Journal of Algebra and Related Topics
Vol. 11, No 1, (2023), pp 43-48

ON DERIVATION ALGEBRA BUNDLE

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Abstract. We show that the radical bundle of an algebra bundle is a characteristic ideal bundle. Further we prove an algebra bundle is semisimple if and only if its derivation algebra bundle is either semisimple or zero.

1. Introduction

J.P. Serre posed the question: does there exist a Hausdorff Lie group bundle whose Lie algebra bundle is isomorphic to a given Lie algebra bundle. A.Douady and M.Lazard have constructed a Lie group bundle $G(\zeta)$ (not necessarily Hausdorff) whose Lie algebra bundle is isomorphic to a given Lie algebra bundle ζ [4, Theorem 3]. They ask whether analogous result still holds locally (around each point of base space) if one requires $G(\zeta)$ to be Hausdorff in analytic case [4, Page 151]. Coppersmith has constructed an example [3] of an analytic Lie algebra bundle over a smooth Hausdorff manifold which does not correspond to the Lie algebra bundle of any Hausdorff Lie group bundle. An associative (Lie) algebra bundle is a vector bundle $\xi = (\xi, p, X)$, together with a morphism $\theta : \xi \oplus \xi \to \xi$ which induces associative (Lie) algebra structure on each fibre ξ_x. A locally trivial associative (Lie) algebra bundle is a vector bundle $\xi = (\xi, p, X)$ in which each fibre is an associative (Lie) algebra and for each x in X there exist an open set U of x in X, a associative (Lie) algebra A and a homeomorphism $\Phi : U \times A \to \bigcup_{x\in U} \xi_x$ such that restriction $\Phi_x : x \times A \to \xi_x$ is an

MSC(2010): 05C25, 05C50
Keywords: algebra bundle, characteristic ideal bundle, radical bundle, semisimple algebra bundle, vector bundle.
Received: 19 August 2022, Accepted: 2 January 2023.

associative (Lie) algebra isomorphism. A subalgebra bundle of an associative (Lie) algebra bundle is a vector subbundle in which each fibre is a subalgebra. Further if each fibre is an ideal then it is called an ideal bundle. A morphism $\varphi : \xi_1 \to \xi_2$ of associative (Lie) algebra bundles ξ_1 and ξ_2 over the same base space X is a continuous map and for each x in X, $\varphi_x : \xi_{1x} \to \xi_{2x}$ is a associative (Lie) algebra homomorphism. A morphism φ is an isomorphism if φ is bijective and φ^{-1} is continuous. Lie algebra bundles [7–9, 12–14] and associative algebra bundles [2, 10] over a field of characteristic zero have been studied.

1.1. **Radical bundle of an algebra bundle.** Local triviality of an algebra bundle ξ is given by

$$\varphi : U \times A \to \bigcup_{x \in U} \xi_x,$$

such that $\varphi_x : A \to \xi_x$ is an algebra isomorphism. Let ξ_x^r be the $(Jacobson)$ radical of ξ_x, $J(A)$ the radical of an algebra A. Then $\varphi_x\big(J(A)\big) \subseteq \xi_x^r$ and $\varphi_x^{-1}\big(\xi_x^r\big) \subseteq J(A)$ [15, Lemma, p.59]. Hence $\varphi|_{U \times J(A)}$ defines an isomorphism between $U \times J(A)$ and $\bigcup_{x \in U} \xi_x^r$. Thus $\Re = \bigcup_{x \in X} \xi_x^r$ is an ideal bundle of ξ. We call $\Re$, the *radical bundle* of ξ.

1.2. **Radical bundle of a Lie algebra bundle.** Let ζ be a locally trivial Lie algebra bundle, and $\Phi : U \times L \to \bigcup_{x \in U} \zeta_x$ be a local triviality of ζ, where L is a Lie algebra. Let R be the radical of L, ζ_x^r be the radical of ζ_x. Then $\Phi|_{U \times R} : U \times R \to \bigcup_{x \in U} \zeta_x^r$ is an isomorphism. We call $\Re = \bigcup_{x \in X} \zeta_x^r$ is the radical bundle of ζ.

Here we show that the radical bundle $\Re$ of an associative algebra bundle ξ is a characteristic ideal bundle when the base field is of characteristic zero. Further we prove that an associative algebra bundle ξ is semisimple if and only if its derivation algebra bundle $\mathcal{D}(\xi)$ is semisimple or $\{0\}$.

Notations and Terminology: All our algebra bundles are associative algebra bundles over a field of characteristic zero unless otherwise mentioned. All our bundles and subbundles and ideal bundles are over the same base space.

2. Derivation algebra bundles

Definition 2.1. Let ξ be an algebra bundle. A vector bundle morphism $D : \xi \to \xi$ is a derivation if $D(u.v) = u.D(v) + D(u).v$, for all $u,\ v \in \xi_x$.

Definition 2.2. A derivation D of ξ is called inner if there is a section S of ξ, such that $D(u) = u.S(x) - S(x).u$, for all u in ξ_x and x in X.

Remark 2.3. We denote set of all derivations of ξ by $\mathcal{D}(\xi)$ and is a locally trivial Lie algebra bundle [11].

Definition 2.4. An ideal bundle η of an algebra bundle ξ is called a characteristic ideal bundle if it is invariant under all derivations D in $\mathcal{D}(\xi)$.

Theorem 2.5. *Let ξ be an algebra bundle over a base space X; let $\Re$ be its radical. Then $\Re$ is a characteristic ideal bundle.*

Proof. Let $\Re \supseteq \Re^2 \supseteq \Re^3 \supseteq \cdots \Re^{p+1} = \{0\}$ be the sequence of the derived algebra bundles of $\Re$. Let D be any derivation of ξ. Suppose that $D^i(\Re^{k+1}) \subseteq \Re$ for all $i = 1, 2, \cdots$; (trivial for k=p). Then we shall show that $D^i(\Re^k) \subseteq \Re$; $i = 1, 2, \cdots$.

Since $\Re^k$ is an ideal bundle of ξ it is easy to see that the set $\Re + D(\Re^k)$ is an ideal bundle of ξ. For, if $\Phi : U \times A \to \bigcup_{x\in U} \xi_x$ is a locally triviality of ξ, then $\Phi|_{U\times J(A)} : U \times J(A) \to \bigcup_{x\in U} \xi_x^r$ gives the local triviality of $\Re$. Then $U \times D(J(A)) \to \bigcup_{x\in U} \Phi D \Phi^{-1}(\xi_x^r)^k$ is an isomorphism for any derivation D of A. Hence

$$U \times J(A) + D((J(A))^k) \to \bigcup_{x\in U} \xi_x^r + \Phi_x D \Phi_x^{-1}(\xi_x^r)^k$$
$$\big(y, u, D(z)\big) \mapsto (y, \Phi_x(u), \Phi_x D(z))$$

is an isomorphism. Then $\bigcup_{x\in X} \xi_x^r + \Phi D \Phi^{-1}(\xi_x^r)^k$ is a locally trivial ideal bundle of ξ. Also each fibre $\xi_x^r + \Phi D \Phi^{-1}(\xi_x^r)^k$ is solvable ideal in ξ_x. Hence $D(\Re^k) \subseteq \Re$ for any derivation D of ξ. Suppose we have already proved that $D^i(\Re^k) \subseteq \Re$ for all $i < n$. It follows from the methods of [6] that $\Re + D^n(\Re^k)$ is an ideal bundle of ξ and $D^n(\Re^k) \subseteq \Re$. Thus we have if $D^i(\Re^{k+1}) \subseteq \Re$ for all i, then also $D^i(\Re^k) \subseteq \Re$ for all i, hence Theorem follows from by induction on k. □

Lemma 2.6. *Every derivation of an algebra bundle ξ over a compact Hausdorff space is the sum of an inner derivation and a derivation which annuls $\mathcal{S}$.*

Proof. Algebra bundle ξ being locally trivial over compact Hausdorff space we have $\xi = \Re + \mathcal{S}$, where $\Re$ is the radical bundle and $\mathcal{S}$ is a subalgebra bundle [10, Theorem 5.1]. Let D be any derivation of ξ. Then by above Theorem (2.5), we have $D(\Re) \subseteq \Re$. On the other hand there exists an element $v_0 = S(x)$ in ξ_x such that $D(s) = sv_0 - v_0 s$ for every s in $\mathcal{S}_x$, where S is a section of ξ. Let D_{v_0} denote the inner derivation effected by v_0. We set $D^{'} = D - D_{v_0}$, then we have $D^{'}(\mathcal{S}) = 0$. Also we have for any $r \in \xi_x^r$, $s \in \mathcal{S}_x$

$$D^{'}(sr) = rD^{'}(s) + sD^{'}(r) = sD^{'}(r),$$

and

$$D^{'}(rs) = D^{'}(r)s + rD^{'}(s) = D^{'}(r)s.$$

Conversely, any derivation D of $\Re$ satisfying above property gives a derivation of ξ if we define $D(\mathcal{S}) = 0$ as continuity of D on ξ follows from pasting Lemma. □

Lemma 2.7. *Let ξ be an algebra bundle over a compact Hausdorff space with $\xi = \Re + \mathcal{S}$ and $\Re \subset Z(\xi)$ (center of ξ). Then there is a non zero abelian ideal bundle in $\mathcal{D}(\xi)$.*

Proof. Let D be any derivation of ξ then by above Theorem (2.5) we have $D(\Re) \subset \Re$. Also $D|_{\mathcal{S}}$ being inner there is a section S with $u_0 = S(x) \in \xi_x$ and $D|_{\mathcal{S}}(s) = u_0 s - s u_0$ for all $s \in \mathcal{S}_x$. Thus $D|_{\mathcal{S}}$ maps $\mathcal{S}$ into its itself for all $D \in \mathcal{D}(\xi)$ since $D|_{\mathcal{S}}$ is inner and $\Re \subset Z(\xi)$. Let $\Re \supset \Re^2 \supset \Re^3 \supset \cdots \supset \Re^k = 0$. It is easily seen by induction on the exponent i that every $\Re^i$ is a characteristic ideal bundle of ξ. Suppose that $\Re \neq 0$. If $\Re^2 = 0$, we can define a derivation of ξ as follows

$$D(r) = r \text{ if } r \in \xi_x^r; \qquad D(s) = 0 \text{ if } s \in \mathcal{S}_x.$$

Then D is a derivation of ξ. Continuity of D follows from pasting Lemma. If D^* is any other derivation of ξ we have for all $r \in \xi_x^r$.

$$[D, D^*](r) = (D^*D - DD^*)(r) = D^*(r) - D(D^*(r)) = 0,$$

since $\Re$ being characteristic $D^*(r) \subset \Re$ and for all $s \in \mathcal{S}_x$

$$[D, D^*](s) = (D^*D - DD^*)(s) = -D(D^*(s)) = 0, \text{ Since } D^*(\mathcal{S}) \subset \mathcal{S}.$$

Hence $[D, D^*] = 0$ for every derivation $D^* \in \mathcal{D}(\xi)$. Thus D is in $Z(\mathcal{D}(\xi))$. For $\Re^2 \neq 0$, and so, in the series above, $k > 2$. If $u_0 \in \Re^{k-2}$ we can define a derivation D_{u_0} of ξ as follows:

$$D_{u_0}(r) = u_0.r \text{ if } r \in \Re; D_{u_0}(s) = 0 \text{ if } s \in \mathcal{S}_x.$$

If D is any other derivation of ξ we have

$$[D_{u_0}, D](r) = D(u_0.r) - u_0.D(r) = D(u_0).r \text{ if } r \in \Re_x$$

and

$$[D_{u_0}, D](s) = -D_{u_0}D(s) = 0 \text{ if } s \in \mathcal{S}_x.$$

Hence $[D_{u_0}, D] = D_{D(u_0)}$, which shows that the derivations of the form D_{u_0}, $u_0 \in \Re^{k-2}$, constitute nonzero abelian ideal bundle in $\mathcal{D}(\xi)$. □

Theorem 2.8. *Let ξ be an algebra bundle over a compact Hausdorff space X. Then ξ is semisimple if and only if $\mathcal{D}(\xi)$ is semisimple or $\{0\}$.*

Proof. Suppose ξ is semisimple then $\mathcal{D}(\xi)$ consists only of inner derivations. Let us denote by ξ^l the Lie algebra bundle obtained from ξ by defining the commutator of two elements as $[u, v] = uv - vu$ for all $u,\ v \in \xi_x$.

Consider the morphism $ad : \xi^l \to \mathcal{D}(\xi)$ defined by $ad(u) = ad_u$, where $ad_u(v) = uv - vu$ for all $u, v \in \xi_x$. Then

$$\begin{aligned} ker(ad)_x &= \{u \in \xi^l_x \mid ad_u(\xi_x) = 0\} \\ &= \{u \in \xi^l_x \mid ad_u(v) = 0 \text{ for all } v \in \xi_x\} \\ &= Z(\xi^l_x) \ = \ \text{center of } \xi^l_x \end{aligned}$$

Thus $ker\ ad = \bigcup_{x\in X} ker(ad)_x = Z(\xi^l)$. Then we have $\xi^l/Z(\xi^l) \cong \mathcal{D}(\xi)$. We know that derived algebra, $(\xi^l)^{(1)} = \bigcup_{x\in X}[\xi^l_x, \xi^l_x]$ of ξ^l is semisimple or $\{0\}$ [16].

Let $\Re(\xi^l)$ be the radical of ξ^l. Then $[\Re(\xi^l), \xi^l]$ is a solvable ideal bundle in $(\xi^l)^{(1)}$ and hence $[\Re(\xi^l), \xi^l] = \{0\}$. This implies that $\Re(\xi^l) = Z(\xi^l)$ since $\Re(\xi^l)$ is the maximal solvable and $Z(\xi^l)$ is solvable ideal bundle of ξ^l,. Hence $\xi^l/Z(\xi^1)$ is semisimple or $\{0\}$.

Suppose now that $\mathcal{D}(\xi)$ is semisimple or $\{0\}$. Algebra bundle ξ being locally trivial over compact Hausdorff space we have $\xi = \Re + \mathcal{S}$ [10, Theorem 5.1]. If $\Re \neq 0$, then consider $\mathcal{D}_{\Re}(\xi)$ be the set of all inner derivation of ξ which are effected by the elements of $\Re$. Then $\mathcal{D}_{\Re}(\xi)$ is an ideal bundle of $\mathcal{D}(\xi)$. Since $\Re$ is solvable, $\mathcal{D}_{\Re}(\xi)$ is solvable ideal bundle of $\mathcal{D}(\xi)$ and hence reduces to zero. Thus $\Re$ is contained in the $Z(\xi)$. Hence by Lemma (2.7) there is a non zero abelian ideal bundle in $\mathcal{D}(\xi)$ which contradicts to the assumption that $\mathcal{D}(\xi)$ is semisimple. Hence the radical bundle $\Re = 0$. Thus ξ is semisimple. □

Acknowledgments

The authors would like to thank the referee for meticulous reading and valuable suggestions. Also authors would like to thank the REVA University for continuous support and encouragement.

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