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SOME PROPERTIES OF PAIR OF n-ISOCLINISM INDUCTION

M. SAJEDI AND H. DARABI*

Abstract. Let (G, M) be a pair of groups, in which M is a normal subgroup of a group G. We study some properties of n-isoclinism of pair of groups. In fact, we show that the subgroups and quotient groups of two n-isoclinism pair of groups are m-isoclinic for all $m \leq n$. Moreover, the properties of π-pair and supersolvable pair of groups which are invariant under n-isoclinism has be studied.

1. Introduction and preliminaries

In 1940, P. Hall [2], introduced the concept of isoclinism between two groups G and H. This concept are equivalence relation among all groups and it is weaker than isomorphism. Two groups G and H are isoclinic if and only if there exist isomorphisms $\alpha : \frac{G}{Z(G)} \longrightarrow \frac{H}{Z(H)}$ and $\beta : G' \longrightarrow H'$ such that β is induced by α, which are compatible. Hekester [6], introduced the concept of nilpotent groups of class at most n and arose the concept of n-isoclinism. Salemkar et al. [8] extended the concept of isoclinism to pairs of groups. Heidarian et al. [3] extended the concept of n-isoclinism extend to the class of all pairs of groups. Hassanzadeh et al. [5], verifies a new notion of nilpotency for pairs of groups. The main goal of this paper is to investigate the properties of subgroup and quotiont groups of pair of groups that are invarint under n-isoclinism.

Now we can define the center of the pair of group as follows:

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$*$Corresponding author.

Definition 1.1. Let (G,M) be a pair of groups, where M is a normal subgroup of G set

$$Z(G,M)=\{m\in M:[m,x]=1, for \quad all \quad x\in G\}.$$

Upper central series, one may define:

$$1=Z_0(G,M)\trianglelefteq Z_1(G,M)\trianglelefteq Z_2(G,M)\trianglelefteq\cdots$$

in which $Z_1(G,M)=Z(G,M)$ and

$$\frac{Z_{n+1}(G,M)}{Z_n(G,M)}=Z(\frac{G}{Z_n(G,M)},\frac{M}{Z_n(G,M)}).$$

A pair (G,M) is called nilpotent of class n, whenever $Z_n(G,M)=M$, for some natural number n.

Now we can also define

$$[G,M]=\langle [g,m]:g\in G, m\in M\rangle,$$

for which having the lower central series for the pair (G,M) as follows:

$$M=\gamma_1(G,M)\geq\gamma_2(G,M)\geq\cdots$$

of (G,M) such that $\gamma_{n+1}(G,M)=[G,\gamma_n(G,M)]$. For arbitrary pair of groups (G,M) and (H,N), let $\alpha:\frac{G}{Z(G,M)}\longrightarrow\frac{H}{Z(H,N)}$, $\beta:[G,M]\longrightarrow[H,N]$ and $\alpha_|:\frac{M}{Z(G,M)}\longrightarrow\frac{N}{Z(H,N)}$, be isomorphism, which are compatible, such that the following diagram is commutative.

$$\begin{array}{ccc} \frac{G}{Z(G,M)}\times\frac{M}{Z(G,M)} & \stackrel{\left((\alpha_|,\alpha),\beta\right)}{\longrightarrow} & \frac{H}{Z(H,N)}\times\frac{N}{Z(H,N)} \\ \downarrow & & \downarrow \\ [G,M] & \stackrel{\beta}{\longrightarrow} & [H,N]\,, \end{array} \tag{1.1}$$

where $\alpha_|$ is the restriction of α on $\frac{M}{Z(G,M)}$. Hence, the pair of isoclinism $((\alpha,\alpha_|),\beta)$ is called an isoclinism between the pairs of groups (G,M) and (H,N), and denoted by $(G,M)\sim(H,N)$. One can generalized this concept to n-isoclinism of the pairs of groups, which will be denoted by $(G,M)\sim_n(H,N)$, where Moghaddam et al. introduced in [7].

For more information on the center and commutators of the groups, we refer the readers to [1, 4].

The following propositions and lemmas are very useful for further investigations.

Proposition 1.2. [3] *Let (G,M) be a pair of groups, H be a subgroup of G and N be a normal subgroup of G with $N\subseteq M$. Then*

(i) $(H, H\cap M)\sim_n(HZ_n(G,M),(H\cap M)Z_n(G,M))$. *In particular, if* $G = HZ_n(G,M)$, *then* $(H, H\cap M)\sim_n(G,M)$. *Conversely, if* $\frac{H}{Z_n(M,H\cap M)}$ *satisfies the descending chain condition on normal subgroups and* $(H, H\cap M)\sim_n(G,M)$, *then* $G = HZ_n(G,M)$.

(ii) *If* $N\cap\gamma_{n+1}(G,M) = 1$, *then* $(G,M)\sim_n(G/N, M/N)$.
Conversely, if $\gamma_{n+1}(G,M)$ *satisfies the ascending chain condition on normal subgroups and* $(G,M)\sim_n(G/N,M/N)$, *then* $N\cap\gamma_{n+1}(G,M) = 1$.

Lemma 1.3. [3] *Let* $((\alpha,\alpha_|),\beta)$ *be an* n*-isoclinism between* (G,M) *and* (H,N).

(a) *If* G_1 *is a subgroup of* G *with* $Z_n(G,M)\subseteq G_1$ *and* $\alpha(\frac{G_1}{Z_n(G,M)}) = \frac{H_1}{Z_n(H,N)}$ *then*
$$(G_1, G_1\cap M)\sim_n(H_1, H_1\cap N).$$

(b) *If* M_1 *is a normal subgroup of* G *with* $M_1\subseteq\gamma_{n+1}(G,M)$, *then*
$$(G/M_1, M/M_1)\sim_n(H/\beta(M_1), N/\beta(M_1)).$$

The following lemma is useful in our investigation.

Lemma 1.4. *Let* (G,M) *and* (H,N) *be two pairs of groups, and let* N *be a normal subgroups of* G *with* $N\subseteq M$, *and* $i,j\geq 1$. *Then*

(a) $Z_i(\frac{(G,M)}{Z_j(G,M)}) = \frac{Z_{i+j}(G,M)}{Z_j(G,M)}$,
(b) $Z_i(\frac{G}{N},\frac{M}{N}) \geq \frac{Z_i(G,M)N}{N}$,
(c) $\gamma_i(\frac{G}{N},\frac{M}{N}) = \frac{\gamma_i(G,M)N}{N}$,
(d) $\gamma_i(G,\gamma_j(G,M)) \leq \gamma_{i+j-1}(G,M)$.

Proof. We use induction on i. If $i = 1$, the result follows by definition. Assume that the result holds for i. We have $Z_{i+1}(\frac{(G,M)}{Z_j(G,M)}) = \frac{Z(Z_i(G,M))}{Z_j(G,M)} = \frac{Z_{1+j}(Z_i(G,M))}{Z_j(G,M)} = \frac{Z(Z_{i+j}(G,M))}{Z_j(G,M)} = \frac{Z_{i+1+j}(G,M)}{Z_j(G,M)}$. So (a) holds. Similarly, we can prove another parts of lemma by induction on i. □

Lemma 1.5. *Let* (G,M) *be a pair of groups,* H *be a subgroup of* G. *If* $\frac{H}{Z_n(M,H\cap M)}$ *satisfies the descending chain condition on normal subgroups and* $\frac{G}{Z_n(G,M)}\cong\frac{H}{Z_n(M,H\cap M)}$, *then*
$$(H, H\cap M)\sim_n(G,M).$$

Proof. Put $K = Z_n(G,M)$, then Proposition 1.2 (i), implies that $(H,H\cap M)\sim_n(K,M)$, in particular $\frac{G}{Z_n(G,M)}\cong\frac{H}{Z_n(M,H\cap M)}\cong\frac{K}{Z_n(K,M)}$. Now, we show $K = G$. Suppose that $\alpha:\frac{G}{Z_n(G,M)}\longrightarrow\frac{K}{Z_n(K,M)}$ is isomorphism, so, clearly $Z_n(G,M)\leq Z_n(K,M)$. Now, we put $\alpha(\frac{K}{Z_n(G,M)}) =$

$\frac{K_1}{Z_n(K,M)}$. Clearly, $Z_n(G,M) \leq Z_n(K,M) \leq K_1 \leq K$. It can be seen that $K_1 = K$ if and only if $K = G$. Again put $\alpha(\frac{K_1}{Z_n(G,M)}) = \frac{K_2}{Z_n(K,M)}$. Clearly $Z_n(G,M) \leq K_2 \leq K_1$ and $K_2 = K_1$ if and only if $K = K_1$, so $K_2 = K_1$ if and only if $K = G$. Continuing this process condition, we have the following ascending chain

$$\frac{K}{Z_n(K,M)} \geq \frac{K_1}{Z_n(K,M)} \geq \frac{K_2}{Z_n(K,M)} \geq \dots$$

of subgroup of $\frac{K}{Z_n(K,M)}$ that $G = K$ if and only if $\frac{K_i}{Z_n(K,M)} = \frac{K_{i+1}}{Z_n(K,M)}$. But $\frac{K}{Z_n(K,M)} \cong \frac{G}{Z_n(G,M)}$ holds in ascending chains of subgroups and the gives result. □

Theorem 1.6. *Let (G,M) be a pair of groups, H be a subgroup and N be a normal subgroup of G with $N \leq M$. Then*

(a) *For each normal subgroup $K \trianglelefteq G$, if $N \cap \gamma_{n+1}(G,M) = 1$, then*

$$(G,M) \sim_n \left(\frac{G}{K \cap N}, \frac{M}{K \cap N}\right).$$

(b) *If $N \cap \gamma_{n+1}(G,M) = 1$, then*

$$(H, H \cap N) \sim_n \left(\frac{H}{H \cap N}, \frac{H \cap N}{H \cap N}\right).$$

(c) *If $G = HZ_n(G,M)$, then*

$$\left(\frac{G}{N \cap \gamma_{n+1}(G,M)}, \frac{M}{N \cap \gamma_{n+1}(G,M)}\right) \sim_n \left(\frac{HN}{N \cap \gamma_{n+1}(G,M)}, \frac{(H \cap N)N}{N \cap \gamma_{n+1}(G,M)}\right).$$

(d) *For each subgroup $K \leq G$, if $G = HZ_n(G,M)$, then*

$$(G,M) \sim_n (\langle H,K\rangle, \langle H,K\rangle \cap M).$$

Proof.

(a) Clearly, $(K \cap N) \cap \gamma_{n+1}(G,M) \leq N \cap \gamma_{n+1}(G,M) = 1$, therefore, by Proposition 1.2 (ii), $(G,M) \sim_n \left(\frac{G}{K \cap N}, \frac{M}{K \cap N}\right)$.

(b) Clearly, $(H \cap N) \cap \gamma_{n+1}(H, H \cap N) \leq N \cap \gamma_{n+1}(G,M) = 1$, hence $(H \cap N) \cap \gamma_{n+1}(H, H \cap N) = 1$. So, by the second isomorphism theorem [8] and Proposition 1.2 (ii) we have:

$$(H, H \cap N) \sim_n \left(\frac{H}{H \cap N}, \frac{H \cap N}{H \cap N}\right) \cong \left(\frac{HN}{N}, \frac{H \cap N}{H \cap N}\right).$$

(c) Consider that $G = HZ_n(G,M)$, implies that $\frac{G}{N} = \frac{HN}{N}.\frac{Z_n(G,M)N}{N}$. On the other hand, Lemma 1.4 (b) implies $\frac{Z_n(G,M)N}{N} \leq Z_n(G/N, M/N)$, that by substitution in previous relation, we have $\frac{G}{N} = \frac{HN}{N}.Z_n(M/N, G/N)$. Now, using Proposition 1.2 (i) obtain the result.

(d) Clearly $H \leq \langle H, K\rangle$, so $G = HZ_n(G,M) \leq \langle H,K\rangle Z_n(G,M)$. Hence
$G = \langle H,K\rangle Z_n(G,M)$. Now, Proposition 1.2 (i) the result is obtained. □

2. Main result

In this section, we prove some properties of subgroup and quotiont groups of pair of groups that are invarint under n-isoclinism. In the following result, we always assume that $H \leq G$ and $N \trianglelefteq G$ with $N \leq M$.

Theorem 2.1. *Let (G,M) and (H,N) two pairs of groups and $\big((\alpha,\alpha_|),\beta\big)$ is pair of $n-$isclinism from (G,M) to (H,N). Then for all $i \geq 0$,*

i) $\frac{\alpha(\gamma_{i+1}(G,N)Z_n(G,M))}{Z_n(G,M)} = \frac{\gamma_{i+1}(H,H\cap N)Z_n(H,H\cap N)}{Z_n(H,H\cap N)}$,

ii) $(\alpha,\alpha_|)\left(\frac{Z_{n+i}(G,M)}{Z_n(G,M)}\right) = \frac{Z_{n+i}(H,H\cap N)}{Z_n(H,H\cap N)}$,

iii) $\frac{(G,M)}{Z_{n+i}(G,M)} \cong \frac{(H,H\cap N)}{Z_{n+i}(H,H\cap N)}$,

iv) $\beta(\gamma_{n+i+1}(G,M)) = \gamma_{n+i+1}(H,H\cap N)$.

Proof. i) From $(\alpha,\alpha_|)$ is pair of isomorphism and using Lemma 1.4 (c), we have

$$\alpha(\frac{\gamma_{i+1}(G,M)Z_n(G,M)}{Z_n(G,M)}) = \alpha(\gamma_{i+1}(\frac{G}{Z_n(G,M)},\frac{M}{Z_n(G,M)}))$$
$$\cong \gamma_{i+1}(\frac{H}{Z_n(H,H\cap N)},\frac{H\cap N}{Z_n(H,H\cap N)}) = \frac{\gamma_{i+1}(H)Z_n(H,H\cap N)}{Z_n(H,H\cap N)}$$

ii) By using definition the upper central series and isomorphism of α we have

$$\alpha(\frac{Z_{n+i}(G,M)}{Z_n(G,M)}) = \alpha\left(Z_i(\frac{G}{Z_n(G,M)},\frac{M}{Z_n(G,M)})\right)$$
$$= Z_i\left(\alpha(\frac{G}{Z_n(G,M)},\frac{M}{Z_n(G,M)})\right)$$
$$\cong Z_i\left(\frac{H}{Z_n(H,H\cap M)},\frac{N}{Z_n(H,H\cap M)}\right) = \frac{Z_{n+i}(H,H\cap N)}{Z_n(H,H\cap M)}.$$

iii) By part(ii),

$$\frac{(G,M)}{Z_{n+i}(G,M)} \cong \frac{\frac{(G,M)}{Z_n(G,M)}}{\frac{Z_{n+i}(G,M)}{Z_n(G,M)}} \cong \frac{\frac{(H,H\cap N)}{Z_n(H,H\cap N)}}{\frac{Z_{n+i}(H,H\cap N)}{Z_n(H,H\cap N)}} \cong \frac{(H,H\cap N)}{Z_{n+i}(H,H\cap N)}.$$

iv) For all $g_1, \ldots, g_{n+1+i} \in G$, $m \in M$ and $h \in \alpha(mZ_n(G, M))$, $h_j \in \alpha(g_j Z_n(G, M))$ which $1 \leq j \leq n+1+i$, we have:

$$\begin{aligned}
&\alpha([m, g_1, \ldots, g_{i+1}]Z_n(G, M)) \\
&= [\alpha(m)Z_n(G, M), \alpha(g_1)Z_n(G, M), \ldots, \alpha(g_{i+1})Z_n(G, M)] \\
&= [hZ_n(H, H\cap M), \alpha(h_1)Z_n(H, H\cap M), \ldots, \alpha(h_{i+1})Z_n(H, H\cap M)] \\
&= [h, h_1, \ldots, h_{i+1}]Z_n(H, H\cap M).
\end{aligned}$$

Now by definition of n-isoclinism, we have

$$\begin{aligned}
&\beta([[m, g_1, \ldots, g_{i+1}], g_{i+2}, \ldots, g_{n+i+1}]) \\
&= [[h, h_1, \ldots, h_{i+1}], h_{i+2}, \ldots, h_{n+i+1}] \\
&= [h, h_1, \ldots, h_{i+1}, h_{i+2}, \ldots, h_{n+i+1}].
\end{aligned}$$

Therefore

$$\beta(\gamma_{n+i+1}(G, M)) = \gamma_{n+i+1}(H, H\cap N).$$

□

Theorem 2.2. *Let $(G, M) \sim_n (H, H\cap N)$. Then for all $k \geq 0$,*

(a) $Z_{n+k}(G, M) \sim_n Z_{n+k}(H, H\cap N)$,

(b) $\frac{(G,M)}{\gamma_{n+1+k}(G,M)} \sim_n \left(\frac{N}{\gamma_{n+1+k}(H,H\cap N)}, \frac{H}{\gamma_{n+1+k}(H,H\cap N)}\right)$.

(a) *Since $Z_n(G, M) \leq Z_{n+k}(G, M) \leq G$, by Lemma 1.3 (a) implies $(\alpha, \alpha_|)\left(\frac{Z_{n+k}(G,M)}{Z_n(G,M)}\right) = \frac{Z_{n+k}(H,H\cap N)}{Z_n(H,H\cap N)}$. By assumption $Z_{n+k}(G, M) = H_1$, according to Lemma 1.3 (a) the result is obtained.*

(b) *Since for all $k \geq 0$, we have $\gamma_{n+1+k}(G, M) \leq \gamma_{n+1}(G, M)$ and the other hand $M_1 = \gamma_{n+1+k}(G, M) \trianglelefteq G$, as a result, by Lemma 1.3 (b) and Theorem 2.1 (iv) the result is obtained.*

Theorem 2.3. *If $(G, M) \sim_n (H, H\cap N)$, then each normal subgroup of G is n-isoclinic with each normal subgroup of H.*

Proof. Let $M_1 \trianglelefteq G$, hence by Lemma 1.3 (a) and Proposition 1.2 (i) we have $(M_1, M_1 \cap M) \sim_n (M_1 Z_n(M, G), (M_1 \cap M)Z_n(M, G)) \sim_n (N_1 Z_n(N_1, N_1 \cap H), (N_1 \cap H)Z_n(N_1, N_1 \cap H)) \sim_n (N_1, N_1 \cap H)$, that $N_1 \trianglelefteq H$ and gives the form following:

$$\alpha\left(\frac{M_1 Z_n(G, M)}{Z_n(G, M)}\right) = \frac{N_1 Z_n(N_1, N_1 \cap H)}{Z_n(N_1, N_1 \cap H)}.$$

□

Theorem 2.4. *Let (G, M) be an arbitrary pair of groups and $n \geq 0$, then for all $i \in \{0, 1, 2, \cdots, n\}$ and each normal subgroup N with*

$N \cap \gamma_{n+1}(G,M) = 1$, we have

$$\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right) \sim_{n-i} \left(\frac{\frac{G}{N}}{Z_i(\frac{G}{N}, \frac{M}{N})}, \frac{\frac{M}{N}}{Z_i(\frac{G}{N}, \frac{M}{N})}\right).$$

Proof. For all $0 \leq i \leq n$, put $Z_i(\frac{G}{N}, \frac{M}{N}) = \frac{M_1}{N}$, so by Lemma 1.4 (b), $M_1 \trianglelefteq G$ and $Z_i(G,M)N \leq M_1$. Therefore $\frac{M_1}{Z_i(M,G)} \trianglelefteq \frac{G}{Z_i(M,G)}$. Now, it is sufficient to show that

$$\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right) \sim_{n-i} \frac{\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right)}{\left(\frac{M_1}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right)}.$$

But, using Theorem 1.6 (a), it is only to show that

$$\frac{M_1}{Z_i(G,M)} \cap \gamma_{n-i+1}\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right) = 1.$$

For all $g \in M_1 \cap \gamma_{n-i+1}(G,M)$ and $x_1, \ldots, x_i \in G$ and attention to $\frac{M_1}{N} = Z_i\left(\frac{G}{N}, \frac{M}{N}\right)$, we have $[g, x_1, \ldots, x_i] \in N$. On the other hand $[g, x_1, \ldots, x_n] \in \gamma_{n+1}(G,M)$, so $[g, x_1, \ldots, x_i] = 1$, that means $g \in Z_i(G,M)$. Thus $M_1 \cap \gamma_{n-i+1}(G,M) \leq Z_i(G,M)$.

By Dedekind's modular law [7],
$M_1 \cap \gamma_{n-i+1}(G,M)Z_i(G,M) = Z_i(G,M)$, as required. □

Theorem 2.5. *Let H be a subgroup of G and (G,M) is a pair of groups and $\frac{H}{Z_n(M,H\cap M)}$ satisfies the descending chain condition on subgroup. If $\frac{G}{Z_i(G,M)} \sim_j \frac{H}{Z_i(H,H\cap M)}$, then $(G,M) \sim_{i+j} (H, H\cap M)$.*

Proof. By Proposition 1.2 (i), if $\frac{H}{Z_i(M,H\cap M)}$ satisfies the descending chain condition on subgroup and $H \leq G$ then $(G,M) \sim_n (H, H\cap M)$. Now, by induction on j, clearly if $j = 0$ and $\frac{G}{Z_i(G,M)} \cong \frac{H}{Z_i(H,H\cap M)}$ then by Lemma 1.4 $(G,M) \sim_i (H, H\cap M)$. Hence, we assumption result hold for j and $\frac{H}{Z_{i+j+1}(H,H\cap M)}$ satisfies the descending chain condition on subgroup and $\frac{(G,M)}{Z_{i+1}(G,M)} \cong \frac{\frac{(G,M)}{Z_i(G,M)}}{Z\left(\frac{(G,M)}{Z_i(G,M)}\right)} \sim_j \frac{\frac{(H,H\cap M)}{Z_i(H,H\cap M)}}{Z\left(\frac{(H,H\cap M)}{Z_i(H,H\cap M)}\right)} \cong \frac{(H,H\cap M)}{Z_{i+1}(H,H\cap M)}$, so $\frac{(G,M)}{Z_{i+1}(G,M)} \sim_j \frac{(H,H\cap M)}{Z_{i+1}(H,H\cap M)}$. But $\frac{H}{Z_{(i+1)+j}(M,H\cap M)}$ satisfies the descending chain condition on subgroup. Hence by assumption induction $(G,M) \sim_{(i+1)+j} (H, H\cap M)$, so $(G,M) \sim_{i+(j+1)} (H, H\cap M)$. Hence the result holds. □

Theorem 2.6. *Let (G,M) be a pair of groups which the qoutient group $\frac{H}{Z_{i+j}(H,H\cap M)}$ satisfies the descending chain condition on subgroup, for*

$i, j > 0$. If the subgroup H contains $Z_i(H, H \cap M)$, then the relation

$$\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right) \sim_j \left(\frac{H}{Z_i(H,H\cap M)}, \frac{H\cap M}{Z_i(H,H\cap M)}\right),$$

implies that

$$(G, M) \sim_{i+j} (H, H \cap M).$$

Proof. Clearly, $\frac{H}{Z_i(H,H\cap M)} \leq \frac{G}{Z_i(G,M)}$. On the other hand, we have $\frac{H}{Z_{i+j}(H,H\cap M)} \cong \frac{\frac{H}{Z_i(H,H\cap M)}}{Z_j\left(\frac{H}{Z_i(H,H\cap M)}\right)}$ satisfies the descending chain condition on subgroup. By using Proposition 1.2 (i) (conversely) the relation

$$\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right) \sim_j \left(\frac{H}{Z_i(H,H\cap M)}, \frac{H\cap M}{Z_i(H,H\cap M)}\right),$$

implies that $\frac{G}{Z_i(G,M)} = \frac{H}{Z_i(H,H\cap M)} \times Z_j\left(\frac{G}{Z_i(G,M)}, \frac{M}{Z_i(G,M)}\right)$.

Since $Z_i(H, H \cap M) \leq Z_i(G, M)$, we have $\frac{G}{Z_i(G,M)} = \frac{H}{Z_i(H,H\cap M)} \times \frac{Z_{i+j}(G,M)}{Z_i(G,M)} = \frac{HZ_{i+j}(G,M)}{Z_i(G,M)}$. Hence $G = HZ_{i+j}(G, M)$, so by Proposition 1.2 (i) we have

$$(G, M) \sim_{i+j} (H, H \cap M).$$

□

Theorem 2.7. *Let (G, M) be a pair of groups, $N \trianglelefteq G$ with $N \leq M$ and $N \cap \gamma_{n+1}(G, M) = 1$. Then for all $0 \leq i \leq n$, we have*

$$\gamma_{i+1}(G, M) \sim_{n-i} \gamma_{i+1}(G/N, M/N).$$

Proof. Put $j = n - i$. Now, we have

$$\gamma_{i+1}(G/N, M/N) = \frac{\gamma_{i+1}(G,M)\,N}{N} = \frac{\gamma_{i+1}(G,M)}{N \cap \gamma_{i+1}(G,M)}.$$

By using $M_1 = N \cap \gamma_{i+1}(G, M)$ it is sufficient to show that $M_1 = N \cap \gamma_{j+1}(G, \gamma_{i+1}(G, M)) = 1$, because by Lemma 1.4 (c), $\gamma_{i+1}(G, M) \sim_j \frac{\gamma_{i+1}(G,M)}{N} = \gamma_{i+1}(G/N, M/N)$. Clearly, $\gamma_{j+1}(G, \gamma_{i+1}(G, M)) \leq \gamma_{i+1}(G, M)$, therefore using Lemma 1.4 (d), we have

$$\begin{aligned} M_1 \cap \gamma_{j+1}(G, \gamma_{i+1}(G,M)) &= N \cap \gamma_{i+1}(G,M) \cap \gamma_{j+1}(G, \gamma_{i+1}(G,M)) \\ &= N \cap \gamma_{j+1}(G, \gamma_{i+1}(G,M)) \leq N \cap \gamma_{i+j+1}(G,M). \end{aligned}$$

Since $j + i + 1 \geq n + 1$, hence $M_1 \cap \gamma_{j+1}(G, \gamma_{i+1}(G, M)) \leq N \cap \gamma_{n+1}(G, M) = 1$. So, the result is obtained. □

Now, we study properties of pair of nilpotent that invariant under n-isoclinism. Let (G, M) be a pair of groups. Then (G, M) is called a nilpotent pair of group if M has a normal series

$$1 = M_0 \leq M_1 \leq \cdots \leq M_t = M,$$

such that M_i is a normal subgroup of G and $\frac{M_{i+1}}{M_i} \leq Z\left(\frac{G}{M_i}, \frac{M}{M_i}\right)$ for all i.

Theorem 2.8. *Let (G, M) be a pair of groups, $N \leq Z(G, M)$ and $\left(\frac{G}{N}, \frac{M}{N}\right)$ be nilpotent. Then (G, M) is nilpotent.*

Proof. Since $\left(\frac{G}{N}, \frac{M}{N}\right)$ is a nilpotent pair of groups, so there exist a series as follows:

$$1 = \frac{M_0}{N} \leq \frac{M_1}{N} \leq \frac{M_2}{N} \leq \cdots \leq \frac{M_n}{N} = \frac{M}{N},$$

such that

$$\frac{\frac{M_{i+1}}{N}}{\frac{M_i}{N}} \leq Z\left(\frac{\frac{G}{N}}{\frac{M_i}{N}}, \frac{\frac{M}{N}}{\frac{M_i}{N}}\right),$$

so, we have $\frac{M_{i+1}}{M_i} \leq Z\left(\frac{G}{M_i}, \frac{M}{M_i}\right)$. On the other hand $N \leq Z(G, M)$. So, the series: $1 = N_0 \leq N \leq M_1 \leq M_2 \cdots \leq M_n = M$, is a central series of G. Hence, (G, M) is nilpotent of groups. $\square$

Theorem 2.9. *Let (G, M) be a nilpotent pair of class m and H be any subgroup of G with $(G, M) \sim_n (H, H \cap M)$ for all $m \leq n$. Then the $(H, H \cap M)$ is a nilpotent pair.*

Proof. By Theorem 2.1 (iii), we have $\frac{G}{Z_{n+i}(G,M)} \cong \frac{H}{Z_{n+i}(H,H\cap M)}$, for all $i \geq 0$. By put $i = m - n$, we have $\frac{G}{Z_m(G,M)} \cong \frac{H}{Z_m(H,H\cap M)}$, so $Z_m(H, H \cap M) = H \cap M$, which gives the result. $\square$

The following is an immediate consequence of Theorem 2.9.

Corollary 2.10. *Let (G, M) be a nilpotent pair of class n and H be an arbitrary. Then $(H, H \cap M)$ is nilpotent pair of class n if and only if $(G, M) \sim_n (H, H \cap M)$. In particular $(G, M) \sim_n (1, 1)$.*

Definition 2.11. Let (G, M) be a pair of groups. Then (G, M) is called supersolvable group if there exist normal series as follow:

$$1 = M_0 \leq M_1 \leq \cdots \leq M_r = M,$$

such that, factors group $\frac{M_{i+1}}{M_i}$ is cyclic for $0 \leq i \leq r - 1$.

Theorem 2.12. *Let $(G, M) \sim_k (H, H \cap N)$. If (G, M) is supersolvable, then $(H, H \cap N)$ is supersolvable.*

Proof. Since (G,M) is supersolvable, so $\frac{(G,M)}{Z_n(G,M)} \cong \frac{(H,H\cap N)}{Z_n(H,H\cap N)}$ is also supersolvable. Therefore $\frac{(H,H\cap N)}{Z_n(H,H\cap N)}$ has a finite normal series with cyclic factor as follows:

$$1 = \frac{N_0}{Z_n\left(H,H\cap N\right)} = \frac{Z_n\left(H,H\cap N\right)}{Z_n\left(H,H\cap N\right)} \leq \frac{N_1}{Z_n\left(H,H\cap N\right)} \leq \frac{N_2}{Z_n\left(H,H\cap N\right)} \leq \cdots \leq \frac{N_m}{Z_n\left(H,H\cap N\right)} = \frac{H}{Z_n\left(H,H\cap N\right)}.$$

Hence, $\frac{N_j}{N_{j-1}} \cong \frac{\frac{N_j}{Z_n(H,H\cap N)}}{\frac{N_{j-1}}{Z_n(H,H\cap N)}}$ for$1 \leq j \leq m$, is cyclic group which $N_{j-1} \leq N_j$. On the other hand $Z_n\left(H,H\cap N\right)$ is finite, so it is solvable. Therefore, each subgroup is solvable. So, there exist the subnormal series

$$Z_i(H,H\cap N) = N_{i,0} \leq N_{i,1} \leq \cdots \leq N_{i,k_i} = Z_{i+1}(H,H\cap N),$$

such that all of factors are cyclic. Therefore,

$$\begin{aligned} &Z_0(H,H\cap N) = N_{0,0} \leq N_{0,1} \leq \cdots \leq N_{0,k_0} \\ &\leq Z_1(H,H\cap N) = N_{1,0} \leq N_{1,1} \leq \cdots \leq N_{1,k_1} \\ &\leq Z_2(H,H\cap N) = N_{2,0} \leq \cdots \leq Z_n(H,H\cap N) \\ &\leq N_1 \leq N_2 \leq \cdots \leq N_m = (H,H\cap N) \end{aligned}$$

is a finite normal series with factors cyclic for $(H,H\cap N)$. So $(H,H\cap N)$ is supersolvable. Now, we study the properties, $\pi-$ separable and $\pi-$solvable of the pair of group (G,M) that preserve under n-isoclinism. □

Let π is the non-empty set of primary number and (G,M) is a pair of arbitrary groups, then the element $m \in M$ is called $\pi-$element of pair (G,M), whenever order m only divide on primary number of π. Also, the finite pair of groups (G,M) is called, $\pi-$group, when order M is so. We may define finite pair of group (G,M),$\pi-$ separable or $\pi-$solvable, whenever there exist a subnormal series that its factors $\pi-$group or $\pi'-$group.

Theorem 2.13. *Let (G,M) and $(H,H\cap M)$ are two finite groups such that $(G,M) \sim_n (H,H\cap N)$. Then, (G,M) is $\pi-$ separable($\pi-$solvable) if and only if $(H,H\cap M)$ is $\pi-$ separable($\pi-$solvable).*

Proof. Suppose that (G,M) is a $\pi-$ seperable group. Hence $\frac{(G,M)}{Z_n(G,M)} \cong \frac{(H,H\cap N)}{Z_n(H,H\cap N)}$ is also seperable. Therefore, there exist the following a subnormal series

$$\frac{N_0}{Z_n(H,H\cap N)} = \frac{Z_n(H,H\cap N)}{Z_n(H,H\cap N)} \trianglelefteq \frac{N_1}{Z_n(H,H\cap N)} \trianglelefteq \dots$$
$$\trianglelefteq \frac{N_k}{Z_n(H,H\cap N)} = \frac{H\cap N}{Z_n(H,H\cap N)},$$

such that its factors $\pi-$ group or $\pi'-$group. On the other hand $Z_n(H, H\cap N)$ is finite nilpotent, and so supersolvable. So, there is following subnormal series

$$M_0 = \langle 1\rangle \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_m = Z_n(H, H\cap N),$$

which its factors $\pi-$ group or $\pi'-$ group. Now, one can easily see the subnormal series

$$M_0 = \langle 1\rangle \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_m = Z_n(H, H\cap N) = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_m = H\cap N,$$

its factors, $\pi-$ group or $\pi'-$ group. So $(H, H\cap N)$ is a $\pi-$ seperable group . □

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M. Sajedi
Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad,

Iran.
Email: m-sajedi1979@yahoo.com

H. Darabi
Department of Mathematics, Esfarayen University of Technology, Esfarayen, Iran.
Email: darabi@esfarayen.ac.ir