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ŁUKASIEWICZ FUZZY IDEALS IN BCK-ALGEBRAS AND BCI-ALGEBRAS

Y. B. JUN

Abstract. The notion of (closed) Łukasiewicz fuzzy ideal is introduced, and several properties are investigated. The relationship between Łukasiewicz fuzzy subalgebra and Łukasiewicz fuzzy ideal is discussed, and characterization of a Łukasiewicz fuzzy ideal is considered. Conditions for a Łukasiewicz fuzzy subalgebra to be a Łukasiewicz fuzzy ideal are provided, and conditions for the $\in$-set, q-set and O-set to be ideals are explored.

1. Introduction

A fuzzy concept, which is introduced by L. A. Zadeh [10], is understood as a concept which is "to an extent applicable" in a situation. That means the concept has gradations of significance or unsharp (variable) boundaries of application. Prior to the emergence of the fuzzy set, the very idea of inferring as an unclear concept faced considerable resistance from the elite in the academic world. They did not want to endorse the use of imprecise concepts in research or argumentation. Yet although people might not be aware of it, the use of fuzzy concepts has risen gigantically in all walks of life from the 1970s onward. That is mainly due to advances in electronic engineering, fuzzy mathematics and digital computer programming. The new technology allows very complex inferences about "variations on a theme" to be anticipated and fixed in a program. As is well known, fuzzy sets have contributed

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significantly to the development of pure and applied mathematics due to their extensive application capabilities. Łukasiewicz logic, which is the logic of the Łukasiewicz t-norm, is a non-classical and many-valued logic. Using the idea of Łukasiewicz t-norm, Jun [4] constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras. He defined the concepts of (strong) Łukasiewicz fuzzy subalgebras, and investigated several properties. He provided conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy subalgebra, and explored the conditions under which Łukasiewicz fuzzy subalgebra becomes strong. He disussed characterizations of Łukasiewicz fuzzy subalgebras. He constructed a three kind of subsets so called $\in$-set, q-set and O-set, and he found the conditions under which they can be subalgebras.

In this paper, we introduce the notion of (closed) Łukasiewicz fuzzy ideal in BCK/BCI-algebras and investigate several properties. We consider characterization of a Łukasiewicz fuzzy ideal. We discuss the relationship between Łukasiewicz fuzzy subalgebra and Łukasiewicz fuzzy ideal. We give a condition for a Łukasiewicz fuzzy subalgebra to be a Łukasiewicz fuzzy ideal. We provide conditions for the $\in$-set, q-set and O-set to be ideals.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [2] and [3]) and was extensively investigated by several researchers. We recall the definitions and basic results required in this paper. See the books [1, 6] for further information regarding BCK-algebras and BCI-algebras.

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I_1) $(\forall a, b, c \in X)\ (((a * b) * (a * c)) * (c * b) = 0)$,
(I_2) $(\forall a, b \in X)\ ((a * (a * b)) * b = 0)$,
(I_3) $(\forall a \in X)\ (a * a = 0)$,
(I_4) $(\forall a, b \in X)\ (a * b = 0,\ b * a = 0 \ \Rightarrow\ a = b)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(K) $(\forall a \in X)\ (0 * a = 0)$,

then X is called a *BCK-algebra*.

The order relation "$\leq$" in a BCK/BCI-algebra X is defined as follows:

$$(\forall a, b \in X)(a \leq b \ \Leftrightarrow\ a * b = 0). \tag{2.1}$$

A subset A of a BCK/BCI-algebra X is called

- a *subalgebra* of X (see [1, 6]) if it satisfies:

$$(\forall a, b \in A)(a * b \in A), \tag{2.2}$$

- an *ideal* of X (see [1, 6]) if it satisfies:

$$0 \in A, \tag{2.3}$$

$$(\forall a, b \in X)(a * b \in A,\, b \in A \;\Rightarrow\; a \in A). \tag{2.4}$$

A fuzzy set f in a set X of the form

$$f(b) := \begin{cases} t \in (0,1] & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $[a/t]$.

For a fuzzy set f in a set X, we say that a fuzzy point $[a/t]$ is

(i) *contained* in f, denoted by $[a/t] \in f$, (see [8]) if $f(a) \geq t$.
(ii) *quasi-coincident* with f, denoted by $[a/t]\, q\, f$, (see [8]) if $f(a) + t > 1$.

If $[a/t]\, \alpha\, f$ is not established for $\alpha \in \{\in, q\}$, it is denoted by $[a/t]\, \overline{\alpha}\, f$.
A fuzzy set f in a BCK/BCI-algebra X is called

- a *fuzzy subalgebra* of X (see [5]) if it satisfies:

$$(\forall a, b \in X)(f(a * b) \geq \min\{f(a), f(b)\}). \tag{2.5}$$

- a *fuzzy ideal* of X (see [5, 9]) if it satisfies:

$$(\forall a \in X)(f(0) \geq f(a)), \tag{2.6}$$

$$(\forall a, b \in X)(f(a) \geq \min\{f(a * b), f(b)\}). \tag{2.7}$$

Definition 2.1 ([4])**.** Let f be a fuzzy set in a set X and let $\varepsilon \in [0,1]$. A function

$$Ł_f^{\varepsilon} : X \to [0,1],\;\; x \mapsto \max\{0, f(x) + \varepsilon - 1\}$$

is called an *ε-Łukasiewicz fuzzy set* of f in X.

Let $Ł_f^{\varepsilon}$ be an ε-Łukasiewicz fuzzy set of a fuzzy set f in X. If $\varepsilon = 1$, then $Ł_f^{\varepsilon}(x) = \max\{0, f(x) + 1 - 1\} = \max\{0, f(x)\} = f(x)$ for all $x \in X$. This shows that if $\varepsilon = 1$, then the ε-Łukasiewicz fuzzy set of a fuzzy set f in X is the classisical fuzzy set f itself in X. If $\varepsilon = 0$, then $Ł_f^{\varepsilon}(x) = \max\{0, f(x) + 0 - 1\} = \max\{0, f(x) - 1\} = 0$ for all $x \in X$, that is, if $\varepsilon = 0$, then the ε-Łukasiewicz fuzzy set is the zero fuzzy set. Therefore, in handling the ε-Łukasiewicz fuzzy set, the value of ε can always be considered to be in $(0,1)$.

Let f be a fuzzy set in a set X and $\varepsilon \in (0,1)$. If $f(x)+\varepsilon \leq 1$ for all $x \in X$, then the ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is the 0-constant function, that is, $Ł_f^\varepsilon(x) = 0$ for all $x \in X$. Therefore, in order for the ε-Łukasiewicz fuzzy set to have a meaningful form, a fuzzy set f in X and $\varepsilon \in (0,1)$ must be set to satisfy the following condition:

$$(\exists x \in X)(f(x)+\varepsilon > 1). \tag{2.8}$$

Definition 2.2 ([4])**.** Let f be a fuzzy set in a BCK/BCI-algebra X and ε an element of $(0,1)$. Then its ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ in X is called an *ε-Łukasiewicz fuzzy subalgebra* of X if it satisfies:

$$[x/t_a] \in Ł_f^\varepsilon,\ [y/t_b] \in Ł_f^\varepsilon \ \Rightarrow [(x*y)/\min\{t_a,t_b\}] \in Ł_f^\varepsilon \tag{2.9}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Let f be a fuzzy set in X. For an ε-Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X and $t \in (0,1]$, consider the sets

$$(Ł_f^\varepsilon, t)_\in := \{x \in X \mid [x/t] \in Ł_f^\varepsilon\},$$

$$(Ł_f^\varepsilon, t)_q := \{x \in X \mid [x/t]\, q\, Ł_f^\varepsilon\},$$

which are called the *$\in$-set* and *q-set*, respectively, of $Ł_f^\varepsilon$ (with value t). Also, consider a set:

$$O(Ł_f^\varepsilon) := \{x \in X \mid Ł_f^\varepsilon(x) > 0\} \tag{2.10}$$

which is called an *O-set* of $Ł_f^\varepsilon$. It is observed that

$$O(Ł_f^\varepsilon) = \{x \in X \mid f(x)+\varepsilon-1 > 0\}.$$

3. Łukasiewicz fuzzy ideals

In what follows, let X be a BCK-algebra or a BCI-algebra, and ε is an element of $(0,1)$ unless otherwise specified. Also, the "ε-Łukasiewicz fuzzy set" is simply called the "Łukasiewicz fuzzy set" by omitting "ε".

Definition 3.1. Let f be a fuzzy set in X. Then its Łukasiewicz fuzzy set $Ł_f^\varepsilon$ in X is called a *Łukasiewicz fuzzy ideal* of X if it satisfies:

$$Ł_f^\varepsilon(0) \text{ is an upper bound of } \{Ł_f^\varepsilon(x) \mid x \in X\}, \tag{3.1}$$

$$[(x*y)/t_a] \in Ł_f^\varepsilon,\ [y/t_b] \in Ł_f^\varepsilon \ \Rightarrow [x/\min\{t_a,t_b\}] \in Ł_f^\varepsilon \tag{3.2}$$

for all $x, y \in X$ and $t_a, t_b \in (0,1]$.

Example 3.2. Let $X = \{0, a_1, a_2, a_3, a_4\}$ be a set with a binary operation "$*$" given by Table 1.

Table 1. Cayley table for the binary operation " $*$ "

$*$	0	a_1	a_2	a_3	a_4
0	0	0	0	0	0
a_1	a_1	0	0	a_1	a_1
a_2	a_2	a_1	0	a_2	a_2
a_3	a_3	a_3	a_3	0	a_3
a_4	a_4	a_4	a_4	a_4	0

Then X is a BCK-algebra (see [6]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x = 0, \\ 0.62 & \text{if } x \in \{a_1, a_2\}, \\ 0.49 & \text{if } x = a_3, \\ 0.72 & \text{if } x = a_4. \end{cases}$$

If we take $\varepsilon := 0.51$, then the Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is given as follows:

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.28 & \text{if } x = 0, \\ 0.13 & \text{if } x \in \{a_1, a_2\}, \\ 0 & \text{if } x = a_3, \\ 0.23 & \text{if } x = a_4, \end{cases}$$

and it is routine to verify that $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X.

Theorem 3.3. *Let f be a fuzzy set in X. Then its Łukasiewicz fuzzy set $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X if and only if it satisfies:*

$$(\forall x \in X)(\forall t_a \in (0,1])\left([x/t_a] \in Ł_f^\varepsilon \ \Rightarrow\ [0/t_a] \in Ł_f^\varepsilon\right), \tag{3.3}$$

$$(\forall x, y \in X)(Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\}). \tag{3.4}$$

Proof. Assume that $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X. Let $x \in X$ and $t_a \in (0,1]$ be such that $[x/t_a] \in Ł_f^\varepsilon$. Using (3.1) leads to $Ł_f^\varepsilon(0) \geq Ł_f^\varepsilon(x) \geq t_a$, and so $[0/t_a] \in Ł_f^\varepsilon$. Note that $[(x * y)/Ł_f^\varepsilon(x * y)] \in Ł_f^\varepsilon$ and $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$ for all $x, y \in X$. It follows from (3.2) that $[x/\min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\}] \in Ł_f^\varepsilon$, and hence $Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\}$ for all $x, y \in X$.

Conversely, suppose that $Ł_f^\varepsilon$ satisfies (3.3) and (3.4). Since $[x/Ł_f^\varepsilon(x)] \in Ł_f^\varepsilon$ for all $x \in X$, we have $[0/Ł_f^\varepsilon(x)] \in Ł_f^\varepsilon$ and so $Ł_f^\varepsilon(0) \geq Ł_f^\varepsilon(x)$ for all $x \in X$ by (3.3). Hence $Ł_f^\varepsilon(0)$ is an upper bound of $\{Ł_f^\varepsilon(x) \mid x \in X\}$. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[(x*y)/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Then $Ł_f^\varepsilon(x * y) \geq t_a$ and $Ł_f^\varepsilon(y) \geq t_b$, which imply from (3.4) that

$Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}$. Thus $[x/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$. Therefore $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X. □

Lemma 3.4. *Every Łukasiewicz fuzzy ideal $Ł_f^\varepsilon$ of X satisfies:*

$$(\forall x, y \in X)(\forall t_a \in (0,1])(x \leq y,\ [y/t_a] \in Ł_f^\varepsilon \ \Rightarrow\ [x/t_a] \in Ł_f^\varepsilon), \tag{3.5}$$

$$(\forall x, y, z \in X)(\forall t_b, t_c \in (0,1]) \left(\begin{array}{l} x * y \leq z,\ [y/t_b] \in Ł_f^\varepsilon,\ [z/t_c] \in Ł_f^\varepsilon \\ \Rightarrow\ [x/\min\{t_b, t_c\}] \in Ł_f^\varepsilon \end{array} \right). \tag{3.6}$$

Proof. Let $x, y \in X$ and $t_a \in (0,1]$ be such that $x \leq y$ and $[y/t_a] \in Ł_f^\varepsilon$. Then $x * y = 0$, and so

$$Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} = \min\{Ł_f^\varepsilon(0), Ł_f^\varepsilon(y)\} = Ł_f^\varepsilon(y) \geq t_a$$

Hence $[x/t_a] \in Ł_f^\varepsilon$ and therefore (3.5) is valid. Let $x, y, z \in X$ and $t_b, t_c \in (0,1]$ be such that $x * y \leq z$, $[y/t_b] \in Ł_f^\varepsilon$ and $[z/t_c] \in Ł_f^\varepsilon$. Then $(x * y) * z = 0$, $Ł_f^\varepsilon(y) \geq t_b$ and $Ł_f^\varepsilon(z) \geq t_c$. Hence

$$\begin{aligned} Ł_f^\varepsilon(x) &\geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} \\ &\geq \min\{\min\{Ł_f^\varepsilon((x * y) * z), Ł_f^\varepsilon(z)\}, Ł_f^\varepsilon(y)\} \\ &= \min\{\min\{Ł_f^\varepsilon(0), Ł_f^\varepsilon(z)\}, Ł_f^\varepsilon(y)\} \\ &= \min\{Ł_f^\varepsilon(z), Ł_f^\varepsilon(y)\} \\ &\geq \min\{t_b, t_c\}, \end{aligned}$$

and so $[x/\min\{t_b, t_c\}] \in Ł_f^\varepsilon$. Therefore (3.6) is valid. □

Proposition 3.5. *If $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X, then (3.5) and (3.6) are equivalent to the following two facts, respectively.*

$$(\forall x, y \in X)(x \leq y \ \Rightarrow\ Ł_f^\varepsilon(x) \geq Ł_f^\varepsilon(y)), \tag{3.7}$$

$$(\forall x, y, z \in X)(x * y \leq z \ \Rightarrow\ Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(y), Ł_f^\varepsilon(z)\}). \tag{3.8}$$

Proof. We first assume that (3.5) is valid and suppose that $x \leq y$ for all $x, y \in X$. Since $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$, it follows from (3.5) that $[x/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$. Hence $Ł_f^\varepsilon(x) \geq Ł_f^\varepsilon(y)$, and so (3.7) is valid. Suppose that (3.6) holds and let $x, y, z \in X$ be such that $x * y \leq z$. Note that $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$ and $[z/Ł_f^\varepsilon(z)] \in Ł_f^\varepsilon$. Thus $[x/\min\{Ł_f^\varepsilon(y), Ł_f^\varepsilon(z)\}] \in Ł_f^\varepsilon$ by (3.6), which implies that

$$Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(y), Ł_f^\varepsilon(z)\}.$$

Conversely, assume that (3.7) is valid. Let $x, y \in X$ and $t_a \in (0,1]$ be such that $x \leq y$ and $[y/t_a] \in Ł_f^\varepsilon$. Then $Ł_f^\varepsilon(x) \geq Ł_f^\varepsilon(y) \geq t_a$, and so $[x/t_a] \in Ł_f^\varepsilon$. Suppose that (3.8) is valid and let $x, y, z \in X$ and

$t_b, t_c \in (0,1]$ be such that $x * y \leq z$, $[y/t_b] \in Ł_f^\varepsilon$ and $[z/t_c] \in Ł_f^\varepsilon$. It follows from (3.8) that

$$Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(y), Ł_f^\varepsilon(z)\} \geq \min\{t_b, t_c\}.$$

Hence $[x/\min\{t_b, t_c\}] \in Ł_f^\varepsilon$. □

Proposition 3.6. *Let $Ł_f^\varepsilon$ be a Łukasiewicz fuzzy ideal of X and assume that $(\cdots((x * y_1) * y_2) * \cdots) * y_n = 0$ and $[y_i/t_{b_i}] \in Ł_f^\varepsilon$ for all $x, y_i \in X$ and $t_{b_i} \in (0,1]$ for $i = 1, 2, \cdots, n$. Then*

$$[x/\min\{t_{b_i} \mid i = 1, 2, \cdots, n\}] \in Ł_f^\varepsilon \tag{3.9}$$

Proof. It can be verified by the induction on n. □

We discuss the relationship between Łukasiewicz fuzzy subalgebra and Łukasiewicz fuzzy ideal.

Theorem 3.7. *In a BCK-algebra, every Łukasiewicz fuzzy ideal is a Łukasiewicz fuzzy subalgebra.*

Proof. Let $Ł_f^\varepsilon$ be a Łukasiewicz fuzzy ideal of a BCK-algebra X. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[x/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Since $x * y \leq x$, we have $[(x * y)/t_a] \in Ł_f^\varepsilon$ by (3.5). Hence $[x/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$ by (3.2), and so $[(x * y)/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$ by (3.5). Therefore $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy subalgebra of X. □

The following example shows that the converse of Theorem 3.7 may not be true.

Example 3.8. Let $X = \{0, a_1, a_2, a_3\}$ be a set with a binary operation "$*$" given by Table 2. Then X is a BCK-algebra (see [6]). Define a

TABLE 2. Cayley table for the binary operation "$*$"

$*$	0	a_1	a_2	a_3
0	0	0	0	0
a_1	a_1	0	0	a_1
a_2	a_2	a_1	0	a_2
a_3	a_3	a_3	a_3	0

fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.62 & \text{if } x = 0, \\ 0.54 & \text{if } x = a_1, \\ 0.41 & \text{if } x = a_2, \\ 0.48 & \text{if } x = a_3. \end{cases}$$

If we take $\varepsilon := 0.75$, then the Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is given as follows:

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.37 & \text{if } x = 0, \\ 0.29 & \text{if } x = a_1, \\ 0.16 & \text{if } x = a_2, \\ 0.23 & \text{if } x = a_3, \end{cases}$$

and it is routine to verify that $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy subalgebra of X for $\varepsilon := 0.75$. Since $Ł_f^\varepsilon(a_2) = 0.16 < 0.29 = \min\{Ł_f^\varepsilon(a_2 * a_1), Ł_f^\varepsilon(a_1)\}$, we know that $Ł_f^\varepsilon$ is not a Łukasiewicz fuzzy ideal of X for $\varepsilon := 0.75$ by Theorem 3.3.

In a BCI-algebra, Theorem 3.7 may not be true as shown in the following example.

Example 3.9. Let $(Y, *, 0)$ be a BCI-algebra and $(\mathbb{Z}, -, 0)$ the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers. Then $(X, \circ, (0,0))$ is a BCI-algebra (see [1]) where $X = Y \times \mathbb{Z}$ and $\circ$ is given as follows:

$$(\forall (x,a), (y,b) \in X)((x,a) \circ (y,b) = (x * y, a - b)).$$

Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = (0,0), \\ 0.7 & \text{if } x \in Y \times \mathbb{N}_0, \\ 0.5 & \text{if } x \in Y \times \{a \in \mathbb{Z} \mid a < 0\}, \\ 0.4 & \text{otherwise} \end{cases}$$

wher $\mathbb{N}_0$ is the set of all nonnegative integes. If we take $\varepsilon := 0.49$, then the Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of f in X is given as follows:

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.39 & \text{if } x = (0,0), \\ 0.19 & \text{if } x \in Y \times \mathbb{N}_0, \\ 0 & \text{if } x \in Y \times \{a \in \mathbb{Z} \mid a < 0\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to verify that $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X. But it is not a Łukasiewicz fuzzy subalgebra of X since

$$Ł_f^\varepsilon((0,2) \circ (0,5)) = Ł_f^\varepsilon((0,-3)) = 0 < 0.19 = \min\{Ł_f^\varepsilon((0,2)), Ł_f^\varepsilon((0,5))\}.$$

Definition 3.10. A Łukasiewicz fuzzy ideal $Ł_f^\varepsilon$ of a BCI-algebra X is said to be *closed* if it is also a Łukasiewicz fuzzy subalgebra of X.

Example 3.11. Let $Ł_f^\varepsilon$ be a Łukasiewicz fuzzy set of a fuzzy set f in a BCI-algebra X which is given as follows:

$$Ł_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.77 & \text{if } x \in \{a \in X \mid 0 \leq a\}, \\ 0.47 & \text{otherwise.} \end{cases}$$

It is routine to check that $Ł_f^\varepsilon$ is a closed Łukasiewicz fuzzy ideal of X.

We give a condition for a Łukasiewicz fuzzy subalgebra to be a Łukasiewicz fuzzy ideal.

Lemma 3.12 ([4])**.** *Every Łukasiewicz fuzzy subalgebra $Ł_f^\varepsilon$ of X satisfies the condition (3.1).*

Theorem 3.13. *If a Łukasiewicz fuzzy subalgebra $Ł_f^\varepsilon$ of a BCK-algebra X satisfies the condition (3.6), then it is a Łukasiewicz fuzzy ideal of X.*

Proof. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[(x*y)/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Since $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.6) that $[x/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$. By combining this and Lemma 3.12, $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X. □

Theorem 3.14. *If f is a fuzzy ideal of X, then its Łukasiewicz fuzzy set $Ł_f^\varepsilon$ in X is a Łukasiewicz fuzzy ideal of X.*

Proof. Let $Ł_f^\varepsilon$ be a Łukasiewicz fuzzy set of a fuzzy ideal f in X. Then

$$Ł_f^\varepsilon(0) = \max\{0, f(0)+\varepsilon-1\} \geq \max\{0, f(x)+\varepsilon-1\} = Ł_f^\varepsilon(x)$$

for all $x \in X$. Hence $Ł_f^\varepsilon(0)$ is an upper bound of $\{Ł_f^\varepsilon(x) \mid x \in X\}$. Let $x, y \in X$ and $t_a, t_b \in (0,1]$ be such that $[(x*y)/t_a] \in Ł_f^\varepsilon$ and $[y/t_b] \in Ł_f^\varepsilon$. Then $Ł_f^\varepsilon(x*y) \geq t_a$ and $Ł_f^\varepsilon(y) \geq t_b$, which imply that

$$\begin{aligned} Ł_f^\varepsilon(x) &= \max\{0, f(x)+\varepsilon-1\} \geq \max\{0, \min\{f(x*y), f(y)\}+\varepsilon-1\} \\ &= \max\{0, \min\{f(x*y)+\varepsilon-1, f(y)+\varepsilon-1\}\} \\ &= \min\{\max\{0, f(x*y)+\varepsilon-1\}, \max\{0, f(y)+\varepsilon-1\}\} \\ &= \min\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\} \geq \min\{t_a, t_b\}. \end{aligned}$$

Hence $[x/\min\{t_a, t_b\}] \in Ł_f^\varepsilon$, and therefore $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X. □

The converse of Theorem 3.14 may not be true as seen in the example below.

Example 3.15. Let $X = \{0, a_1, a_2, a_3, a_4\}$ be a set with a binary operation "$*$" given by Table 3.

Table 3. Cayley table for the binary operation "$*$"

$*$	0	a_1	a_2	a_3	a_4
0	0	0	0	a_3	a_3
a_1	a_1	0	a_1	a_4	a_3
a_2	a_2	a_2	0	a_3	a_3
a_3	a_3	a_3	a_3	0	0
a_4	a_4	a_3	a_4	a_1	0

Then X is a BCI-algebra (see [1]). Define a fuzzy set f in X as follows:

$$f : X \to [0,1],\ x \mapsto \begin{cases} 0.87 & \text{if } x = 0, \\ 0.43 & \text{if } x = a_1, \\ 0.79 & \text{if } x = a_2, \\ 0.66 & \text{if } x = a_3, \\ 0.52 & \text{if } x = a_4. \end{cases}$$

If we take $\varepsilon := 0.48$, then the Łukasiewicz fuzzy set Ł_f^ε of f in X is given as follows:

$$\text{Ł}_f^\varepsilon : X \to [0,1],\ x \mapsto \begin{cases} 0.35 & \text{if } x = 0, \\ 0 & \text{if } x = a_1, \\ 0.27 & \text{if } x = a_2, \\ 0.14 & \text{if } x = a_3, \\ 0 & \text{if } x = a_4. \end{cases}$$

and it is routine to verify that Ł_f^ε is a Łukasiewicz fuzzy ideal of X. But f is not a fuzzy ideal of X since $f(a_1) = 0.43 \ngeq 0.52 = \min\{f(a_1 * a_4), f(a_4)\}$.

Let f be a fuzzy set in X. For the Łukasiewicz fuzzy set Ł_f^ε of f in X and $t \in (0,1]$, consider the sets

$$(\text{Ł}_f^\varepsilon, t)_\in := \{x \in X \mid [x/t] \in \text{Ł}_f^\varepsilon\},$$

$$(\text{Ł}_f^\varepsilon, t)_q := \{x \in X \mid [x/t]\, q\, \text{Ł}_f^\varepsilon\},$$

which are called the $\in$-*set* and the q-*set*, respectively, of Ł_f^ε (with value t).

We explore the conditions under which the $\in$-set of Łukasiewicz fuzzy set can be an ideal.

Theorem 3.16. *Let Ł_f^ε be the Łukasiewicz fuzzy set of a fuzzy set f in X. Then the $\in$-set $(\text{Ł}_f^\varepsilon, t)_\in$ of Ł_f^ε with value $t \in (0.5, 1]$ is an ideal* *of X if and only if the following assertions are valid.*

$$(\forall x \in X)\left(Ł_f^\varepsilon(x) \leq \max\{Ł_f^\varepsilon(0), 0.5\}\right), \tag{3.10}$$

$$(\forall x, y \in X)\left(\min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} \leq \max\{Ł_f^\varepsilon(x), 0.5\}\right). \tag{3.11}$$

Proof. Assume that $(Ł_f^\varepsilon, t)_\in$ is an ideal of X for $t \in (0.5, 1]$. If

$$Ł_f^\varepsilon(a) > \max\{Ł_f^\varepsilon(0), 0.5\}$$

for some $a \in X$, then $Ł_f^\varepsilon(a) \in (0.5, 1]$ and $Ł_f^\varepsilon(a) > Ł_f^\varepsilon(0)$. If we take $t = Ł_f^\varepsilon(a)$, then $[a/t] \in Ł_f^\varepsilon$, that is, $a \in (Ł_f^\varepsilon, t)_\in$, and $0 \notin (Ł_f^\varepsilon, t)_\in$. This is a contradiction, and so $Ł_f^\varepsilon(x) \leq \max\{Ł_f^\varepsilon(0), 0.5\}$ for all $x \in X$. Now, suppose that the condition (3.11) is not valid. Then there exist $a, b \in X$ such that

$$\min\{Ł_f^\varepsilon(a * b), Ł_f^\varepsilon(b)\} > \max\{Ł_f^\varepsilon(a), 0.5\}.$$

If we take $s := \min\{Ł_f^\varepsilon(a * b), Ł_f^\varepsilon(b)\}$, then $s \in (0.5, 1]$ and $[(a * b)/s]$, $[b/s] \in (Ł_f^\varepsilon, s)_\in$, i.e., $a * b$, $b \in (Ł_f^\varepsilon, s)_\in$. Since $(Ł_f^\varepsilon, s)_\in$ is an ideal of X, we have $a \in (Ł_f^\varepsilon, s)_\in$. But $Ł_f^\varepsilon(a) < s$ implies $a \notin (Ł_f^\varepsilon, s)_\in$, a contradiction. Hence the condition (3.11) is valid.

Conversely, suppose that $Ł_f^\varepsilon$ satisfies (3.10) and (3.11). Let $t \in (0.5, 1]$. For every $x \in (Ł_f^\varepsilon, t)_\in$, we have

$$0.5 < t \leq Ł_f^\varepsilon(x) \leq \max\{Ł_f^\varepsilon(0), 0.5\}$$

by (3.10). Thus $0 \in (Ł_f^\varepsilon, t)_\in$. Let $x, y \in X$ be such that $x * y \in (Ł_f^\varepsilon, t)_\in$ and $y \in (Ł_f^\varepsilon, t)_\in$. Then $Ł_f^\varepsilon(x * y) \geq t$ and $Ł_f^\varepsilon(y) \geq t$, which imply from (3.11) that

$$0.5 < t \leq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} \leq \max\{Ł_f^\varepsilon(x), 0.5\}.$$

Hense $[x/t] \in Ł_f^\varepsilon$, i.e., $x \in (Ł_f^\varepsilon, t)_\in$. Therefore $(Ł_f^\varepsilon, t)_\in$ is an ideal of X for $t \in (0.5, 1]$. □

Theorem 3.17. *If the Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of a fuzzy set f in X is a Łukasiewicz fuzzy ideal of X, then the q-set $(Ł_f^\varepsilon, t)_q$ of $Ł_f^\varepsilon$ with value $t \in (0, 1]$ is an ideal of X.*

Proof. Assume that the Łukasiewicz fuzzy set $Ł_f^\varepsilon$ of a fuzzy set f in X is a Łukasiewicz fuzzy ideal of X and let $t \in (0, 1]$. If $0 \notin (Ł_f^\varepsilon, t)_q$, then $[0/t]\, \overline{q}\, Ł_f^\varepsilon$, that is, $Ł_f^\varepsilon(0) + t \leq 1$. Since $Ł_f^\varepsilon(0) \geq Ł_f^\varepsilon(x)$ for $x \in (Ł_f^\varepsilon, t)_q$, it follows that $Ł_f^\varepsilon(x) \leq Ł_f^\varepsilon(0) \leq 1 - t$. Hence $[x/t]\, \overline{q}\, Ł_f^\varepsilon$, and so $x \notin (Ł_f^\varepsilon, t)_q$. This is a contadiction, and thus $0 \in (Ł_f^\varepsilon, t)_q$. Let $x, y \in X$ be such that $x * y \in (Ł_f^\varepsilon, t)_q$ and $y \in (Ł_f^\varepsilon, t)_q$. Then $[(x * y)/t]\, q\, Ł_f^\varepsilon$ and $[y/t]\, q\, Ł_f^\varepsilon$, that is, $Ł_f^\varepsilon(x * y) > 1 - t$ and $Ł_f^\varepsilon(y) > 1 - t$. It follows from (3.4) that $Ł_f^\varepsilon(x) \geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} > 1 - t$. Thus $[x/t]\, q\, Ł_f^\varepsilon$ and so $x \in (Ł_f^\varepsilon, t)_q$. Therefore $(Ł_f^\varepsilon, t)_q$ is an ideal of X. □

Corollary 3.18. *Let* $Ł_f^\varepsilon$ *be the Lukasiewicz fuzzy set of a fuzzy set* f *in* X*. If* f *is a fuzzy ideal of* X*, then the q-set* $(Ł_f^\varepsilon, t)_q$ *of* $Ł_f^\varepsilon$ *with value* $t \in (0, 1]$ *is an ideal of* X*.*

Theorem 3.19. *Let* f *be a fuzzy set in* X*. For the Łukasiewicz fuzzy set* $Ł_f^\varepsilon$ *of* f *in* X*, if the q-set* $(Ł_f^\varepsilon, t)_q$ *of* $Ł_f^\varepsilon$ *is an ideal of* X*, then the following assertions are valid.*

$$0 \in (Ł_f^\varepsilon, t_a)_\in, \tag{3.12}$$

$$[(x * y)/t_a]\, q\, Ł_f^\varepsilon,\ [y/t_b]\, q\, Ł_f^\varepsilon \ \Rightarrow\ x \in (Ł_f^\varepsilon, \max\{t_a, t_b\})_\in \tag{3.13}$$

for all $x, y \in X$ *and* $t_a, t_b \in (0, 0.5]$*.*

Proof. Let $x, y \in X$ and $t_a, t_b \in (0, 0.5]$. If $0 \notin (Ł_f^\varepsilon, t_a)_\in$, then $[0/t_a]\, \overline{\in}\, Ł_f^\varepsilon$ and so $Ł_f^\varepsilon(0) < t_a \leq 1 - t_a$ since $t_a \leq 0.5$. Hence $[0/t_a]\, \overline{q}\, Ł_f^\varepsilon$ and thus $0 \notin (Ł_f^\varepsilon, t_a)_q$. This is a contradiction, and therefore $0 \in (Ł_f^\varepsilon, t_a)_\in$. Let $[(x * y)/t_a]\, q\, Ł_f^\varepsilon$ and $[y/t_b]\, q\, Ł_f^\varepsilon$. Then $x * y \in (Ł_f^\varepsilon, t_a)_q \subseteq (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$ and $y \in (Ł_f^\varepsilon, t_b)_q \subseteq (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$. Hence $x \in (Ł_f^\varepsilon, \max\{t_a, t_b\})_q$, and so

$$Ł_f^\varepsilon(x) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\},$$

that is, $[x/\max\{t_a, t_b\}] \in Ł_f^\varepsilon$. Therefore $x \in (Ł_f^\varepsilon, \max\{t_a, t_b\})_\in$. □

Theorem 3.20. *Given a fuzzy set* f *in* X*, let* $Ł_f^\varepsilon$ *be the Łukasiewicz fuzzy set of* f *in* X*. If* f *is a fuzzy ideal of* X*, then the O-set* $O(Ł_f^\varepsilon)$ *of* $Ł_f^\varepsilon$ *is an ideal of* X*.*

Proof. Assume that f is a fuzzy ideal of X. Then $Ł_f^\varepsilon$ is a Łukasiewicz fuzzy ideal of X by Theorem 3.14. It is clear that $0 \in O(Ł_f^\varepsilon)$. Let $x, y \in X$ be such that $x * y \in O(Ł_f^\varepsilon)$ and $y \in O(Ł_f^\varepsilon)$. Then $f(x * y) + \varepsilon - 1 > 0$ and $f(y) + \varepsilon - 1 > 0$. It follows from Theorem 3.3 that

$$\begin{aligned} Ł_f^\varepsilon(x) &\geq \min\{Ł_f^\varepsilon(x * y), Ł_f^\varepsilon(y)\} \\ &= \min\{f(x * y) + \varepsilon - 1, f(y) + \varepsilon - 1\} > 0. \end{aligned}$$

Hence $x \in O(Ł_f^\varepsilon)$, and therefore $O(Ł_f^\varepsilon)$ is an ideal of X. □

Theorem 3.21. *Let* $Ł_f^\varepsilon$ *be the Łukasiewicz fuzzy set of a fuzzy set* f *in* X*. If the image of* X *under* $Ł_f^\varepsilon$ *is positive and* $Ł_f^\varepsilon$ *satisfies:*

$$[(x * y)/t_a] \in Ł_f^\varepsilon,\ [y/t_b] \in Ł_f^\varepsilon \ \Rightarrow\ [x/\max\{t_a, t_b\}]\, q\, Ł_f^\varepsilon \tag{3.14}$$

for all $x, y \in X$ *and* $t_a, t_b \in (0, 1]$*, then the O-set* $O(Ł_f^\varepsilon)$ *of* $Ł_f^\varepsilon$ *is an ideal of* X*.*

Proof. Assume that $Ł_f^\varepsilon(x) > 0$ for all $x \in X$ and the condition (3.14) is valid for all $x, y \in X$ and $t_a, t_b \in (0, 1]$. It is clear that $0 \in O(Ł_f^\varepsilon)$. Let $x, y \in X$ be such that $x * y \in O(Ł_f^\varepsilon)$ and $y \in O(Ł_f^\varepsilon)$. Then

$f(x*y)+\varepsilon-1>0$ and $f(y)+\varepsilon-1>0$. Since $[(x*y)/Ł_f^\varepsilon(x*y)] \in Ł_f^\varepsilon$ and $[y/Ł_f^\varepsilon(y)] \in Ł_f^\varepsilon$, it follows from (3.14) that

$$[x/\max\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\}]\, q\, Ł_f^\varepsilon. \tag{3.15}$$

If $x \notin O(Ł_f^\varepsilon)$, then $Ł_f^\varepsilon(x) = 0$ and so

$$\begin{aligned} &Ł_f^\varepsilon(x) + \max\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\} = \max\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\} \\ &= \max\{\max\{0, f(x*y)+\varepsilon-1\}, \max\{0, f(y)+\varepsilon-1\}\} \\ &= \max\{f(x*y)+\varepsilon-1, f(y)+\varepsilon-1\} \\ &= \max\{f(x*y), f(y)\} + \varepsilon - 1 \\ &\le 1+\varepsilon-1 = \varepsilon \le 1, \end{aligned}$$

that is, $[x/\max\{Ł_f^\varepsilon(x*y), Ł_f^\varepsilon(y)\}]\, \overline{q}\, Ł_f^\varepsilon$. This is impossible, and thus $x \in O(Ł_f^\varepsilon)$. Therefore $O(Ł_f^\varepsilon)$ is an ideal of X. □

Theorem 3.22. *Let $Ł_f^\varepsilon$ be the Łukasiewicz fuzzy set of a fuzzy set f in X. If it satisfies $[0/\varepsilon]\, q\, f$ and the condition (3.13) for all $x, y \in X$ and $t_a, t_b \in (0,1]$, then the O-set $O(Ł_f^\varepsilon)$ of $Ł_f^\varepsilon$ is an ideal of X.*

Proof. It is obvious that $0 \in O(Ł_f^\varepsilon)$ by the condition $[0/\varepsilon]\, q\, f$. Let $x, y \in X$ be such that $x*y \in O(Ł_f^\varepsilon)$ and $y \in O(Ł_f^\varepsilon)$. Then $f(x*y)+\varepsilon-1>0$ and $f(y)+\varepsilon-1>0$. Hence

$$\begin{aligned} Ł_f^\varepsilon(x*y)+1 &= \max\{0, f(x*y)+\varepsilon-1\}+1 \\ &= f(x*y)+\varepsilon-1+1 \\ &= f(x*y)+\varepsilon > 1 \end{aligned}$$

and $Ł_f^\varepsilon(y)+1 = \max\{0, f(y)+\varepsilon-1\}+1 = f(y)+\varepsilon-1+1 = f(y)+\varepsilon > 1$, that is, $[(x*y)/1]\, q\, Ł_f^\varepsilon$ and $[y/1]\, q\, Ł_f^\varepsilon$. It follows from (3.13) that $x \in (Ł_f^\varepsilon, \max\{1,1\})_\in = (Ł_f^\varepsilon, 1)_\in$. Hence $x \in O(Ł_f^\varepsilon)$ because if not, then $f(x)+\varepsilon-1 \le 0$ and so $f(x) \le 1-\varepsilon < 1$, which is a contradiction. Therefore $O(Ł_f^\varepsilon)$ is an ideal of X. □

4. Conclusion

In mathematics and philosophy, Łukasiewicz logic is a non-classical, many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued modal logic. Triangular norm (abbreviated, t-norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. Łukasiewicz t-norm is an example of t-norms, and its name comes from the fact that the t-norm is the standard semantics for strong conjunction in Łukasiewicz fuzzy logic. Using the idea of Łukasiewicz t-norm, Jun [4] have constructed the concept of

Łukasiewicz fuzzy sets based on a given fuzzy set and have applid it to BCK-algebras and BCI-algebras. In this paper, we have introduced the notion of (closed) Łukasiewicz fuzzy ideal in BCK/BCI-algebras and have investigated several properties. We have considered characterization of a Łukasiewicz fuzzy ideal, and have discussed the relationship between Łukasiewicz fuzzy subalgebra and Łukasiewicz fuzzy ideal. We have provided a condition for a Łukasiewicz fuzzy subalgebra to be a Łukasiewicz fuzzy ideal, and have explored conditions for the $\in$-set, q-set and O-set to be ideals. In the future, we will use Łukasiewicz fuzzy set to study the substructures of various algebraic systems based on the ideas and results of this paper. In particular, we will study the Łukasiewicz fuzy set theory for the (n-fold) filters in EQ-algebras studied by Paad and Jafari (see [7]).

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Young Bae Jun
Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea.
Email: skywine@gmail.com