**Journal of Mathematical Modeling**
Vol. 11, No. 2, 2023, pp. 375–394. Research Article

JMM

# Stability and bifurcation of stochastic chemostat model

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**Abstract.** The main purpose of this paper is to study dynamics of stochastic chemostat model. In this order, Taylor expansions, polar coordinate transformation and stochastic averaging method are our main tools. The stability and bifurcation of the stochastic chemostat model are considered. Some theorems provide sufficient conditions to investigate stochastic stability, $D$-bifurcation and $P$-bifurcation of the model. As a final point, to show the effects of the noise intensity and illustrate our theoretical results, some numerical simulations are presented.

*Keywords*: Stochastic chemostat model; Lyapunov exponent; $D$-bifurcation; $P$-bifurcation
*AMS Subject Classification 2010*: 60H10, 34K50, 35B32

## 1 Introduction

Continuous culture is a system that is designed for long-term operation which can be operated over the long term because it is an open system with a continuous feed of influent solution that contains nutrients and substrate. Chemostat, as a growth container, is a type of continuous bioreactor that functions for the cultivation of microalgae and other microorganisms that are usually used in laboratory and industrial scales [2, 9]. The trivial ways for competition is whenever two or more populations are competing for unchanged resource. In order to study the modeling of competition in nature, the chemostat is a type of competition that refers to a laboratory device [15]. It is used in mathematical biology for growing microorganisms. Because of the possibility of relevant experiments and mathematically tractable, this device is significant in ecological studies. In the engineering literature, it is known as a continuous stir tank reactor (CSTR) which considers both substrate utilization and the cell growth. It has played a central place in mathematical ecology [10, 19, 22]. Firstly, a short outline of deterministic chemostat model is provided. This simple deterministic chemostat model is based on the two standard assumptions: (a) the availability of the nutrient and its supply rate are fixed and (b) the tendency of microorganism to adhere to surfaces is not taken into account. Denoting by $x(t)$ the concentration of the microorganism at any

*Corresponding author
Received: 02 April 2023/ Revised: 28 April 2023/ Accepted: 29 April 2023
DOI: 10.22124/jmm.2023.24214.2165

 

specific time $t$ and $y(t)$ the concentration of the growth-limiting nutrient. These assumptions lead to the following growth model [6]

$$\begin{cases} \dot{x} = -qx + \frac{Rxy}{K+y}, \\ \dot{y} = q(c-y) - \frac{Rxy}{a(K+y)}, \end{cases} \tag{1}$$

where $R, K, q, c > 0$ and $a \neq 0$.

Chemostat model has an essential role in ecological theory and often observed as a model of simple lake or an ocean system, for instance (see [2, 9, 15, 17]). In [13, 14, 22], it is used for studying recombinant problems relevant to genetically altered microorganisms. Also in order to analysis of antibiotic and in some problems of waste water treatment, this model is applicable (see [10, 18, 19]). Some studies show that at microscopic scale the accumulation of small perturbations in the chemostat could not be neglected and increasing environmental noise may lead to extinction in scenarios where the deterministic model predicts persistence [7, 16]. To make this model more realistic some authors investigated the stochastic version of it by influencing white noise (see [7, 23, 26]). Caraballo, etal. consider the chemostat models with random nutrient supplying rate or random input nutrient concentration, with or without wall growth. The results arises from this study proved the existence of a unique random attractor to the random chemostat models, by constructing sufficient conditions for extinction and persistence of the microorganism [5]. Sun and Zhang showed that the solutions of a stochastic chemostat model with time delay will oscillate around the equilibriums of the corresponding deterministic model and under small noise, when the time delay is small, microorganism is persistent; when the time delay is large, microorganism will be extinct [24]. Also in [6] by simplifying the chemostat models, the authors investigated the existence and uniqueness of solutions and existence of a random attractor by the random dynamical system via the solution.

In [20], Luo and Guo investigated the stability and bifurcation of a two-dimensional stochastic differential equations with multiplicative excitations. They provided some conditions on drift and diffusion coefficients of a two-dimensional nonlinear stochastic system to obtain $P$-bifurcation and $D$-bifurcation.

In this paper, we study stability and bifurcation stochastic chemostat model. In Section 2, an overview of dynamical behaviour in two-dimensional stochastic systems with multiplicative excitations, which provided by Luo and Guo in [20], is presented. Particularly, this section focused on sufficient conditions on drift and diffusion coefficients for stability, $D$-bifurcation and $P$-bifurcation in two dimensional stochastic dynamical systems. Then, we consider deterministic and stochastic chemostat model. Sections 3 and 4 are the main part of our paper that are devoted to study stability of stochastic chemostat model by largest Lyapunov exponent, $D$-bifurcation and $P$-bifurcation. We consider several conditions on diffusion and drift coefficients that the model undergoes $P$-bifurcation. Also, using Euler-Maruyama method, we demonstrate some numerical simulation to validate the theoretical results.

## 2 Preliminaries

In this section firstly, we present the general form of a stochastic differential equation and the types of it's stability adopted from [1]. Consider the following stochastic differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \tag{2}$$

where $W_t = (W_t^1, \dots, W_t^m)^T$ is a $m$ dimensional standard Brownian motion where $(W(0) = 0)$, $f = (f_1, \dots, f_n)$ is an $\mathbb{R}^n$ valued function, $g = (g_{ij})$ a $n \times m$ real matrix and $t$ denote the time $t \geq 0$. The functions $f$ and $g$ satisfy the conditions of the existence of the solutions of this SDE with initial conditions $X_0 = x_0 \in \mathbb{R}^n$, and also $f(t,0) = 0$ and $g(t,0) = 0$. The process $X_t$ is centered and the aim is to study the stability of this process in the neighborhood of the zero solution.

The zero solution is stochastically stable if for every pair of $\varepsilon > 0$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r) > 0$ such that

$$P\{\omega : |X(t,x_0)| < r \text{ for all } t \geq t_0\} \geq 1 - \varepsilon,$$

whenever $\|x_0\| < \delta$. Otherwise, it is said to be stochastically unstable.

The zero solution is said to be stochastically asymptotically stable if it is stochastically stable and, moreover, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$P\{\omega : \lim_{t\to\infty} X(t,x_0) = 0\} \geq 1 - \varepsilon,$$

whenever $\|x_0\| < \delta$.

The zero solution is said to be global stochastically asymptotically stable if it is stochastically stable and, moreover, for all $x_0 \in \mathbb{R}^d$

$$P\{\omega : \lim_{t\to\infty} x(t,x_0) = 0\} = 1.$$

Secondly, as we mentioned before, in this paper we focus on the following two-dimensional stochastic differential equations with multiplicative excitations

$$\begin{cases} dx = f_1(x,y)dt + g_1(x,y)dW_1(t), \\ dy = f_2(x,y)dt + g_2(x,y)dW_2(t), \end{cases} \tag{3}$$

where $f_i \in C^3(R \times R, R)$, $g_i \in C^1(R \times R, R)$ $(i = 1,2)$ and $dW_i(t)$ $(i = 1,2)$ are mutually independent standard real-valued Wiener processes on the complete probability space $(\Omega, F, P)$.

In [20] using Taylor expansion, polar coordinate transformation and stochastic averaging method, a general framework for the stability and bifurcation analysis of the stochastic System (3) is provided. Suppose that $f_i(0,0) = 0$ and $g_i(0,0) = 0$ $(i = 1,2)$. If in the Taylor expansion of $f_i$ and $g_i$ at the point $O(0,0)$ we ignore the terms higher than third order and rescaling the system as presented in [12], then we obtain the following system

$$\begin{cases} dx = & \varepsilon[a_{110}x + a_{101}y + a_{120}x^2 + a_{111}xy + a_{102}y^2 + a_{130}x^3 + a_{121}x^2y \\ & \quad + a_{112}xy^2 + a_{103}y^3]dt + \sqrt{\varepsilon}[b_{110}x + b_{101}y]dW_1(t), \\ dy = & \varepsilon[a_{210}x + a_{201}y + a_{220}x^2 + a_{211}xy + a_{202}y^2 + a_{230}x^3 + a_{221}x^2y \\ & \quad + a_{212}xy^2 + a_{203}y^3]dt + \sqrt{\varepsilon}[b_{210}x + b_{201}y]dW_2(t). \end{cases} \tag{4}$$

In [21], by combining polar coordinate transformation, the authors rewrote System (4) to Ito stochastic differential equations

$$\begin{cases} dr = [(\varphi_1 + \frac{1}{16}\varphi_2)r + \frac{1}{8}\varphi_3 r^3]dt + (\frac{\varphi_4}{8}r^2)^{\frac{1}{2}}dW_r(t), \\ d\theta = [\frac{1}{4}\varphi_5 + \frac{1}{8}\varphi_6 r^2]dt + (\frac{\varphi_2}{8})^{\frac{1}{2}}dW_\theta(t), \end{cases} \tag{5}$$

with the following notations

$$\begin{aligned}
\varphi_1 =& \frac{1}{2}(a_{110}+a_{201}),\\
\varphi_2 =& b_{110}^2+b_{201}^2+b_{101}^2+3b_{210}^2,\\
\varphi_3 =& 3a_{130}+a_{112}+a_{221}+a_{203},\\
\varphi_4 =& 3b_{110}^2+b_{101}^2+b_{210}^2+b_{201}^2,\\
\varphi_5 =& -2a_{101}+2a_{210}+b_{110}b_{101}-b_{210}b_{201},\\
\varphi_6 =& -a_{103}+a_{212}-a_{121}+3a_{230}.
\end{aligned} \tag{6}$$

Taking account of the existence of random factors, we assume that $\varphi_2$ and $\varphi_4$ are positive numbers, in the sequel. Since the modulus equation is uncoupled with the phase equation, we only need the averaging modulus equation

$$dr = [(\varphi_1+\frac{1}{16}\varphi_2)r+\frac{1}{8}\varphi_3 r^3]dt+(\frac{\varphi_4}{8}r^2)^{\frac{1}{2}}dW_r(t), \tag{7}$$

to investigate the stability and bifurcation of System (5).

In the following theorems, by using Eq. (7), we investigate the stability conditions of the equilibrium point of System (3).

**Theorem 1** ([20]). *When $16\varphi_1+\varphi_2-\varphi_4<0$ and $2\varphi_3<\varphi_4$, the stochastic system (7) is globally stable at the equilibrium point O.*

These theorems describe the changes in qualitative behavior of the stochastic system, depend on drift and diffusion parameters changes. The next two theorems, investigate some conditions that System (7) undergoes stochastic phenomenological bifurcation or $P$-bifurcation.

**Theorem 2** ([20]). *If $\varphi_3<0$ and $\varphi_4>0$, System (7) undergoes stochastic phenomenological bifurcations as the parameter $\varphi_4$ passes through the values of $8\varphi_1+\frac{1}{2}\varphi_2$, $\frac{16\varphi_1+\varphi_2}{3}$ and $\frac{16\varphi_1+\varphi_2}{4}$.*

## 2.1 Stochastic chemostat model

In mathematical biology models, noise can be arose from by different reasons and it may appears by various sources. There are a lot of different ways to vary deterministic chemostat model into stochastic one and investigating the dynamical behaviour of it (see [3,4,16,25]).

The dynamical model (1) has two equilibriums $e_1=(0,c)$ and

$$e_2=(\frac{a(K+\frac{qK}{R-q})(c-\frac{qK}{R-q}))(R-q)}{RK},\frac{qK}{R-q}).$$

In order to investigate of stability and bifurcation of stochastic version it, we change coordinates to transfer the equilibriums $e_1$ and $e_2$ into origin. By taking $u=x, v=y-c$ the equilibrium $e_1$ change into $(0,0)$ and System (1) becomes

$$\begin{cases}\dot{u}=-qu+\frac{Ru(v+c)}{K+v+c},\\ \dot{v}=-vq-\frac{Ru(v+c)}{a(K+v+c)}.\end{cases} \tag{8}$$

Similarity by taking

$$u = x - \frac{a(K+\frac{qK}{R-q})(c-\frac{qK}{R-q}))(R-q)}{RK}, \quad \text{and} \quad v = y - \frac{qK}{R-q},$$

the equilibrium $e_2$ vary into $(0,0)$ and System (1) becomes

$$\begin{cases} \dot{u} = -\frac{(R-q)(Kaq-Rac+acq-Ru+qu)v}{(KR+Rv-qv)}, \\ \dot{v} = -\frac{(Kaq^2v+R^2acv-2Racqv+Raqv^2+acq^2v-aq^2v^2+KRqu+R^2uv-Rquv)}{(KR+Rv-qv)a}. \end{cases} \tag{9}$$

Firstly, we consider the stochastic model of System (8) by the following form

$$\begin{cases} du = (-qu + \frac{Ru(v+c)}{K+v+c})dt + \sigma_1 u\, dW_1(t), \\ dv = (-qv - \frac{Ru(v+c)}{a(K+v+c)})dt + \sigma_2 v\, dW_2(t), \end{cases} \tag{10}$$

where, $\sigma_1, \sigma_2$ measure the noise intensity in the system due to the environment and $W_1(t), W_2(t)$ denote the independent standard Wiener processes.

By considering Taylor expansions of deterministic parts of System (10) at origin, the following equivalent system is obtained

$$\begin{cases} du = \big((-q+\frac{Rc}{(K+c)})u + (\frac{RK}{(K+c)^2})vu - (\frac{RK}{(K+c)^3})uv^2 + o(4)\big)dt + \sigma_1 u dW_1(t), \\ dv = \big(-qv - \frac{Rcu}{a(K+c)} + (\frac{RK}{a(K+c)^2})vu + (\frac{RK}{a(K+c)^3})uv^2 + o(4)\big)dt + \sigma_2 v dW_2(t), \end{cases} \tag{11}$$

where $o(4)$ show the high order terms. We consider truncated equations of System (11) and assume $u = \overline{u}, v = \overline{v}, t = \overline{t}$ and $a_{jis} = \varepsilon \overline{a_{jis}}$, $b_{jis} = \sqrt{\varepsilon}\overline{a_{jis}}$ for all $j, i, s$. For simplicity that we drop the bars from the scaled variables. Then obtain

$$\begin{cases} du = \varepsilon[\big((-q+\frac{Rc}{K+c})u + (\frac{RK}{(K+c)^2})vu - (\frac{RK}{(K+c)^3})uv^2\big)]dt + \sqrt{\varepsilon}\sigma_1 u dW_1(t), \\ dv = \varepsilon[\big(-qv - \frac{Rcu}{a(K+c)} + (\frac{RK}{a(K+c)^2})vu + (\frac{RK}{a(K+c)^3})uv^2\big)]dt + \sqrt{\varepsilon}\sigma_2 v dW_2(t). \end{cases} \tag{12}$$

Now, by Khasminskii limiting theorem, System (12) can be transformed into the following limiting Ito averaging equations via polar coordinate $x = r\cos\theta$ and $y = r\sin\theta$ with the Ito formula, we have

$$\begin{cases} dr = [(\varphi_1 + \frac{1}{16}\varphi_2)r + \frac{1}{8}\varphi_3 r^3]dt + (\frac{\varphi_4}{8}r^2)^{\frac{1}{2}} dW_r(t), \\ d\theta = [\frac{1}{4}\varphi_5]dt + (\frac{\varphi_2}{8})^{\frac{1}{2}} dW_\theta(t), \end{cases} \tag{13}$$

where the parameters $\varphi_i$ arises from Equations (6) and given as follows:

$$\begin{array}{ll} \varphi_1 = -q + \frac{1}{2}\frac{Rc}{(K+c)}, & \varphi_2 = \sigma_1^2 + \sigma_2^2, \\ \varphi_3 = \frac{-RK}{(K+c)^3}, & \varphi_4 = 3\sigma_1^2 + \sigma_2^2, \\ \varphi_5 = \frac{-2Rc}{a(K+c)}, & \varphi_6 = \frac{-RK}{a(K+c)^3}. \end{array}$$

As we mentioned in Section 2 for investigating the stability and bifurcation of System (10), the following averaging modulus equation is considered.

$$dr = [(-q + \frac{1}{2}\frac{Rc}{(K+c)} + \frac{\sigma_1^2+\sigma_2^2}{16})r - \frac{RK}{8(K+c)^3}r^3]dt + (\frac{(3\sigma_1^2+\sigma_2^2)r^2}{8})^{\frac{1}{2}} dW_r(t). \tag{14}$$

In similar way, the stochastic version of System (9) is in the following form

$$\begin{cases} du = (-\frac{(R-q)v(Kaq-Rac+acq-Ru+qu)}{KR+(R-q)v})dt + \sigma_1 u dW_1(t), \\ dv = (-\frac{(Kaq^2+R^2ac-2Racq+acq^2)v+(Raq-aq^2)v^2+KRqu+(R^2-Rq)uv}{(KR+(R-q)v)a})dt + \sigma_2 v dW_2(t), \end{cases} \tag{15}$$

By considering Taylor expansions of deterministic parts of System (15) at origin, we have the following system

$$\begin{cases} du = & \big(\frac{-v(R-q)(Kaq-Rac+acq)}{RK} - \frac{(R-q)(-R+q)vu}{RK} + \frac{(R-q)^2a(Kq-Rc+cq)v^2}{R^2K^2} \\ & -\frac{(R-q)^3uv^2}{R^2K^2} - \frac{(R-q)^3(Kaq-Rac+acq)v^3}{R^3K^3} + o(4)\big)dt + \sigma_1 u dW_1(t), \\ dv = & \big(\frac{-qu}{a} - \frac{(Kaq^2+R^2ac-2Racq+acq^2)v}{RKa} - \frac{(R^2-Rq-q(R-q))vu}{RKa} \\ & -\frac{(Raq-aq^2-\frac{(Kaq^2+R^2ac-2Racq+acq^2)(R-q))v^2}{KR}}{KRa} + \frac{(KR^3-2KR^2q+KRq^2)(R-q)uv^2}{R^3K^3a} \\ & +\frac{(KR^2aq-2KRaq^2+Kaq^3-R^3ac+3R^2acq-3Racq^2+acq^3)(R-q)v^3}{R^3K^3a} + o(4)\big)dt + \sigma_2 v dW_2(t). \end{cases}$$

Hence, we obtain the following system

$$\begin{cases} dr = [(\varphi_1 + \frac{1}{16}\varphi_2)r + \frac{1}{8}\varphi_3 r^3]dt + (\frac{\varphi_4}{8}r^2)^{\frac{1}{2}} dW_r(t), \\ d\theta = [\frac{1}{4}\varphi_5]dt + (\frac{\varphi_2}{8})^{\frac{1}{2}} dW_\theta(t), \end{cases} \tag{16}$$

where

$$\begin{aligned} \varphi_1 &= -\frac{1}{2}\frac{Kaq^2+R^2ac-2Racq+acq^2}{RKa}, \\ \varphi_2 &= \sigma_1^2 + \sigma_2^2, \\ \varphi_3 &= \frac{(KR^2aq-2KRaq^2+Kaq^3-R^3ac+3R^2acq-3Racq^2+acq^3)(R-q)}{(R^3K^3a)} - \frac{(R-q)^3}{R^2K^2}, \\ \varphi_4 &= 3\sigma_1^2 + \sigma_2^2, \\ \varphi_5 &= \frac{2(R-q)(Kaq-Rac+acq)}{RK} - \frac{2q}{a}, \\ \varphi_6 &= \frac{(R-q)^3(Kaq-Rac+acq)}{R^3K^3} + \frac{(KR^3-2KR^2q+KRq^2)(R-q)}{R^3K^3a}. \end{aligned}$$

In order to investigate the stability and bifurcation of System (15), we consider the following averaging modulus equation

$$\begin{aligned} dr = {} & [((\frac{-1}{2})\frac{Kaq^2+R^2ac-2Racq+acq^2}{RKa} + \frac{\sigma_1^2+\sigma_2^2}{16})r + (-\frac{(R-q)^3}{R^2K^2} \\ & + \frac{(KR^2aq-2KRaq^2+Kaq^3-R^3ac+3R^2acq-3Racq^2+acq^3)(R-q)}{R^3K^3a})\frac{r^3}{8}]dt \\ & + [(\frac{\sigma_1^2+\sigma_2^2}{8})r^2]^{\frac{1}{2}} dW_r(t). \end{aligned} \tag{17}$$

## 3 The dynamic behavior of the stochastic system

In this section, we focus on System (10) and investigate its stability and stochastic bifurcation.

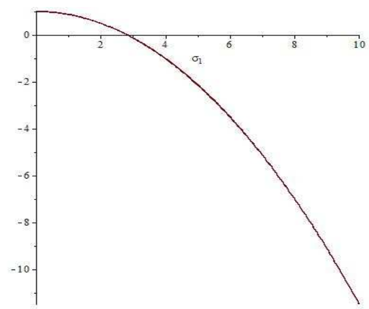


Figure 1: Largest Lyapunov exponent of System (10) where $q = c = K = 1, R = 8$ and $0 \leq \sigma_1 \leq 10$.

### 3.1 Largest Lyapunov exponent and stability

Let $\lambda$ be the largest Lyapunov exponent of System (3). Oseledec multiplicative ergodic theorem [1] shows that $\lambda < 0$ implies the asymptotically stability of the trivial solution of linearized equation and $\lambda > 0$ implies that our stochastic system is unstable at the equilibrium $(0,0)$. In Theorem 3.1 of [20], the authors prove that

$$\lambda = \lim_{t \to +\infty} \frac{1}{t} \ln \|r(t)\| = \varphi_1 + \frac{1}{16}\varphi_2 - \frac{1}{16}\varphi_4,$$

where $r(t)$ is solution of Eq. (13). Then we have the following theorem

**Theorem 3.** *$(i)$ If $8[-q + (\frac{1}{2})\frac{Rc}{(K+c)}] < \sigma_1^2$, the trivial solution of the linear Ito stochastic differential equation (14) is asymptotically stable with probability 1, then the stochastic System (10) is stable at the equilibrium point $O$.*
*$(ii)$ If $8[-q + (\frac{1}{2})\frac{Rc}{(K+c)}] > \sigma_1^2$, the trivial solution of the linear first Ito stochastic differential equation (14) is unstable with probability 1, which implies that the stochastic system (10) is unstable at the equilibrium point $O$.*

In Figure 1 we plot the largest Lyapunov exponent of System (10), where $\sigma_1$ is variable and $q = c = K = 1, R = 8$. Then for every $\sigma_1 > 2\sqrt{2}$, the largest Lyapunov exponent is negative. Due to Theorem 3 the stochastic system (10) is stable at the equilibrium point $O$. In Figure 2 we plot the largest Lyapunov exponent where $q$ and $\sigma_1$ are variable.

**Remark 1.** *Because $\varphi_3 = -\dfrac{RK}{(K+c)^3}$ and $\varphi_4 = 3\sigma_1^2 + \sigma_2^2$, Theorem 1 implies that if $\dfrac{4Rc}{K+c} < \sigma_1^2 + 8q$, then the stochastic system (10) is globally stable at the equilibrium point $O$.*

### 3.2 Stochastic bifurcation

In this section, by using concepts and theorems in Section 1, we investigate dynamical bifurcation and phenomena bifurcation of System (10). The definitions of $D$ and $P$ bifurcations are presented.

**Definition 1** ([11])**.** *(**D-bifurcation**) Dynamical bifurcation is concerned with a family of random dynamical systems which is differential and has the invariant measure $\varphi_\theta$. If there exists a constant $\theta_0$*

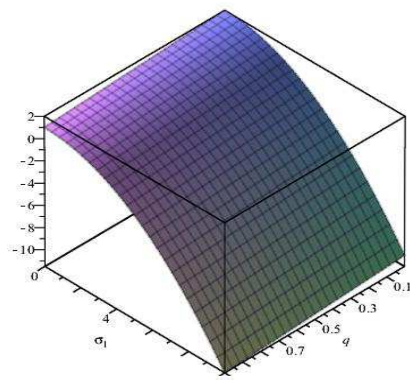


Figure 2: Largest Lyapunov exponent of System (10) where $c = K = 1, R = 8$ and $0 \le \sigma_1 \le 10$, $0.1 < q < 1$

*satisfying in any neighbourhood of $\theta_0$, there exists another constant $\theta$ and the corresponding invariant measure $\nu_\theta \neq \varphi_\theta$ satisfying $\nu_\theta \to \varphi_\theta$ as $\theta \to \theta_0$. Then, the constant $\theta_0$ is a point of dynamical bifurcation.*

Base on Theorem 4.1 of [20] and Section 3 of [11], when $\varphi_4 = 16\varphi_1 + \varphi_2$, i.e., $Rc = \frac{1}{4}(K+c)\sigma_1^2 + 8q$, the stochastic system (14) undergoes a $D-$bifurcation.

The stochastic $P$-bifurcation is a type of stochastic bifurcation that occurs in a stochastic system. This bifurcation describes the mode of the stationary probability density function or the invariant measure of the stochastic process. Stochastic systems undergoes the stochastic $P$-bifurcation when the mode of the stationary probability density function changes in nature. It indicates the jump of the distribution of the random variable in probability sense. There is no direct relation between $D$-bifurcation and $P$-bifurcation [27]. To investigate the $P$-bifurcation of stochastic System (10) and its polar coordinate transformation (14), we use probability density functions.

According to Section 4 of [20], the stationary probability density function $p(r)$ of random variable $r$ can be given by

$$P(r) = \begin{cases} \delta(r), & \text{when } Rc \le \frac{1}{4}(\sigma_1^2 + 8q)(K+c), \\ \dfrac{r^{\frac{-2(K\sigma_1^2 + c\sigma_1^2 + 8Kq - 4Rc + 8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}} \exp\left(\frac{-RK}{(3\sigma_1^2+\sigma_2^2)(K+c)^3} r^2\right)}{\Gamma\left(\frac{-(K\sigma_1^2 + c\sigma_1^2 + 8Kq - 4Rc + 8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}\right)\left(\frac{(3\sigma_1^2+\sigma_2^2)(K+c)^3}{RK}\right)^{\frac{-(K\sigma_1^2 + c\sigma_1^2 + 8Kq - 4Rc + 8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}}}, & \text{when } Rc > \frac{1}{4}(\sigma_1^2 + 8q)(K+c), \end{cases} \tag{18}$$

This is clear that the extreme value point of $p(r)$ is $r_0 = 0$ or

$$r_1 = \sqrt{\frac{-[5\sigma_1^2 + \sigma_2^2 + 16q - \frac{8Rc}{K+c}](K+c)^3}{2RK}},$$

when $\frac{5\sigma_1^2+\sigma_2^2}{2} < -8q + \frac{4RC}{K+C}$. Consequently, we have the following three type of conditions

(i) If $-8q+\frac{4Rc}{K+c} < \frac{1}{2}(5\sigma_1^2+\sigma_2^2) < -16q+\frac{8Rc}{K+c}+\dfrac{\sigma_1^2+\sigma_2^2}{2}$, then $\lim_{r\to 0^+} P(r)=\infty$ and the random trajectories of System (14) centralized in a neighborhood of the point $r_0=0$.

(ii) If $\frac{1}{3}(-16q+\frac{8Rc}{K+c}) < \frac{1}{3}(8\sigma_1^2+2\sigma_2^2) < -8q+\dfrac{4Rc}{K+c}+\frac{\sigma_1^2+\sigma_2^2}{6}$, then $P(r)$ has the minimum value at the point $r_0$ and the maximum value at the point $r_1$, but the derivative of $P(r)$ at $r_0$ does not exist. Moreover, the random trajectories of System (14) centralized in a neighborhood of the point $r_1$.

(iii) If $8\sigma_1^2+2\sigma_2^2 < -16q+\frac{8Rc}{K+c}$, then $P(r)$ has the minimum value at the point $r_0$ and the maximum value at the point $r_1$. In this case, the probability density function $P(r)$ becomes a smooth function at the point $r_1$.

We can summarize these results to the following theorem.

**Theorem 4.** *(I) System* (14) *undergoes stochastic phenomenological bifurcations as the parameter $q$ passes through the values of*

$$\frac{Rc}{2(K+c)}-\frac{5\sigma_1^2+\sigma_2^2}{16}, \quad \textit{and} \quad \frac{Rc}{2(K+c)}-\frac{4\sigma_1^2+\sigma_2^2}{2}.$$

*(II) System* (14) *undergoes stochastic phenomenological bifurcations as the parameter $R$ passes through the values of*

$$(\frac{K+c}{4c})(8q+\frac{5\sigma_1^2+\sigma_2^2}{2}) \quad \textit{and} \quad \frac{(K+c)^3(16q-\sigma_1^2-\sigma_2^2)}{K[3\sigma_1^2+\sigma_2^2+\frac{8Kc(K+c)^3}{K+c}]}.$$

**Remark 2.** *It is note that when the parameters $q$ and $R$ passes through the value of $Rc/(2(K+c)-\frac{\sigma_1^2}{8})$ and $((K+c)(2\sigma_1^2+16q))/8c$, respectively, the probability density function $P(r)$ varies from Dirac function $\delta(r)$ to the other function in* (18)*, which means that System* (14) *undergoes a P-bifurcation in a generalized sense.*

**Example 1.** As an example, we take $\sigma_1=\sigma_2=\frac{1}{2}$, $R=K=c=1$. By varying parameter $q$, we can see qualitative changes of density function $P(r)$. Simple calculation implies that

(i) If $-8q+2<\frac{3}{4}<-16q+\frac{17}{4}$, then $\lim_{r\to 0^+} P(r)=\infty$.

(ii) If $\frac{-16q+4}{3}<\frac{5}{6}<-8q+\frac{23}{12}$, then $P(r)$ has the minimum value at the point $r_0=0$ and the maximum value at the point $r_1$, but the derivative of $P(r)$ at $r_0$ does not exist.

(iii) If $\frac{5}{2}<-16q+4$, then $P(r)$ has the minimum value at the point $r_0=0$ and the maximum value at the point $r_1$. (see Figure 3 from left to right respectively).

**Example 2.** In this example, we plot the effect of noise on probability density $P(r)$ of System (14) by fixed parameters $q=\frac{1}{5}, R=K=c=1$ and it is considered $\sigma_1=\sigma_2$. (see Figure 4)

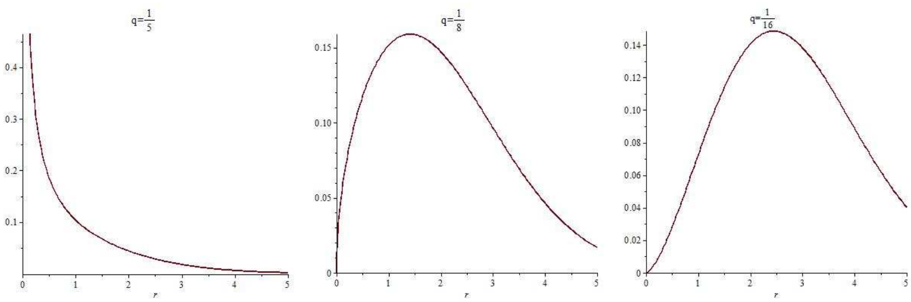


Figure 3: Variations of probability density $P(r)$ of System (14) by changing $q$ for parameters $\sigma_1 = \sigma_2 = \frac{1}{2}, R = K = c = 1$

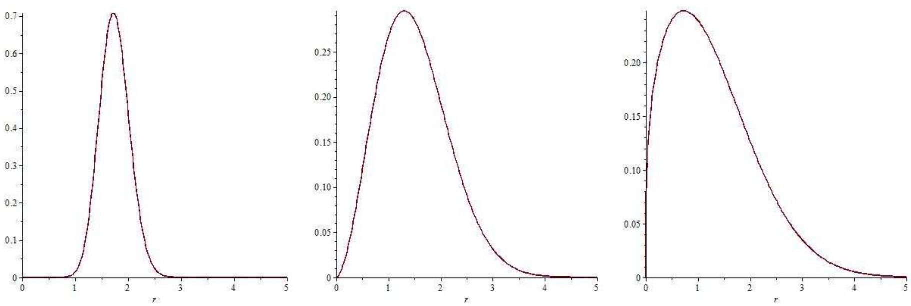

Figure 4: The effect of noise on probability density $P(r)$ of System (14) for $q = \frac{1}{5}, R = K = c = 1$ and $\sigma_1 = \frac{1}{10}, \frac{1}{4}, \frac{1}{3}$ from left to right.

By transforming $P(r)$ to the probability density $\rho(u,v)$ of the stationary distribution in terms of Cartesian coordinates $u$ and $v$ (for more details see [20]), we have

$$P(u,v) = \begin{cases} \delta(\sqrt{u^2+v^2}), & \text{when } Rc \leq \frac{1}{4}(\sigma_1^2+8q)(K+c), \\ \dfrac{(u^2+v^2)^{\frac{-2(K\sigma_1^2+c\sigma_1^2+8Kq-4Rc+8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}} \exp\left(\frac{-RK}{(3\sigma_1^2+\sigma_2^2)(K+c)^3}(u^2+v^2)\right)}{\pi\Gamma\left(\frac{-(K\sigma_1^2+c\sigma_1^2+8Kq-4Rc+8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}\right)\left(\frac{(3\sigma_1^2+\sigma_2^2)(K+c)^3}{RK}\right)^{\frac{-(K\sigma_1^2+c\sigma_1^2+8Kq-4Rc+8cq)}{(K+c)(3\sigma_1^2+\sigma_2^2)}}}, & \text{when } Rc > \frac{1}{4}(\sigma_1^2+8q)(K+c). \end{cases}$$

Similar to the above argument for $P(r)$, the extremal value point of $\rho(u,v)$ may be obtained. In this way we need to calculate the gradient of $\rho(u,v)$ in $\mathbb{R}^2$. Hence, we reach the following results

(i) If
$$2\sigma_1^2 + \frac{1}{2}\sigma_2^2 \geq -4q + \frac{2Rc}{K+c},$$
then $\rho(u,v)$ tends to infinite as $u \to 0$ and $v \to 0$.

(ii) If
$$-4q + \frac{2Rc}{K+c} + \frac{\sigma_1^2+\sigma_2^2}{4} < 3\sigma_1^2 + \sigma_2^2 < \frac{-16}{3}q + \frac{8Rc}{3(K+c)} + \frac{\sigma_1^2+\sigma_2^2}{3},$$

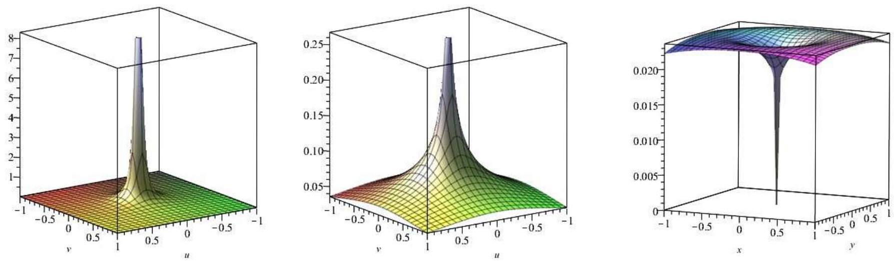

Figure 5: Variations of probability density $P(u,v)$ of System (10) by changing $q$ for fixed parameters $\sigma_1 = \sigma_2 = \frac{1}{2}, R = K = c = 1$ and $q = \frac{1}{5}, \frac{1}{8}, \frac{1}{16}$ from left to right.

then $\rho(u,v)$ has a minimum value point at the origin, but its partial derivatives at the origin is not continuous. Moreover, It has a maximum value at the points of the stable limit cycle

$$u^2+v^2 = \frac{8\sigma_1^2+2\sigma_2^2+16q-\frac{8Rc}{K+c}}{2(\sigma_1^2+\sigma_2^2)}.$$

(iii) If

$$\frac{11}{4}\sigma_1^2+\frac{3}{4}\sigma_2^2 < -4q+\frac{2Rc}{K+c},$$

then $\rho(u,v)$ has a minimum value point at the origin, and a maximum value at the points of the stable limit cycle

$$u^2+v^2 = \frac{8\sigma_1^2+2\sigma_2^2+16q-\frac{8Rc}{K+c}}{2(\sigma_1^2+\sigma_2^2)}.$$

Moreover, $\rho(u,v)$ has continuous partial derivatives.

We can summarize these results to the following theorem.

**Theorem 5.** *The stochastic system* (10) *undergoes phenomenological bifurcations as the parameter $q$ passes through the values of*

$$\frac{Rc}{2(K+c)} - \frac{4\sigma_1^2+\sigma_2^2}{8} \quad \textit{and} \quad -11\sigma_1^2-3\sigma_2^2+\frac{Rc}{2(K+c)}.$$

**Example 3.** Similar to Example 1, we take $\sigma_1 = \sigma_2 = \frac{1}{2}$, $R = K = c = 1$. By varying parameter $q$, we can see qualitative changes of density function $P(u,v)$, (see Figure 5).

**Example 4.** To see the effect of noise on probability density $P(u,v)$ of System (10) we choose parameters and noise the same as Example 2 (see Figure 6).

**Remark 3.** *The line graphs and surface graphs in Figure 3 and Figure 5 represent the changes of probability density $P(r)$ and $P(u,v)$ of System* (14) *respectively for parameters $\sigma_1 = \sigma_2 = \frac{1}{2}$, $R = K = c = 1$ by varying parameter $q$. For instance, the probability density $P(r)$ in the left line graph has its*

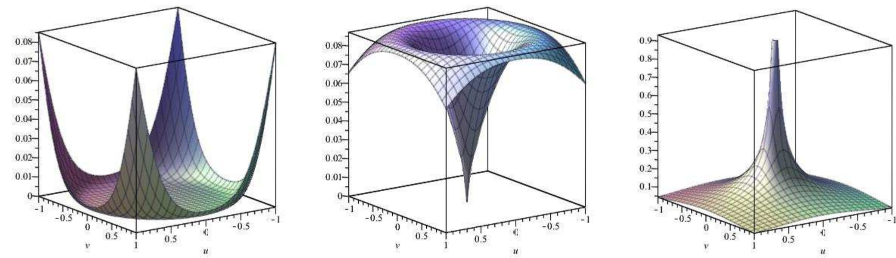

Figure 6: Variations of probability density $P(u,v)$ of System (10) for $q=\frac{1}{5}, R=K=c=1$ and $\sigma_1=\frac{1}{10},\frac{1}{4},\frac{1}{3}$ from left to right.

*maximum when r is near the origin. On the other hand, this density of probability is completely visible in the left surface graph in Figure 5 near origin. For middle and right graphs the Figures 3 and 5 depict the same probability density as well. Also by considering $q=\frac{1}{4}, R=K=c=1$ and changing $\sigma_1=\frac{1}{10},\frac{1}{4},\frac{1}{3}$, the effect of noise on probability density $P(r)$ and $P(u,v)$ of System (10) is shown in Figures 4 and 6 respectively. As we can see from left graphs in both Figure 4 and 6 the probability density $P(r)$ and $P(u,v)$ has its minimum near the origin and for others, the probability density $P(r)$ and $P(u,v)$ behave similarly.*

### 3.3 Numerical simulation of the stochastic chemostat model

In order to confirm the analytical results, we numerically simulate the solution of stochastic chemostat model.

We apply the Euler-Maruyama method to System (10) and obtain discrete system [8]

$$\begin{cases} u(i+1)=u(i)+[-qu(i)+\dfrac{Ru(i)(v(i)+c)}{K+v(i)+c}]\Delta t+\sigma_1 u(i)\sqrt{\Delta t}N(0,1); \\ v(i+1)=v(i)+[-qv(i)-\dfrac{Ru(i)(v(i)+c)}{a(K+v(i)+c)}]\Delta t+\sigma_2 v(i)\sqrt{\Delta t}N(0,1); \end{cases} \tag{19}$$

where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance. According to Theorem 3, if $q>\frac{1}{2}\frac{Rc}{K+c}-\frac{1}{8}\sigma_1^2$, then the origin is stable and if $q<\frac{1}{2}\frac{Rc}{K+c}-\frac{1}{8}\sigma_1^2$, it is unstable. Here, we choose $a=c=K=R=1, \sigma_1=\sigma_2=\dfrac{1}{2}$. So, if $q>\frac{7}{32}$, the origin is stable.

In Figure 7, we plot the phase portrait of System (10). This figure shows that for $q=\frac{1}{4}$ and initial condition $(u_0,v_0)=(0.1,0.1)$ the random trajectory goes to the origin. For $q=\frac{1}{32}$ origin is unstable because the Lyapunov exponent is positive. This fact, verifies the Theorem 3.

In Figure 8, we investigate the effect of the noise in System (10) with the fixed parameters $q=\frac{1}{4}, c=R=K=a=1$ and initial condition $(u_0,v_0)=(0.1,0.1)$. This figure shows that if the intensity of the noise be increased, then the trajectories tend to chaotic behavior. If Figure 9, we plot the time series of System (10) for fixed parameters $q=\frac{1}{4}, c=R=K=a=1$ and initial condition $(u_0,v_0)=(0.1,0.1)$. We consider $\sigma_1=\sigma_2=0,0.1,0.25$ and $0.33$ in figures $(a),(b),(c)$ and $(d)$, respectively.

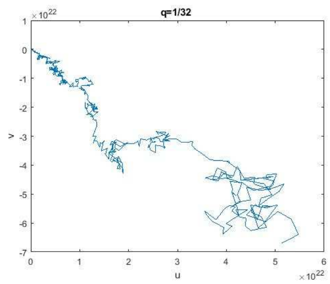


Figure 7: Phase portrait of System (10) for $a = c = K = R = 1, \sigma_1 = \sigma_2 = \frac{1}{2}$ with initial condition $(u_0, v_0) = (0.1, 0.1)$.

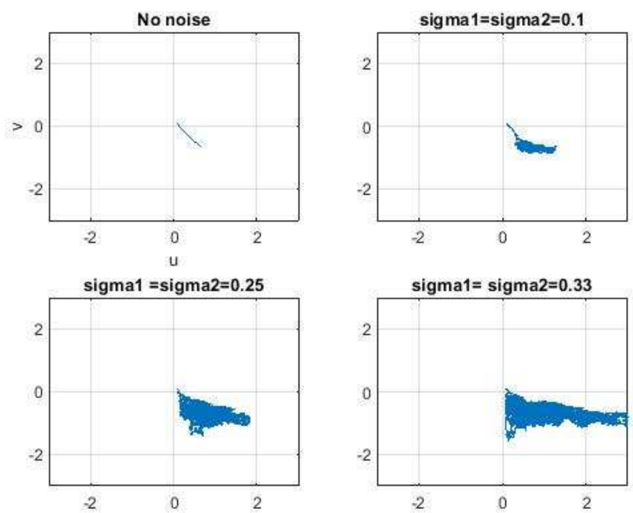


Figure 8: Phase portrait for System (10) for $q = \frac{1}{4}, c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$.

**Remark 4.** *By comparing Figure 8 with Figure 9, it is clear when there are no noise System* (10) *behave almost linear. When the noise increases more and more the behavior of System* (10) *is not predictable.*

Also, we plot 20 trajectories of System (10) in Figure 10 for $q = \frac{1}{4}$, $c = R = K = 1$ and $\sigma_1 = \sigma_2 = 0.1$.

## 4 The dynamic behavior of the chemostat stochastic system

In this section we consider System (15) and investigate its stability and stochastic bifurcation by similar procedures in Section 3. The following theorem determines the stability of System (15) at the equilibrium point $O$.

**Theorem 6.** *Suppose $a > 0$. The trivial solution of the linear Ito stochastic differential Eq.* (17)*, is*

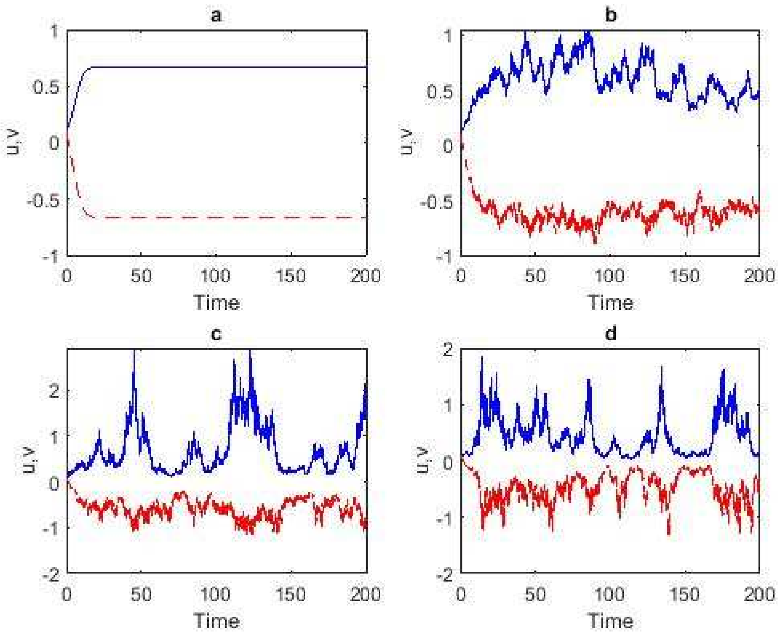


Figure 9: Time series of System (10) for $q = \frac{1}{4}$, $c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$.

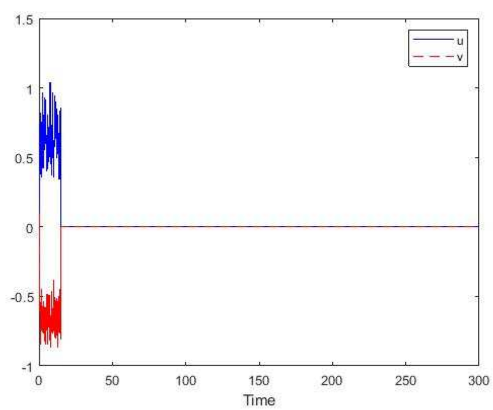


Figure 10: 20 trajectories of System (10) for $q = \frac{1}{4}$, $c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$ which shows the random trajectory goes to the origin that confirms the stability of the origin.

*asymptotically stable with probability* 1*, for all parameters* $K, R, q, c$ *and* $a$*. Then the stochastic System* (15) *is stable at the equilibrium point O.*

*Proof.* The largest Lyapunov exponent equal to

$$\lambda = -\frac{1}{2}[\frac{(Kaq^2 + R^2ac - 2Racq + acq^2)}{RKa}] - \frac{1}{8}\sigma_1^2.$$

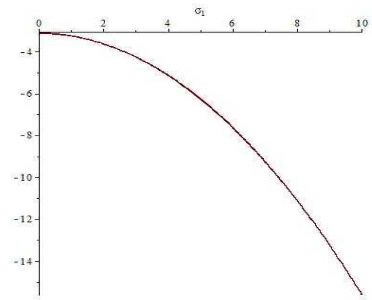


Figure 11: Largest Lyapunov exponent of System (15) where $a = q = c = K = 1, R = 8$ and $0 \leq \sigma_1 \leq 10$. For every $\sigma_1$ the Largest Lyapunov exponent is negative and then the stochastic System (15) is stable at the equilibrium point $O$. The larger $\sigma_1$ gets, the more the Largest Lyapunov exponent is negative.

We assume $R \neq q$. So

$$\begin{aligned}(R-q)^2 > 0 \;&\Rightarrow\; R^2+q^2 > 2Rq \\ &\Rightarrow\; c(R^2+q^2) > 2Rqc \\ &\Rightarrow\; Kq^2 + c(R^2+q^2) > 2Rqc \\ &\Rightarrow\; aKq^2 + R^2ca + q^2ca > 2Rqca \\ &\Rightarrow\; aKq^2 + R^2ca + q^2ca - 2Rqca > 0 \\ &\Rightarrow\; \lambda < 0,\end{aligned}$$

for all parameters $K, R, q, c$ and $a > 0$. If $R = q$ the proof is clear. □

In Figure 11, we plot largest Lyapunov exponent of System (15), where $\sigma_1$ is variable and $a = q = c = K = 1, R = 8$. In Figure 12, we plot largest Lyapunov exponent where $q$ and $\sigma_1$ are variable. As it is seen in Figure 13 the random trajectory goes to the origin.

**Remark 5.** *Assume $a > 0$. Due to Theorem 1 for global stability of System (15) at the equilibrium O, the following conditions must be satisfied*
*(i)* $\frac{-4(Kaq^2+R^2ac-2Racq+acq^2)}{KRa} - \sigma_1^2 < 0$, *and*
*(ii)* $\frac{(KR^2aq-2KRaq^2+Kaq^3-R^3ac+3R^2acq-3Racq^2+acq^3)(R-q)}{R^3K^3a} < \frac{3\sigma_1^2+\sigma_2^2}{2} + \frac{(R-q)^3}{R^2K^2}$.
*The condition $(i)$ is satisfied for all parameters $K, R, q, c$ and $a$.*

## 4.1 Stochastic bifurcation

In this subsection, we investigate some conditions which System (17) undergoes $D$-bifurcation and $P$-bifurcation. As a matter of fact for System (17) neither $D$-bifurcation nor $P$-bifurcation happens if $a > 0$. We mentioned in Section 3 when $\varphi_4 = 16\varphi_1 + \varphi_2$ System (17) possesses $D$-bifurcation. So by simple computation we obtain

$$-\frac{8(Kaq^2 + R^2ac - 2Racq + acq^2)}{RKa} = \sigma_1^2,$$

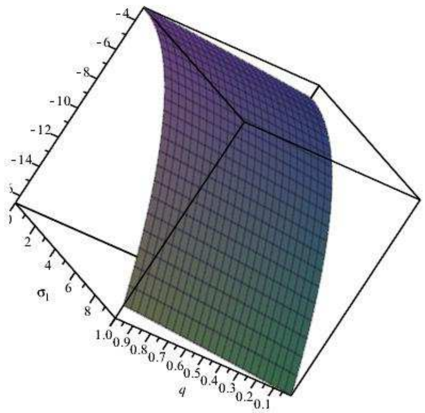


Figure 12: Largest Lyapunov exponent of System (15) where $a = c = K = 1, R = 8$ and $0 \leq \sigma_1 \leq 10$, $0.1 < q < 1$.

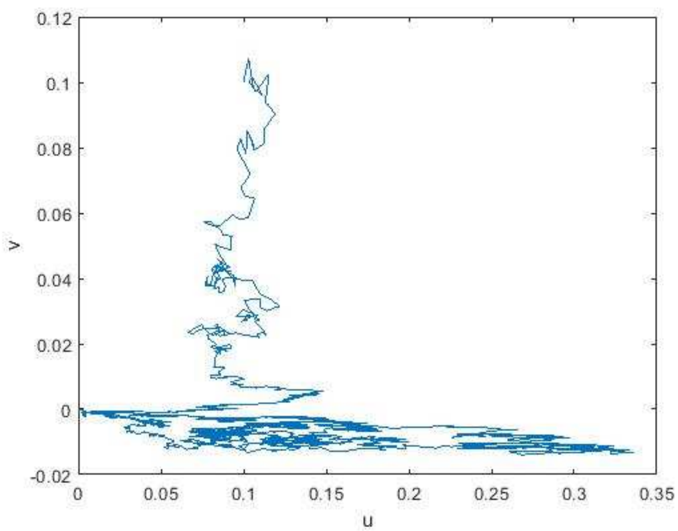


Figure 13: Phase portrait of System (15) for $q = \frac{1}{16}, a = c = K = R = 1, \sigma_1 = \sigma_2 = \frac{1}{2}$ with initial condition $(u_0, v_0) = (0.1, 0.1)$. For $0 < u < 0.35$ and $-0.1 < v < 0$ there is the most density of trajectories.

which is a contradiction. Because $Kaq^2 + R^2ac - 2Racq + acq^2 > 0$ for all parameters $K, R, q, c$ and $a > 0$. Hence $D$-bifurcation does not happen for System (17).

In order to investigate $P$-bifurcation, we should compute probability density function $p(r)$ of random variable $r$. But it is not possible, since the domain of Gamma function equal to

$$-\frac{4(Kaq^2 + R^2ac - 2Racq + acq^2) + RKa\sigma_1^2}{3\sigma_1^2 + \sigma_2^2},$$

which is negative for all parameters $K, R, q, c$ and $a > 0$.

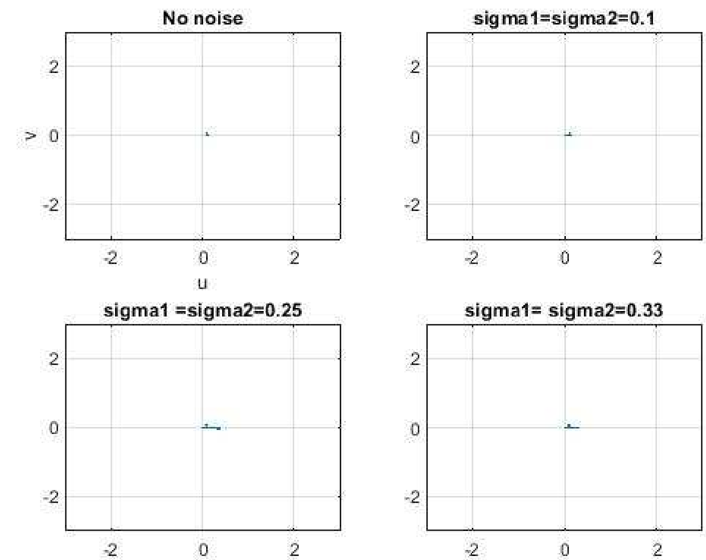


Figure 14: Phase portrait for System (15) for $q = \frac{1}{16}, c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$.

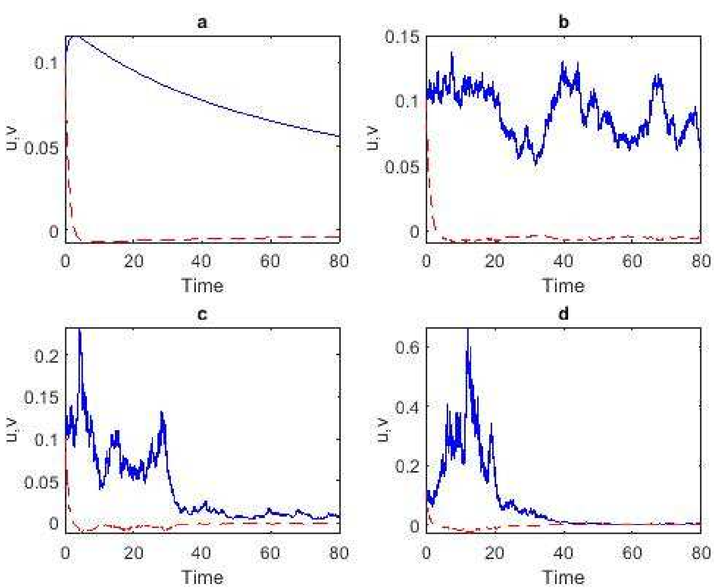


Figure 15: Time series of System (15) for $q = \frac{1}{16}, c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$.

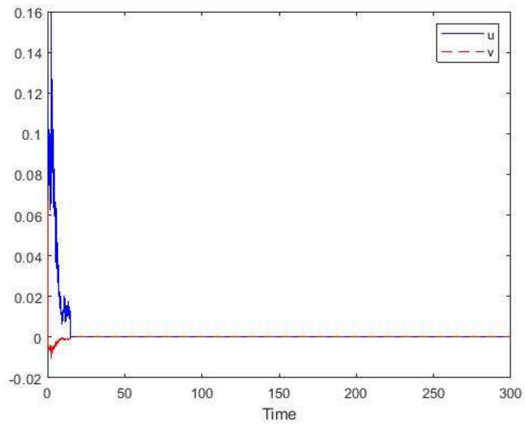


Figure 16: 20 trajectories of System (15) for $q = \frac{1}{16}, a = c = R = K = 1$ and $\sigma_1 = \sigma_2 = 0.1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$.

### 4.2 Numerical simulation of the stochastic chemostat model model

By applying the Euler-Maruyama method to System (15) the following discrete system is obtained.

$$\begin{cases} u(i+1) = & u(i) + [-\frac{(R-q)v(i)(Kaq-Rac+acq-(R+q))u(i)}{KR+(R-q)v(i)}]\Delta t + \sigma_1 u(i)\sqrt{\Delta t}N(0,1), \\ v(i+1) = & v(i) + [-\frac{(Kaq^2+R^2ac-2Racq+acq^2)v(i)+(Raq-aq^2)v(i)^2+KRqu(i)+(R^2-Rq)u(i)v(i)}{(KR+(R-q)v(i))a}]\Delta t \\ & +\sigma_2 v(i)\sqrt{\Delta t}N(0,1), \end{cases} \tag{20}$$

where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance. In Figure 13, we plot the phase portrait of System (15) for $a = c = K = R = 1, \sigma_1 = \sigma_2 = \frac{1}{2}$ and $q = \frac{1}{16}$. Figs. 14, 15 and 16 show the effect of the noise, time series and 20 trajectories of System (15) for fixed parameters $q = \frac{1}{16}$, $c = R = K = a = 1$ and initial condition $(u_0, v_0) = (0.1, 0.1)$ respectively. In Figure 15 we take $\sigma_1 = \sigma_2 = 0, 0.1, 0.25$ and $0.33$ in figures $(a), (b), (c)$ and $(d)$.

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