B are any two subsets of X, then $\psi(A \cup B) = \psi(A) \cup \psi(B)$ and $\psi(A \cap B) = \psi(A) \cap \psi(B)$.

Proposition 3.27. *Let X be a positive implicative BCK-algebra, then the map $\psi : X \to L(X)$ is a BCK-isomorphism.*

Proof. It is clear that ψ is a bijection. We have $\psi(x * y) = L_{x*y}$ and $L_{x*y}(z) = (x * y) * z$. Since X is positive implicative, we have $(x * y) * z = (x * z) * (y * z)$. Therefore, $L_{x*y}(z) = L_x(z) \circ L_y(z) = (L_x \circ L_y)(z)$. Hence, $\psi(x * y) = \psi(x) \circ \psi(y)$ for all $x, x \in X$, so ψ is a BCK-isomorphism.

Proposition 3.28. *Let X be a positive implicative BCK-algebra and τ be a topology on X, then the following statements are true:*

(1) *The family $\sigma = \{\psi(G) \subseteq L(X) : G \in \tau\}$ is a topology on $L(X)$.*
(2) *For any subset A of X, $L_{\overline{A}} = \overline{L_A}$.*
(3) *If A is any pre-open set in (X, τ), then $\psi(A)$ is a pre-open set in $(L(X), \sigma)$.*

Proof. (1) The proof of σ is a topology is obvious.

(2) For any subset A of X, we have $A \subseteq \overline{A}$. Hence, $L_A \subseteq L_{\overline{A}}$ and $\overline{A}$ is closed in X, so by definition of σ, we have $L_{\overline{A}}$ is closed in $L(X)$. Therefore, we obtain $\overline{L_A} \subseteq \overline{L_{\overline{A}}} = L_{\overline{A}}$. To prove $L_{\overline{A}} \subseteq \overline{L_A}$, let $L_x \in L_{\overline{A}}$, then $x \in \overline{A}$ and let L_G be any open set containing L_x. Hence G is an open set containing x, hence $A \cap G \neq \phi$. Therefore, $L_A \cap L_G \neq \phi$.

Implies that $L_x \in \overline{L_A}$, so $L_{\overline{A}} \subseteq \overline{L_A}$ and hence $L_{\overline{A}} = \overline{L_A}$.

(3) Let A be any pre-open set in X, so there exists an open set O in X such that $A \subseteq O \subseteq \overline{A}$. Hence $L_A \subseteq L_O \subseteq L_{\overline{A}}$ and by (2), $L_A \subseteq L_O \subseteq \overline{L_A}$. Hence, L_A is pre-open in $L(X)$.

Proposition 3.29. *Let X be a positive implicative PBCK-algebra. Then $(L(X), \circ, \sigma)$ is a PBCK-algebra.*

Proof.
Let L_W be an open set containing $L_x \circ L_y = L_{x*y}$. Hence W is an open set containing $x * y$ in X and since X is a PBCK-algebra, so there exist an open set U and a pre-open set V containing x and y respectively and $U * V \subseteq W$. Therefore, $L_{U*V} \subseteq L_W$. Since X is positive implicative, so $L_{U*V} = L_U \circ L_V \subseteq L_W$. By Proposition 3.28, L_V is a pre-open in $L(X)$ containing L_y, hence the proof.
Recalling that a function $f : X \to Y$ is pre-continuous [13] if the inverse image of each open set in Y is a pre-open set in X and it is called pre-open if the image of each open set is pre-open.

Proposition 3.30. Let X be a PBCK-algebra, then every left map on X is pre-continuous.

Proof. Let $a \in X$, define a left map $L_a : X \to X$ by $L_a(x) = a * x, \forall x \in X$. Let W be any open set containing $L_a(x) = a * x$. Since X is a PBCK-algebra, so there exists an open set U containing a and a pre-open set V containing x such that $U * V \subseteq W$. clearly, $a * V \subseteq U * V \subseteq W$. Hence, $L_a(V) \subseteq W$. This implies that L_a is pre-continuous.

Definition 3.31. A BCK-algebra X is called p-transitive (resp.,p-open) if for each $a \in X \setminus \{0\}$, the left map L_a is pre-continuous (resp., pre-open) and it is transitive open if the right map R_a is both continuous and open.

Remark 3.32. From Proposition 3.30, if X is a PBCK-algebra such that for each $a \in X \setminus \{0\}$, the left map L_a is pre-open, then X is p-transitive and p-open.

Proposition 3.33. Let X be a PBCK-algebra such that for each $a \in X \setminus \{0\}$, the left map L_a is pre-open. If U is an open subset of X, then the following statements are true:

(1) The set $a * U$ is pre-open.

(2) $L_a^{-1}(U) = \{x \in X : a * x \in U\}$ is pre-open.
(3) The set $A * U$ is pre-open for each $A \subseteq X$.

Proof. Since L_a is pre-open and U is open, so $L_a(U) = a * U$ is pre-open. By Proposition 3.30 L_a is pre-continuous. Hence $L_a^{-1}(U) = \{x \in X : a * x \in U\}$ is pre-open. Lastly, we have

$$A * U = \bigcup_{a \in A} (a * U)$$

is the union of pre-open sets, so $A * U$ is pre-open.

Proposition 3.34. *Let X be a PBCK-algebra, then every right map on X is continuous.*

Proof. Let $a \in X$, define a right map $R_a : X \to X$ by $R_a(x) = x * a, \forall x \in X$. Let W be any open set containing $R_a(x) = x * a$. Since X is a PBCK-algebra, so there exists an open set U containing x and a pre-open set V containing a such that $U * V \subseteq W$. clearly, $U * a \subseteq U * V \subseteq W$. Hence, $R_a(U) \subseteq W$. This implies that R_a is continuous.

Proposition 3.35. *Let U be an open subset of a transitive open PBCK-algebra X and let $a \in X$. Then the following statements are true:*

(1) *The set $U * a$ is open.*
(2) $R_a^{-1}(U) = \{x \in X : x * a \in U\}$ *is open.*
(3) *The set $U * A$ is open for each $A \subseteq X$.*

Proof. Since R_a is open and U is open, so $L_a(U) = U * a$ is open. By Proposition 3.34, R_a is continuous hence $R_a^{-1}(U) = \{x \in X : a * x \in U\}$ is open. Also we have

$$U * A = \bigcup_{a \in A} (U * a)$$

which is the union of open sets, so $U * A$ is open.

Definition 3.36. A topological space X is called P_x-space if any finite intersection of pre-open sets containing the point $x \in X$ is pre-open.

Proposition 3.37. *Let F and E be two disjoint subsets of a PBCK-algebra $(X, *, \tau)$ which is P_0-space. If F is compact and E is closed and for any $a \in X$, $L_a(x) = a * x$ is a pre-open map from X into X, then there is a pre-open set V containing* 0 *such that* $(F * V) \cap E = \phi$

Proof. Let $x \in F \subseteq X \setminus E$. Since $(x * 0) * 0 = x \in X \setminus E \in \tau$ and X is a PBCK-algebra, then there exists an open set W in X containing $(x*0)$ and a pre-open set V_0 in X containing 0 such that $W * V_0 \subseteq X \setminus E$. Also, there is a pre-open set V_1 containing 0 such that $x * V_1 \subseteq W$. Since X is P_0-space, so if $V_x = V_0 \cap V_1$, then V_x is a pre-open set containing 0 and $(x * V_x) * V_x \subseteq W * V_0 \subseteq X \setminus E$. Since L_a is a pre-open map, so $C = \{x * V_x : x \in F\}$ is an open cover of the compact set F. Hence, there are $x_1 * V_{x_1}, ..., x_n * V_{x_n} \in C$ such that

$$F \subseteq \bigcup\nolimits_{i=1}^{n} (x_i * V_{x_i}).$$

Suppose that

$$V = \bigcap\nolimits_{i=1}^{n} (V_{x_i}).$$

Since X is P_0-space, so V is a pre-open set containing 0 such that for each $y \in F, y \in x_i * V_{x_i}$, for some x_i, and $y * V \subseteq (x_i * V_{x_i}) * V \subseteq (x_i * V_{x_i}) * V_{x_i} \subseteq W * V_0 \subseteq X \setminus E$. This proves that $(F * V) \cap E = \phi$. Recalling that a topological space X is called submaximal [3], if every dense subset of X is open.

Corollary 3.38. *Let X be a submaximal PBCK-algebra. If $L_a : X \to X$ is pre-open and $F \cap E = \phi$ where F is compact and E is closed, then there is a pre-open set V containing 0 such that $(F * V) \cap E = \phi$.*

Proof. The proof follows from Proposition 3.37 and the fact that when X is submaximal, then $PO(X)$ forms a topology on X [5].

Definition 3.39. [15] A subset $\mathcal{I}$ of X is called a BCC-ideal if the following conditions hold:

(1) $0 \in \mathcal{I}$,
(2) If $y \in \mathcal{I}$ and $(x * z) * y \in \mathcal{I}$, then $x * y \in \mathcal{I}$.

It is proved in [15], that each BCC-ideal is an ideal.

Theorem 3.40. *Let $(X, *, 0)$ be a BCK-algebra and $\mathcal{I}$ be a BCC-ideal. Suppose that τ is the topology generated by the family $\{\mathcal{I}[x] : x \in X\}$, then $(X, *, 0)$ with the topology τ is a PBCK-algebra.*

Proof. First we shall prove that $\mathcal{I}[0] \subseteq \mathcal{I}[x]$ for all $x \in X$. For this, let $a \in \mathcal{I}[0]$ so we have, $a*0 = a \in \mathcal{I}$. Now $(a*x)*a = (a*a)*x = 0 \in \mathcal{I}$, implies $a*x \in \mathcal{I}$ and hence $a \in \mathcal{I}[x]$. Therefore, $\mathcal{I}[0] \subseteq \mathcal{I}[x]$. Since $\mathcal{I}[0]$ is an open set contained in each $\mathcal{I}[x]$ for all $x \in X$. Therefore, every open set in τ is a super set of $\mathcal{I}[0]$. Hence for each $x \in X$, we have $\mathcal{I}[0] \cup \{x\}$ is a pre-open set containing x. To prove that $(X, *, 0)$ with the topology τ is a PBCK-algebra, let $x, y \in X$ and U be any open set containing $x * y$, then by definition of τ, we have $x * y \in \mathcal{I}[x * y] \subseteq U$. Since $\mathcal{I}[x]$ is an open set containing x and $\mathcal{I}[0] \cup \{y\}$ is a pre-open set containing y, so it is enough to show that

$$\mathcal{I}[x] * (\mathcal{I}[0] \cup \{y\}) \subseteq \mathcal{I}[x * y]$$

For this, suppose that $a * b \in \mathcal{I}[x] * (\mathcal{I}[0] \cup \{y\})$ implies that $a \in \mathcal{I}[x]$ and $b \in (\mathcal{I}[0] \cup \{y\})$. Hence, we get the following two cases:

Case 1. $a \in \mathcal{I}[x]$ and $b \in (\mathcal{I}[0]$ which implies that $a * x \in \mathcal{I}$ and $b * 0 = b \in \mathcal{I}$, so we have $[(a * x) * b] * (a * x) = [(a * x) * (a * x)] * b = 0 * b = 0 \in \mathcal{I}$ implies that $(a * x) * b \in \mathcal{I}$. Since $\mathcal{I}$ is a BCC-ideal, $b \in \mathcal{I}$, $(a * x) * b \in \mathcal{I}$, implies that $a * b \in \mathcal{I}$. Consider $[(a * b) * (x * y)] * (a * b) = [(a * b) * (a * b)] * (x * y) = 0 * (x * y) = 0 \in \mathcal{I}$. This proves that $(a * b) * (x * y) \in \mathcal{I}$ and hence $(a * b) \in \mathcal{I}[(x * y)]$ which implies that

$$\mathcal{I}[x] * (\mathcal{I}[0]) \subseteq \mathcal{I}[x * y] \tag{3.1}$$

Case 2. $a \in \mathcal{I}[x]$ and $b = y$, so we have $a * x \in \mathcal{I}$. Consider $[(a * b) * (x * y)] * (a * x) = [(a * b) * (a * x)] * (x * y) \leq (x * b) * (x * y)$ and since $b = y$, so $[(a * b) * (x * y)] * (a * x) = 0 \in \mathcal{I}$. again we have $a * x \in \mathcal{I}$ and by definition of $a * x \in \mathcal{I}$, we obtain that $(a * b) * (x * y) \in \mathcal{I}$, so $(a * b) \in \mathcal{I}[(x * y)]$ which implies that

$$\mathcal{I}[x] * y \subseteq \mathcal{I}[x * y] \tag{3.2}$$

Hence, from inclusion 3.1 and inclusion 3.2, we get

$$\mathcal{I}[x] * (\mathcal{I}[0] \cup \{y\}) \subseteq \mathcal{I}[x * y] \subseteq U$$

This shows that $(X, *, 0)$ with the topology τ is a PBCK-algebra.

4. Conclusion

After introducing the notion of BCK-algebra, some mathematicians payed attention to the concept of topological BCK-algebras similar to the concept of topological groups, topological rings and topological vector spaces. The goal of this paper, is to define other type of topological BCK-algebra by using of pre-open sets. Several properties of this type are investigated.

References

[1] T. Aho and T. Nieminen, *Spaces in which preopen subsets are semiopen*, Ric. Mat. (1) **43** (1994), 45-60.

[2] R. A. Alo, *Topologies of BCK-algebras*, Math. Japon. (6) **31** (1986), 841–853.

[3] A. V Arhangel'skiĭ and P. J. Collins, *On submaximal spaces*, Topology Appl. (3) **64** (1995), 219–241.

[4] K. H. Dar, *A characterization of positive implicative BCK-algebras by self-maps*, Math. Japonica, (2) **31** (1986), 197–199.
[5] J. Dontchev, *Survey on preopen sets*, arXiv preprint math/9810177 1998.
[6] A. Dvurecenskij and S. Pulmannová, *New trends in quantum structures*, Springer Science & Business Media, **516**, 2013.
[7] R. Engelking, *General topology*, PWN–Polish Scientific Publishers, Warsaw, 1977.
[8] Y. Imai and K. Iséki, *On axiom systems of propositional calculi XIV*, Proc. Japan Acad. (1) **42** (1966), 19–22.
[9] K. Iséki, *An algebra related with a propositional calculus*, Proc. Japan Acad. (1) **42** (1966), 26–29.
[10] Y. B. Jun, X. L. Xin and D. S. Lee, *On topological BCI-algebras*, Inform. Sci. (2-4) **116** (1999), 253–261.
[11] A. Kar and P. Bhattacharyya, *Some weak separation axioms*, Bull. Calcutta Math. Soc. (5) **82** (1990), 415-422.
[12] D. S. Lee and D. N. Ryu, *Notes on topological BCK-algebras*, Sci. Math, **1** (1998), 231-235.
[13] A. S. Mashhour, I. A. Hasanein and S. N El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt, **53** (1982), 47–53.
[14] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa Company, 1994.
[15] F. R. Setudeh and N. Kouhestani, *On (Semi) Topological BCC-algebras*, The 46^{th} Annual Iranian Mathematics Conference, 2015.
[16] M. A. Yousef and A. B. Khalaf, *Semi-topological UP-algebras*, J. Algebra Relat. Topics, (2) **9** (2021), 1–22.

A. B. Khalaf
Department of Mathematics, University of Duhok, Duhok, Iraq.
Email: `aliasbkhalaf@uod.ac`

N. Ahmed
Department of Mathematics, College of Education), Salahaddin University Erbil, Iraq.
Email: `nehmat.ahmed@su.edu.krd`