Journal of Algebra and Related Topics
Vol. 10, No 1, (2022), pp 143-158

ON SPECIAL AMALGAMS AND CLOSED VARIETIES OF POSEMIGROUPS

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Abstract. In this paper, we extend a result of Scheiblich by showing that variety of po-normal bands is closed. We also extend the well known results to posemigroups namely, that pogroups and inverse posemigroups have special amalgamation property in the category of all posemigroups and commutative posemigroups, respectively. Finally, we find some varieties of posemigroups which are closed if they are self convex.

1. Introduction and Summary

In [9], Scheiblich had shown that the variety of normal bands was closed. Therefore, the special semigroup amalgam $[U; \{S, S'\}; \{i, \alpha|_U\}]$ is embeddable, where U and S are any normal bands. In section 3, we extend this result to the variety of po-normal bands. In [7], Howie and Isbell had shown that inverse semigroups were absolutely closed in the category of semigroups. In this section, we partially generalize this result by showing that the special posemigroup amalgam $\{U; \{S, S'\}; \{i, \alpha|_U\}\}$ is poembeddable in the category of commutative posemigroups, where U is any inverse posemigroup. Further, we show that the special posemigroup amalgam $[U; \{S, S'\}; \{i, \alpha|_U\}]$ is poembeddable in the category of all posemigroups, where U is any pogroup, i.e., pogroups have special amalgamation property in the category of all posemigroups.

MSC(2010): Primary: 06F05; Secondary: 20M07, 20M10.
Keywords: Posemigroup, Dominion, Zigzag Inequalities, Permutative, Variety.
Received: 10 August 2021, Accepted: 29 December 2021.
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In [6], Higgins has shown that the variety of right normal bands was not absolutely closed, hence, so is the case with variety of po-right normal bands. So, it is of worth interest to find the subvarieties of the variety of all posemigroups in which the variety of po-left [po-right] normal bands is closed. In section 4, we show some varieties, which contain the variety of po-left [po-right] normal bands, are closed.

2. PRELIMINARIES

A *partially ordered semigroup*, briefly *posemigroup*, is a pair $(S, \leq)$ comprising a semigroup S and a partial order $\leq$ on S that is compatible with its binary operation, i.e. for all $s_1, s_2, t_1, t_2 \in S$ ($s_1 \leq t_1$ and $s_2 \leq t_2$) $\implies$ $s_1 s_2 \leq t_1 t_2$. If S is a monoid, then we call $(S, \leq)$ a *partially ordered monoid*, in short, a *pomonoid*. We call $(U, \leq_U)$ a *subposemigroup* of a posemigroup $(S, \leq_S)$ if U is a subsemigroup of the semigroup S and $\leq_U = \leq_S \cap (U \times U)$. The corresponding notion of subpomonoid is defined analogously. In whatever follows we shall denote a posemigroup $(S, \leq)$ simply by S if there is no confusion about the order relation. A posemigroup morphism $f : S \to T$, where S and T are posemigroups, is a monotone map (i.e., $x \leq y \implies f(x) \leq f(y)$) that is also a morphism of the underlying semigroups.

In this article, we shall always treat a posemigroup $(S, \leq)$ as a semigroup by simply disregarding the order. In the later case we shall denote it by S. Let $\mathcal{A}$ be a class of posemigroups. Then by $\mathcal{A}'$, we shall denote the class of semigroups obtained by disregarding the order in $\mathcal{A}$ (that is $\mathcal{A}' = \{S : (S, \leq) \in \mathcal{A}\}$).

Let S and T be posemigroups and $f : S \to T$ be any posemigroup morphism. Then f is said be an *epimorphism (epi, for short)* if for any posemigroup W and any posemigroup morphisms $\alpha,\ \beta : T \to W$, $\alpha f = \beta f$ implies $\alpha = \beta$. One can easily observe that the forgetful functor from the category of all pomonoids to the category of all monoids preserves epis.

Let U be a subposemigroup of a posemigroup S and $d \in S$. We say that U dominates d if for all posemigroup morphisms $\alpha,\ \beta : S \to T$, whenever $\alpha(u) = \beta(u)$ for all $u \in U$ then $\alpha(d) = \beta(d)$. The set of all elements of S that are dominated by U is called the *posemigroup dominion* of U in S and is denoted by $\widehat{Dom}(U, S)$. One can easily verify that $\widehat{Dom}(U, S)$ is a subposemigroup of S containing U. Let U be a subposemigroup of a posemigroup S. We say U is *closed* in S if $\widehat{Dom}(U, S) = U$ and U is said to be *absolutely closed* if it is closed in every containing posemigroup. A posemigroup U is said to be *saturated* if $\widehat{Dom}(U, S) \neq S$ for every properly containing posemigroup

S. A variety $\mathcal{V}$ of posemigroups is absolutely closed [saturated] if each member of $\mathcal{V}$ is absolutely closed [saturated].

A class of posemigroups is called a variety of posemigroups if it is closed under cartesian products endowed with component-wise binary operation and order, morphic images and subposemigroups. It is also possible to describe posemigroup varieties alternatively with the help of inequalities using Birkhoff type characterization; we refer the readers to [4] for all such details.

A variety $\mathcal{V}$ of posemigroups is closed if for any posemigroups $U,\ S \in \mathcal{V}$ with U as a subposemigroup of S, $\widehat{Dom}(U,S) = U$. Let U and S be two posemigroups such that U is a subposemigroup of S. Then U is said to be *convex in* S if for any $u, v \in U$ and $s \in S$ with $u \leq s \leq v$, implies $s \in U$ and we say posemigroup U is convex if it is convex in every containing posemigroup. A variety $\mathcal{V}$ of posemigroups is said to be convex if each member of the variety is convex and it is said to be $\mathcal{V}-$*convex* if for any posemigroups $U, S \in \mathcal{V}$, U is convex in S.

The following characterization of posemigroup dominion is provided by Sohail and Tart called the Zigzag Theorem for posemigeroups which is as follows.

Theorem 2.1. *([10], Theorem 5) Let U be a subposemigroup of a posemigroup S. Then $d \in \widehat{Dom}(U,S)$ if and only if $d \in U$ or*

$$\begin{aligned} d &\leq x_1u_0, & u_0 &\leq u_1y_1 \\ x_iu_{2i-1} &\leq x_{i+1}u_{2i}, & u_{2i}y_i &\leq u_{2i+1}y_{i+1} \ (i = 1, 2, \ldots, m-1) \\ x_mu_{2m-1} &\leq u_{2m}, & u_{2m}y_m &\leq d; \ \textit{and} \end{aligned} \tag{2.1}$$

$$\begin{aligned} v_0 &\leq s_1v_1, & d &\leq v_0t_1 \\ s_jv_{2j} &\leq s_{j+1}v_{2j+1}, & v_{2j-1}t_j &\leq v_{2j}t_{j+1} \ (j = 1, 2, \ldots, m'-1) \\ s_{m'}v_{2m'} &\leq d, & v_{2m'-1}t_{m'} &\leq v_{2m'}; \end{aligned} \tag{2.2}$$

where $u_0, v_0, \ldots, u_{2m}, v_{2m'} \in U, x_1, y_1, \ldots, x_m, y_m, s_1, t_1, \ldots, s_{m'}, t_{m'} \in S$.

The above inequalities are called zigzag ineqalities in S over U with value d of length (m, m'). We say that the above zigzag inequalites are of minimal lenght (m, m') if m and m' are the smallest positive integers. The zigzag inequalities (2.1) give:

$$d \leq x_1u_0 \leq x_1u_1y_1 \leq x_2u_2y_1 \leq \cdots \leq x_mu_{2m-1}y_m \leq u_{2m}y_m \leq d.$$

Therefore

$$d = x_1u_0 = x_1u_1y_1 = x_2u_2y_1 = \cdots = x_mu_{2m-1}y_m = u_{2m}y_m = d. \tag{2.3}$$

Similarly the zigzag inequalities (2.2) give:

$$d \leq v_0 t_1 \leq s_1 v_1 t_1 \leq s_1 v_2 t_2 \leq \cdots \leq s_{m'} v_{2m'-1} t_{m'} \leq s_{m'} v_{2m'} \leq d.$$

Thus

$$d = v_0 t_1 = s_1 v_1 t_1 = s_1 v_2 t_2 = \cdots = s_{m'} v_{2m'-1} t_{m'} = s_{m'} v_{2m'} = d. \quad (2.4)$$

The next theorems are very instrumental in our investigations.

Theorem 2.2. *([1], Lemma 3.2) Let* $d \in \widehat{Dom}(U, S) \setminus U$ *and let (2.1) and (2.2) be zigzag inequalities for* d *of minimal length* (m, m')*. Then* $x_i, y_i, s_j, t_j \in S \setminus U$ *for all* $i = 1, 2, \ldots, m, j = 1, 2, \ldots, m'$*.*

Theorem 2.3. *([1], Lemma 3.3) Let* U *be a subposemigroup of a posemigroup* S *such that* $\widehat{Dom}(U, S) = S$*. Then, for any* $d \in S \setminus U$ *and for any positive integers* k *and* k'*, there exist* $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_{k'}$ $\in U$ *and* $d', d'' \in S \setminus U$ *such that* $d = u_1 u_2 \cdots u_k d' = d'' v_{k'} v_{k'-1} \cdots v_2 v_1$*.*

Theorem 2.4. *([1], Lemma 2.1) Let* U *be subposemigroup of a posemigroup* S*. Then* $\widehat{Dom}(Z(U), S)$ *is central in* U*.*

The bracketed statements wherever used shall mean the dual to the other statements.

3. Poembeddable Posemigroup Amalgams

A posemigroup S is said to be *po-band* if $x^2 = x$ for all $x \in S$; i.e., every element of S is an idempotent. A po-band is said to be a *po-normal band* if it satisfies the identity $xyzx = xzyx$.

A *posemigroup amalgam* $\mathcal{A} = [U; S_i, \phi_i]$ consists of a posemigroup U, called the *core*, a family $\{S_i : i \in I\}$ of posemigroups and a family $\{\phi_i : i \in I\}$ of order embeddings, $\phi_i : U \to S_i$. The posemigroup amalgam $\mathcal{A} = [U; S_i, \phi_i]$ is said to be *weakly embeddable [poembeddable]* in a posemigroup T if there exist posemigroup monomorphisms [order-embeddings] $\psi_i : S_i \to T$ such that for all $i \neq j$ in I, $\psi_i \phi_i = \psi_j \phi_j$. If in addition $\psi_i(S_i) \cap \psi_j(S_j) = \psi_i \phi_i(U)$, the amalgam is *embeddable [poembeddable]* in T. Hence it can easily be deduced that

$$\begin{array}{ccc} \textit{poembeddable} & \Longrightarrow & \textit{embeddable} \\ \Downarrow & & \Downarrow \\ \textit{weakly poembeddable} & \Longrightarrow & \textit{weakly embeddable.} \end{array}$$

A posemigroup amalgam $[U; S, S'; i, \pi_U]$ consisting of a posemigroup S, a subposemigroup U of S, an order isomorphism $\pi : S \to S'$ and the inclusion $i : U \to S$ is called a *special posemigroup amalgam*.

Theorem 3.1. *[[8], Theorem 8.3.2] Let U be a subposemigroup of a posemigroup S and let $\pi : S \to S'$ be an order isomorphism. Then $\widehat{Dom}(U,S) = \Theta_1^{-1}(\Theta_1(S) \cap \Theta_2(S'))$, where $\Theta_1 : S \to S *_U S'$ and $\Theta_2 : S' \to S *_U S'$ are posemigroup order embeddings.*

Therefore one can easily verify from Theorem 3.1 the following:

Corollary 3.2. *Let U be a subposemigroup of a posemigroup S. Then U is closed in S if and only if the special posemigroup amalgam, $[U; S, S'; i, \pi_U]$ is poembeddable.*

In the next theorem, we show that the special posemigroup amalgam $[U; \{S, S'\}; \{i, \alpha|_U\}]$, where U and S are po-normal bands such that U is a sub po-band of S is poembeddable. To prove our theorem, we first prove the following lemmas.

Lemma 3.3. *Let U and S be po-normal bands such that U be a sub po-band of S. Let $d \in \widehat{Dom}(U,S) \setminus U$ and let (2.1) and (2.2) be zizag inequalities in S over U with value d of length (m, m'). Then for all $k = 1, 2, \ldots, m$, $d \geq x_k u_{2k-1} y_k (\prod_{i=1}^{k} u_{2i-1}) u_0$.*

Proof. We use induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1u_1y_1 \text{ (by zigzg equations (2.3))} \\ &= x_1u_1y_1u_1y_1 \text{ (as } S \text{ is a po-band)} \\ &\geq x_1u_1y_1u_0 \text{ (by zigzag inequalities (2.1))} \\ &= x_1u_1u_1y_1u_0 \text{ (as } U \text{ is a po-band)} \\ &= x_1u_1y_1(\prod_{i=1}^{1} u_i)u_0 \text{ (as } S \text{ is a po-normal band).} \end{aligned}$$

Therefore the result holds for $k = 1$. Assume inductively that the result holds for $k = l < m$. We show the that the result holds for $k = l + 1$. Now

$$\begin{aligned} d &\geq x_l u_{2l-1} y_l (\prod_{i=1}^{l} u_{2i-1}) u_0 \text{ (by inductive hypothesis)} \\ &= x_{l+1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l} u_{2i-1}) u_0 \text{ (by zigzag equations (2.3))} \\ &= x_{l+1} u_{2l+1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l} u_{2i-1}) u_0 \text{ (as } U \text{ is a po-band)} \end{aligned}$$

$$= x_{l+1}u_{2l+1}y_{l+1}(\prod_{i=1}^{l+1} u_{2i-1})u_0 \text{ (as } S \text{ is a po-normal band).}$$

Thus the result holds for $k = l + 1$, as required. □

Lemma 3.4. *Let U and S be po-normal bands such that U is a sub po-band of S. Let $d \in \widehat{Dom}(U, S) \setminus U$ and let (2.1) and (2.2) be zizag inequalities in S over U with value d of length (m, m'). Then for all $k = 1, 2, \ldots, m,\ d \leq u_{2m}(\prod_{i=1}^{k} u_{2i-1})x_k u_{2k-1} y_k$.*

Proof. The proof follows on the similar lines as the proof of the Lemma 3.3. □

Theorem 3.5. *The variety of po-normal bands is closed.*

Proof. Let U and S be any two po-normal bands with U as a subpo-band of S. We show that $\widehat{Dom}(U, S) = U$. Assume on contrary and take any $d \in \widehat{Dom}(U, S) \setminus U$. Let (2.1) and (2.2) be the zigzag inequalities in S over U with value d of length (m, m'). Now

$$\begin{aligned}
d &\geq x_m u_{2m-1} y_m (\prod_{i=1}^{m} u_{2i-1})u_0 \text{ (by Lemma 3.3)} \\
&= u_{2m} y_m (\prod_{i=1}^{m} u_{2i-1})u_0 \text{ (by zigzag equations (2.3))} \\
&= u_{2m} u_{2m-1} y_m (\prod_{i=1}^{m-1} u_{2i-1})u_0 \text{ (as } S \text{ is a po-normal band)} \\
&\geq (\prod_{i=0}^{1} u_{2m-2i}) y_{m-1} (\prod_{i=1}^{m-1} u_{2i-1})u_0 \text{ (by zigzag inequalities (2.1)).}
\end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned}
d &\geq (\prod_{i=0}^{m-1} u_{2m-2i}) y_1 (\prod_{i=1}^{1} u_{2i-1})u_0 \\
&= (\prod_{i=0}^{m-1} u_{2m-2i}) u_1 y_1 u_0 \text{ (as } S \text{ is a po-normal band)} \\
&\geq (\prod_{i=0}^{m} u_{2m-2i}) u_0 \text{ (by zigzag inequalities (2.1))} \\
&= \prod_{i=0}^{m} u_{2m-2i} \text{ (as } U \text{ is a band).}
\end{aligned}$$

Now on similar lines, by Lemma 3.4 and zigzag inequalities (2.1), we obtain $d \leq \prod_{i=0}^{m} u_{2m-2i}$. Thus $d = \prod_{i=0}^{m} u_{2m-2i} \in U$, a contradiction, as required. □

The next corollary follows from Corollary 3.2 and Theorem 3.5.

Corollary 3.6. *Let U and S be po-norml bands such that U is a subpoband of S. Then the special posemigroup amalgam $[U; \{S, S'\}; \{i, \alpha|_U\}]$ is poembeddable.*

Let S be a semigroup and $x \in S$. Then $x^{-1} \in S$ is said to be an inverse of x if $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. A semigroup S is said to be regular if every element of S has an inverse and an inverse semigroup if every element of S has a unique inverse. A posemigroup S is said to be regular [inverse] posemigroup if it is such as a semigroup. One can easily verify that a regular semigroup is an inverse semigroup if and only if its idempotents commute.

In [2], Al Subaiei had shown that the pogroups were absolutely closed. In the next theorem we extend this result by proving the following.

Theorem 3.7. *Let U be any regular posemigroup and S be any posemigroup containing U as a subposemigroup. If*

(a) *idempotents of U are central in U;*
(b) *any $d \in \widehat{Dom}(U, S) \setminus U$ has zigzag inequalities of the type (2.1) and (2.2);*
(c) *for any idempotents e and f in U, $x_i u_{2i-1} y_i e = x_i u_{2i-1} e y_i$ and $f x_i u_{2i-1} y_i = x_i f u_{2i-1} y_i$ for all $i = 1, 2, \ldots, m$,*

then $d \in U$.

Proof. For all $k = 1, 2, \ldots, m$, we show that

$$d = x_k u_{2k-1} y_k (\prod_{i=1}^{k} u_{2i-1}^{-1} u_{2i-1}). \tag{3.1}$$

To prove (3.1), we use induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1 u_1 y_1 \text{ (by zigzag equations (2.3))} \\ &= x_1 u_1 u_1^{-1} u_1 y_1 \text{ (as } U \text{ is an inverse posemigroup)} \\ &= x_1 u_1 y_1 (\prod_{i=1}^{1} u_{2i-1}^{-1} u_{2i-1}) \text{ (as } u_1^{-1} u_1 \text{ is an idempotent of } U). \end{aligned}$$

Therefore the result holds for $k=1$. Assume inductively that the result holds for $k=l<m$. We show that the result also holds for $k=l+1$. Now

$$\begin{aligned}
d &= x_l u_{2l-1} y_l (\prod_{i=1}^{l} u_{2i-1}^{-1} u_{2i-1}) \text{ (by inductive hypothesis)}\\
&= x_{l+1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l} u_{2i-1}^{-1} u_{2i-1}) \text{ (by zigzag equations (2.3))}\\
&= x_{l+1} u_{2l+1} u_{2l+1}^{-1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l} u_{2i-1}^{-1} u_{2i-1}) \text{ (as } U \text{ is an inverse posemigroup)}\\
&= x_{l+1} u_{2l+1} y_{l+1} u_{2l+1}^{-1} u_{2l+1} (\prod_{i=1}^{l} u_{2i-1}^{-1} u_{2i-1}) \text{ (as } u_{2l+1}^{-1} u_{2l+1} \text{ is an idempotent)}\\
&= x_{l+1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l+1} u_{2i-1}^{-1} u_{2i-1}) \text{ (as idempotents commute in } U).
\end{aligned}$$

Thus the result also holds for $k=l+1$, as required. Since the idempotents are central in U, therefore by Theorem 2.4, they are central in $\widehat{Dom}(U,S)$; i.e.,

$$de = ed \text{ for all } d \in \widehat{Dom}(U,S) \text{ and for any idempotent } e \text{ of } U. \quad (3.2)$$

Now, we have

$$\begin{aligned}
d &= x_m u_{2m-1} y_m (\prod_{i=1}^{m} u_{2i-1}^{-1} u_{2i-1}) \text{ (by equation (3.1))}\\
&= u_{2m} y_m (\prod_{i=1}^{m} u_{2i-1}^{-1} u_{2i-1}) \text{ (by zigzag equations (2.3))}\\
&= (\prod_{i=1}^{m} u_{2i-1}^{-1} u_{2i-1}) u_{2m} y_m \text{ (by equations (3.2) as } (\textstyle\prod_{i=1}^{m} u_{2i-1}^{-1} u_{2i-1})\\
&\qquad \text{is an idempotent and } d = u_{2m} y_m)\\
&= (\prod_{i=1}^{m-1} u_{2i-1}^{-1} u_{2i-1}) u_{2m} u_{2m-1}^{-1} u_{2m-1} y_m \text{ (as } u_{2m-1}^{-1} u_{2m-1} \text{ is an idempotent}\\
&\qquad \text{and idempotents are central in } U)\\
&\geq (\prod_{i=1}^{m-1} u_{2i-1}^{-1} u_{2i-1}) u_{2m} u_{2m-1}^{-1} u_{2m-2} y_{m-1} \text{ (by zigzag inequalities (2.1))}
\end{aligned}$$

$$= \Big(\prod_{i=1}^{m-2} u_{2i-1}^{-1}u_{2i-1}\Big)u_{2m}u_{2m-1}^{-1}u_{2m-2}u_{2m-3}^{-1}u_{2m-3}y_{m-1} \text{ (as } u_{2m-3}^{-1}u_{2m-3} \text{ is an idempotent and idempotents are central in } U)$$

$$= \Big(\prod_{i=1}^{m-2} u_{2i-1}^{-1}u_{2i-1}\Big)\Big(\prod_{i=0}^{1} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_{2m-3}y_{m-1}.$$

Proceeding like wise, we obtain

$$d \geq \Big(\prod_{i=1}^{1} u_{2i-1}^{-1}u_{2i-1}\Big)\Big(\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_3y_2$$

$$\geq u_1^{-1}u_1\Big(\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_2y_1 \text{ (by zigzag inequalities (2.1))}$$

$$= \Big(\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_2u_1^{-1}u_1y_1 \text{ (as } u_1^{-1}u_1 \text{ is an idempotent and idempotents are central in } U)$$

$$\geq \Big(\prod_{i=0}^{m-1} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_0 \text{ (by zigzag inequalities (2.1)).}$$

By a similar token, we obtain $d \leq (\prod_{i=0}^{m-1} u_{2m-2i}u_{2m-2i-1}^{-1})u_0$ and hence

$$d = \Big(\prod_{i=0}^{m-1} u_{2m-2i}u_{2m-2i-1}^{-1}\Big)u_0 \in U,$$

as required. □

The next corollaries easily follows from the Theorem 3.7.

Corollary 3.8. *Let U be an inverse posemigroup and S be any posemigroup with U as a subposemigroup such that the idempotents of U are central in S. Then U is closed in S.*

Corollary 3.9. *Inverse posemigroups are absolutely closed in the category of all commutative posemigroups.*

Theorem 3.10. *Pogroups are absolutely closed in the category of all posemigroups.*

Proof. Let G be any pogroup and S be any posemigroup containing G as a subposemigroup. Let e be the identity of G. Then e is the only idempotent of G and clearly e is central in G. Take any $d \in$

$\widehat{Dom}(G,S)\setminus G$ and let (2.1) and (2.2) be the zigzag inequalities with value d. Then the zigzag equation (2.3) gives

$$de = x_1u_0e = d \text{ and } ed = eu_{2m}y_m = d.$$

Now, one can easily verify that

$$x_iu_{2i-1}y_ie = x_iu_{2i-1}ey_i = x_ieu_{2i-1}y_i = ex_iu_{2i-1}y_i \text{ for all } i = 1,2,\ldots,m.$$

Thus, by Theorem 3.7, G is closed in S and hence pogroups are absolutely closed in the category of all posemigroups. □

The next corollary follows from Corollary 3.2 and Theorem 3.10.

Corollary 3.11. *Let U be a subpogroup of a posemigroup S. Then the special posemigroup amalgam $[U;\{S,S'\};\{i,\alpha|_U\}]$ is poembeddable.*

The next corollary follows from the Corollary 3.2 and the Corollary 3.8.

Corollary 3.12. *Let U be a subposemigroup of a posemigroup S such that U is an inverse posemigroup. The special posemigroup amalgam $[U;\{S,S'\};\{i,\alpha|_U\}]$ is poembeddable in the category of all commutative posemigroups.*

4. Closed Varieties of Posemigroups

A po-band is said to be a *po-left [po-right]* normal band if it satisfies the identity $xyz = xzy$ $[xyz = yxz]$.

In this section, we show that the varieties $\mathcal{V}_1 = [xyz = xxzy]$ and $\mathcal{U}_1 = [xyz = xzyy]$ of posemigroups are closed if they are $\mathcal{V}_1$-convex and $\mathcal{U}_1$-convex, respectively. In particular, it will imply that the variety of po-left normal bands is closed in both the above mentioned varieties.

Theorem 4.1. *The variety $\mathcal{V}_1$ of posemigroups satisfying the identity $xyz = xxzy$ is closed if it is $\mathcal{V}_1$-convex.*

Proof. Let U, $S \in \mathcal{V}_1$ such that U is convex subposemigroup of S. We show that $\widehat{Dom}(U,S) = U$. Assume on contrary and take any $d \in \widehat{Dom}(U,S)\setminus U$. Let (2.1) and (2.2) be the zigzag inequalities in S over U with value d of length (m,m'). In order to prove the theorem, we first prove the following lemmas.

Lemma 4.2. *For all $j = 1,2,\ldots,m'$,*

$$d = (\prod_{i=1}^{j} s_is_iv_{2i-1}v_{2i-1})s_jv_{2j-1}t_j.$$

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= xxzy \text{ (as } S \in \mathcal{V}_1) \\ &= xxyxz \text{ (as } S \in \mathcal{V}_1) \\ &= xxyyzx \text{ (as } S \in \mathcal{V}_1) \\ &= xxyyz \text{ (as } S \in \mathcal{V}_1) \\ &= xxyyzx \text{ (as } S \in \mathcal{V}_1) \\ &= xxyyxyz \text{ (as } S \in \mathcal{V}_1). \end{aligned} \tag{4.1}$$

Now, to prove the lemma, we use induction on j. For $j = 1$, we have

$$\begin{aligned} d &= s_1 v_1 t_1 \text{ (by zigzag equations (2.4))} \\ &= s_1 s_1 v_1 v_1 s_1 v_1 t_1 \text{ (by equation (4.1))}. \end{aligned}$$

Therefore the result holds for $j = 1$. Assume inductively that the result is true for $j = l < m$. We now show that the result also holds for $j = l + 1$. Now

$$\begin{aligned} d &= (\prod_{i=1}^{l} s_i s_i v_{2i-1} v_{2i-1}) s_l v_{2l-1} t_l \text{ (by inductive hypothesis)} \\ &= (\prod_{i=1}^{l} s_i s_i v_{2i-1} v_{2i-1}) s_{l+1} v_{2l+1} t_{l+1} \text{ (by zigzag equations (2.4))} \\ &= (\prod_{i=1}^{l} s_i s_i v_{2i-1} v_{2i-1}) s_{l+1} s_{l+1} v_{2l+1} v_{2l+1} s_{l+1} v_{2l+1} t_{l+1} \\ &\qquad \text{(by equation (4.1))} \\ &= (\prod_{i=1}^{l+1} s_i s_i v_{2i-1} v_{2i-1}) s_{l+1} v_{2l+1} t_{l+1}. \end{aligned}$$

Therefore the result also holds for $j = l + 1$, as required. □

Lemma 4.3. *For all $k = 1, 2, \ldots, m$,*

$$d = (\prod_{i=1}^{k} x_{(k-i)+1} x_{(k-i)+1} u_{2(k-i)+1} u_{2(k-i)+1}) x_1 u_1 y_1.$$

Proof. It follows on similar lines as the proof of the Lemma 4.2, by using the zigzag equations (2.3). □

We now return to the proof of the theorem and let $j = m'$ in Lemma 4.2, we get

$$\begin{aligned}
d &= (\prod_{i=1}^{m'} s_i s_i v_{2i-1} v_{2i-1}) s_{m'} v_{2m'-1} t_{m'} \\
&= (\prod_{i=1}^{m'} s_i s_i v_{2i-1} v_{2i-1}) s_{m'} v_{2m'} \text{ (by zigzag equations (2.4))} \\
&= (\prod_{i=1}^{m'-1} s_i s_i v_{2i-1} v_{2i-1}) s_{m'} s_{m'} v_{2m'-1} v_{2m'-1} s_{m'} v_{2m'} \\
&= (\prod_{i=1}^{m'-1} s_i s_i v_{2i-1} v_{2i-1}) s_{m'} s_{m'} v_{2m'-1} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&= (\prod_{i=1}^{m'-1} s_i s_i v_{2i-1} v_{2i-1}) s_{m'} v_{2m'-1} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&\geq (\prod_{i=1}^{m'-1} s_i s_i v_{2i-1} v_{2i-1}) s_{m'-1} v_{2m'-2} v_{2m'-1} v_{2m'} \\
&\qquad\qquad\qquad\qquad \text{(by zigzag inequalities (2.2))} \\
&\vdots \\
&\geq (\prod_{i=1}^{2} s_i s_i v_{2i-1} v_{2i-1}) s_2 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \\
&= s_1 s_1 v_1 v_1 s_2 s_2 v_3 v_3 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&= s_1 s_1 v_1 v_1 s_2 v_3 v_3 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&\geq s_1 s_1 v_1 v_1 s_1 v_2 v_3 v_4 \cdots v_{2m'-2} v_{2m'} \text{ (by zigzag inequalities (2.2))} \\
&= s_1 s_1 v_1 v_1 v_2 v_3 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&= s_1 v_1 v_2 v_3 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \text{ (as } S \in \mathcal{V}_1) \\
&\geq v_0 v_1 v_2 v_3 v_4 \cdots v_{2m'-2} v_{2m'-1} v_{2m'} \text{ (by zigzag inequalities (2.2))} \\
&= (\prod_{i=0}^{m'-1} v_{2_i} v_{2_i+1}) v_{2m'}.
\end{aligned}$$

Now on similar lines, by using the Lemma 4.3 and the zigzag inequalities (2.1), we obtain $d \leq (\prod_{i=0}^{m-1} u_{2(m-i)} u_{2(m-i)-1}) u_0$. Therefore $d \in U$, as U is convex in S, a contradiction as required. □

Dually we can prove the following theorem.

Theorem 4.4. *The variety $\mathcal{V}_2$ of posemigroups satisfying the identity $xyz = yxzz$ is closed if it is $\mathcal{V}_2$-convex.*

From Theorems 4.1 and 4.4, we have the following immediate corollaries.

Corollary 4.5. *The variety of po-left [po-right] normal bands is closed in the variety $\mathcal{V}_1$ [$\mathcal{V}_2$] of posemigroups if it is $\mathcal{V}_1$-convex [$\mathcal{V}_2$-convex].*

Corollary 4.6. *The variety $\mathcal{V} = \mathcal{V}_1 \cap \mathcal{V}_2$ of posemigroups is closed.*

Theorem 4.7. *The variety $\mathcal{U}_1$ of posemigroups satisfying the identity $xyz = xzyy$ is closed if it is $\mathcal{U}_1$-convex.*

Proof. Let $U,\ S \in \mathcal{U}_1$ such that U is convex subposemigroup of S. We show that $\widehat{Dom}(U,S) = U$. Assume on contrary and take any $d \in \widehat{Dom}(U,S) \setminus U$. Let (2.1) and (2.2) be the zigzag inequalities in S over U with value d of length (m,m'). In order to prove the theorem, we first prove the following lemmas.

Lemma 4.8. *For all $k = 1, 2, \cdots, m$, $d = x_k u_{2k-1} y_k (\prod_{i=1}^{k} u_{2(k-i)+1} u_{2(k-i)+1})$.*

Proof. For any $x, y, z \in S$, we have

$$\begin{aligned} xyz &= xzyy \text{ (as } S \in \mathcal{U}_1) \\ &= xyyzz \text{ (as } S \in \mathcal{U}_1) \\ &= xyzzyy \text{ (as } S \in \mathcal{U}_1) \\ &= xzyyy \text{ (as } S \in \mathcal{U}_1) \\ &= xyyzyzy \text{ (as } S \in \mathcal{U}_1) \\ &= xyzyy \text{ (as } S \in \mathcal{U}_1). \end{aligned} \tag{4.2}$$

To prove the lemma we use induction on k. For $k = 1$, we have

$$\begin{aligned} d &= x_1 u_1 y_1 \text{ (by zigzag equations (2.3)} \\ &= x_1 u_1 y_1 u_1 u_1 \text{ (by equation (4.2)).} \end{aligned}$$

Therefore the result holds for $k = 1$. Assume inductively that the result is true for $k = l < m$. We now show that the result also holds for $k = l + 1$. Now

$$\begin{aligned} d &= x_l u_{2l-1} y_l (\prod_{i=1}^{l} u_{2(l-i)+1} u_{2(l-i)+1}) \text{ (by inductive hypothesis)} \\ &= x_{l+1} u_{2l+1} y_{l+1} (\prod_{i=1}^{l} u_{2(l-i)+1} u_{2(l-i)+1}) \text{ (by zigzag equations (2.3))} \end{aligned}$$

$$= x_{l+1}u_{2l+1}y_{l+1}u_{2l+1}u_{2l+1}(\prod_{i=1}^{l} u_{2(l-i)+1}u_{2(l-i)+1}) \text{ (by equation (4.2))}$$

$$= x_{l+1}u_{2l+1}y_{l+1}(\prod_{i=1}^{l+1} u_{2(l-i)+1}u_{2(l-i)+1}).$$

Therefore the result also holds for $k = l+1$, as required. □

Lemma 4.9. *For all* $k = 1, 2, \cdots, m$,

$$d = x_1u_1y_1(\prod_{i=1}^{j} u_{2i-1}u_{2i-1}).$$

Proof. It follows on similar lines as the proof of the Lemma 4.8 and using the zigzag equations (2.3). □

We now return to the proof of the theorem and let $k = m$ in Lemma 4.8, we get

$$d = x_mu_{2m-1}y_m(\prod_{i=1}^{m} u_{2(m-i)+1}u_{2(m-i)+1})$$

$$= u_{2m}y_mu_{2m-1}u_{2m-1}(\prod_{i=2}^{m} u_{2(m-i)+1}u_{2(m-i)+1})$$

(by zigzag equations (2.3))

$$= u_{2m}u_{2m-1}y_m(\prod_{i=2}^{m} u_{2(m-i)+1}u_{2(m-i)+1}) \text{ (as } S \in \mathcal{U}_1))$$

$$\geq u_{2m}u_{2m-2}y_{m-1}(\prod_{i=2}^{m} u_{2(m-i)+1}u_{2(m-i)+1}) \text{ (by zigzag inequalities (2.1))}$$

$$= (\prod_{i=0}^{1} u_{2(m-i)})y_{m-1}u_{2m-3}u_{2m-3}(\prod_{i=3}^{m} u_{2(m-i)+1}u_{2(m-i)+1})$$

$$\vdots$$

$$= (\prod_{i=0}^{m-2} u_{2(m-i)})y_2u_3u_3u_1u_1$$

$$= (\prod_{i=0}^{m-2} u_{2(m-i)})u_3y_2u_1u_1 \text{ (as } S \in \mathcal{U}_1)$$

$$\geq (\prod_{i=0}^{m-2} u_{2(m-i)})u_2y_1u_1u_1 \text{ (by zigzag inequalities (2.1))}$$

$$= (\prod_{i=0}^{m-1} u_{2(m-i)})u_1 y_1 \text{ (as } S \in \mathcal{U}_1)$$

$$\geq (\prod_{i=0}^{m} u_{2(m-i)} \text{ (by zigzag inequalities (2.1))}.$$

Now on similar lines by using the Lemma 4.9 and the reverse zigzag inequalities (2.1), we obtain $d \leq \prod_{i=0}^{m} u_{2i}$. Therefore $d \in U$, as U is convex in S, a contradiction as required. $\square$

Dually we can prove the following theorem.

Theorem 4.10. *The variety $\mathcal{U}_2$ of posemigroups satisfying the identity $xyz = yyxz$ is closed if it is $\mathcal{U}_2$-convex.*

From Theorems 4.7 and 4.10, we have the following immediate corollaries.

Corollary 4.11. *The variety of po-left [po-right] normal bands is closed in the variety $\mathcal{U}_1$ [$\mathcal{U}_2$] of posemigroups if it is $\mathcal{U}_1$-convex [$\mathcal{U}_2$-convex].*

Corollary 4.12. *The variety $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$ of posemigroups is closed.*

Acknowledgments

The authors would like to thank the anonymous referee for his careful reading and making critical suggestions which helped in improving the presentation of this paper.

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