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JORDAN ALGEBRA BUNDLES AND JORDAN RINGS

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Abstract. In this paper, We define Jordan algebra bundles of finite type and we give one-one correspondence between Jordan algebra bundles of finite type and Jordan rings.

1. Introduction

In modern mathematics, an important notion is that of non-associative algebra. This variety of algebras is characterized by the fact the product of elements verifies a more general law than the associativity law.

There are two important classes of non-associative algebras: Lie algebras (introduced in 1870 by the Norwegian mathematician Sophus Lie in his study of the groups of transformations) and Jordan algebras (introduced in 1932-1933 by the German physicist Pasqual Jordan (1902-1980) in his algebraic formulation of quantum mechanics [2, 3, 4]). These two algebras are interconnected, as it was remarked for instance by Kevin McCrimmon [10, p.622]:

> "We are saying that if you open up a Lie algebra and look inside, 9 times out of 10 there is a Jordan algebra (of pair) which makes it work".

J.P. Serre [14] has shown that there is a one to one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its co-ordinate ring.

Richard G. Swan [15] has shown that a similar correspondence exists between topological vector bundles over a compact Hausdorff space

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X and finitely generated projective modules over the ring of continuous real valued functions on X. Later Goodearl [8] observed that the equivalence holds in the more general case of para compact Hausdorff space X if one restricts to the bundle of finite type (i.e. there exists a finite open covering T of X such that the restriction of the bundle to each $U \in T$ is trivial). This restriction excludes vector bundles of unbounded dimension which cannot come from the category of finitely generated $C(X)-$modules.

In 1986, L.N. Vaserstein [16] has extended this result to an arbitrary topological space X, with a appropriate definition of finite type. According to Vaserstein, the bundle over an arbitrary space X is of finite type if there is a finite partition S of 1 on X (that is a finite set S of nonnegative continuous functions on X whose sum is 1) such that the restriction of bundle to the set $\{s \in X | f(s) \neq 0\}$ is trivial for each f in S.

For example, a bundle over a compact Hausdorff space X is of finite type always. The definitions of finite type of Vaserstein and Goodearl are equivalent if the space X is normal.

A Jordan algebra bundle is a vector bundle $\xi = (\xi, p, X)$ together with a vector bundle morphism $\theta : \xi \oplus \xi \to \xi$ inducing a Jordan algebra structure on each fibre ξ_x, $x \in X$. By a trivial Jordan algebra bundle we mean a trivial vector bundle $(X \times J, p, X)$, where J is a Jordan algebra. A locally trivial Jordan algebra bundle ξ is a vector bundle $\xi = (\xi, p, X)$ in which each fibre is a Jordan algebra and for each $x \in X$, there exists an open neighborhood U of x, a Jordan algebra J and a homeomorphism $\phi : U \times J \to p^{-1}(U)$ such that ϕ restricted to each fibre is a Jordan algebra isomorphism.

In this paper, We define Jordan algebra bundles of finite type and we give one-one correspondence between Jordan algebra bundles of finite type and Jordan rings which is also finitely generated projective modules.

Notations and Terminologies: Our base field is field of real numbers and all fibres of underlying bundles are finite dimensional. We denote Jordan algebra bundle (ξ, p, X, θ) by ξ, where $\theta : \xi \oplus \xi \to \xi$ is the morphism giving the Jordan multiplication on each fibre. All bundles, subbundles, ideal bundles, all have the same topological space X as base space unless otherwise stated. All bundles, subbundles and ideal bundles are locally trivial bundles unless otherwise mentioned.

2. Jordan Algebra Bundles and Jordan Rings

Definition 2.1. A Jordan algebra bundle ξ over a base space X is said to be of finite type if there is a finite partition S of 1 of X such that the restriction of the bundle to the set $\{x \in X | f(x) \neq 0\}$ is trivial Jordan algebra bundle for each function $f \in S$.

First, we show that given a Jordan ring P over $C(X)$, the ring of all continuous real valued functions on X, we can find a Jordan algebra bundle ξ over X of finite type such that P is isomorphic to the Jordan ring of the set of all sections of ξ denoted by $\Gamma(\xi)$, where ξ is a Jordan algebra bundle if and only if P is a finitely generated projective module over $C(X)$.

Remark 2.2. Let us observe that the set of all sections of ξ denoted by $\Gamma(\xi)$, where ξ is a Jordan algebra bundle is a Jordan ring : Given two sections $S_1, S_2 : X \to \xi$ we define $S_1 * S_2 : X \to \xi$ by $(S_1 * S_2)(x) = \theta(S_1(x), S_2(x))$ in ξ_x, $x \in X$. The mapping $S_1 * S_2 : X \to \xi$ is continuous since it is the following composition

$$X \xrightarrow{S_1 \oplus S_2} \xi \oplus \xi \xrightarrow{\theta} \xi.$$

Theorem 2.3. *Every Jordan ring is isomorphic to the Jordan ring of all sections of a Jordan algebra bundle of finite type when the given Jordan ring is finitely generated and is a projective module over the ring of all real valued continuous functions defined on the base space of the bundle.*

Proof. Since P is a finitely generated projective module over $\mathcal{C}(X)$, P is isomorphic to the column space of a square hermitian idempotent matrix $e = e^2 = e^*$ over $\mathcal{C}(X)$ [16]. Every finitely generated projective module P over $\mathcal{C}(X)$ gives an vector bundle ξ over X whose fibre at x is just $e(x)F^N$.

$\xi = \bigcup_{x \in X} e(x)F^N$ is locally trivial.

Let $x,\ y \in X$ and call $g(x,y) = e(x)e(y) + (I - e(x))(I - e(y))$, where I denotes the identity. Since $e(x),\ e(y)$ are idempotent, we can have

$$e(x)g(x,y) = e(x)e(y) = g(x,y)e(y).$$

Now fix x, since $g(x,x) = I$, there is an open neighbourhood U of x such that $g(x,y)$ is invertible for every $y \in U$.
For any $y \in U$ we have $e(y) = g(x,y)^{-1}e(x)g(x,y)$. So over U we get vector bundle isomorphism

$$U \times e(x)F^N \xrightarrow{\Phi} \xi|_U; \qquad (y,v) \xrightarrow{\Phi_y} (y, g(x,y)^{-1}v)$$

We now show that $\xi = \bigcup_{x \in X} e(x)F^N$ is vector bundle of finite type.

Let consider the set $Y = \{p \in M_n(F) |\ p = p^2 = p^*\}$ which is compact. Therefore there is a finite partition of unity $f_1, f_2, \cdots, f_n$ such that $|p - q| < \frac{1}{3}$ whenever $f_i(p)f_i(q) \neq 0$ for some i.

The matrix e above can be considered as a continuous map $X \to Y$, the $f_i \circ e$ make up a finite partition of unity on X.

Now we will show that ξ is trivial on every $U_i = \{x \in X | f_i \circ e(x) \neq 0\}$.

For any $x,\ y \in X$ we consider as above $g(x,y) = e(x)e(y) + (I - e(x))(I - e(y))$ with $e(x)g(x,y) = g(x,y)e(y)$. Also

$$\begin{aligned} g(x,y) &= e(x)e(y) + (I - e(x))(I - e(y)) \\ &= 2e(x)e(y) + I - e(x) - e(y) \\ &= I + (e(x) - e(y))(I - 2e(y)). \end{aligned}$$

Now, if $x, y \in U_i$ by definition of f_i and U_i we have

$$|g(x,y) - I| \leq |e(x) - e(y)||I - 2e(y)| < \frac{1}{3}.3 = 1$$

Thus $g(x,y)$ is invertible for $x, y \in U_i$.

Also for any $x, y \in U_i$, we have $e(y) = g(x,y)^{-1}e(x)g(x,y)$. As above we have the triviality on U_i.

We can give a Jordan algebra bundle structure on ξ in the following way:

Let $I_x = \{\alpha \in \mathcal{C}(X) | \alpha(x) = 0\}$ be the maximal ideal of $\mathcal{C}(X)$ attached to $x \in X$. Then P/I_xP is isomorphic to $e(x)(F^N)$ given by the mapping $G_x : P/I_xP \to e(x)F^N$ defined by $G_x[e(f_1, f_2, \cdots, f_N) + I_xP] = e(x)(f_1(x), f_2(x), \cdots, f_N(x))$ which is an isomorphism of vector spaces [15].

Given two elements $e(x)(s),\ e(x)(t) \in e(x)F^N$ we can define the multiplication

$$\theta_x(e(x)(s), e(x)(t)) = G_x(G_x^{-1}(e(x)(s)) * G_x^{-1}(e(x)(t))),$$

where " $*$ " is the Jordan multiplication on P.

Hence $e(x)(F^N)$ has the structure of a Jordan algebra as it inherits the Jordan multiplication which we denote by θ_x from P and is having a vector space structure over F. Now let us define $\theta : \xi \to \xi$ as $\theta(u,v) = \theta_x(u,v)$, if u, v belong to $e(x)(F^N)$.

The continuity of θ follows from the commutative diagram

$$\begin{array}{ccc} U \times (e(x)F^N \times e(x)F^N) & \xrightarrow{\Phi\times\Phi} & \bigcup_{y\in U}(e(y)F^N \times e(y)F^N) \\ {\scriptstyle (I,\theta_x)}\downarrow & & \downarrow{\scriptstyle \theta} \\ U \times e(x)F^N & \xrightarrow{\Phi} & \bigcup_{y\in U} e(y)F^N \end{array}$$

Hence the theorem.

□

Theorem 2.4. *For every Jordan algebra bundle there corresponds a Jordan ring which is a finitely generated and is projective module over the ring of all real valued continuous functions defined on the base space of the bundle.*

Proof. Let ξ be a Jordan algebra bundle of finite type. By [16, Lemma 7], the set of sections $\Gamma(\xi)$ is finitely generated projective $\mathcal{C}(X)-$module. By the Remark 2.2, $\Gamma(\xi)$ possesses a Jordan multiplication. Hence it is a Jordan ring. □

Theorem 2.5. *Every Jordan algebra bundle homomorphism gives rise to a Jordan ring module homomorphism between the associated Jordan rings and vice versa.*

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