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GENERALIZED ORTHOGONAL GRAPHS OF CHARACTERISTIC A POWER OF 2

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ABSTRACT. Let R be a finite local ring of characteristic a power of 2 with the residue field k. In this paper, we define a generalized orthogonal graph on a module of rank at least 2 over R. Then we study its graph properties via the same graph over k. The number of vertices and the valency of each vertex in this graph over R are computed. We also prove that this graph is arc transitive and find its diameter. Moreover, the first subconstituent of this orthogonal graph is considered. We show that it is a generalized strongly regular graph.

1. INTRODUCTION

Graphs defined from the geometry of classical groups over finite fields and rings have been widely studied recently. The collinearity graphs of finite classical palar spaces are well-known strongly regular graphs [2, 9]. For more details about strongly regular graphs see [4]. Orthogonal graphs over finite fields of odd characteristic were studied in [5, 6, 7, 14] by using the geometry of orthogonal group. Li et al. [13] initiated the study of orthogonal graphs over finite Galois rings of odd characteristic. In 2016, Meemark and Sriwongsa [16] extended this work to the orthogonal graphs over finite commutative rings of odd characteristic which has an application in a construction of cartesian authentication codes [12]. For the case of even characteristic, orthogonal graphs over finite fields of characteristic 2 were studied in

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[8, 20]. Recently, the graphs over $\mathbb{Z}_{2^n}$ were investigated in [18]. As a development, many authors have considered orthogonal graphs in more general cases. Huo and Zhang presented orthogonal graphs of type $(m, m-1, 0)$ over finite fields of odd characteristic in [10]. Later on, generalized orthogonal graphs over finite commutative rings were introduced and analyzed by Sirisuk and Meemark [17]. The authors defined the graphs by using a rank of matrices over commutative rings, namely McCoy rank [15]. This rank generalizes the usual rank of matrices over fields. Most recently, orthogonal graphs of type $(m, m-1, 0)$ over finite fields of characteristic 2 were studied in [11]. Note that, recently, many researchers have studied graphs from modules over commutative rings (see for example [3]). Motivated by the above research, in this paper, we consider the analogous problem of generalized orthogonal graphs over finite local rings of characteristic a power of 2 by using the McCoy rank described as follows.

The rank of matrices over commutative rings with identity was introduced by McCoy [15]. It is defined via the annihilators of ideals as follows. For any ideal I of a commutative ring R with identity, the *annilhilator* of I is given by

$$\mathrm{Ann}_R(I) := \{r \in R \mid \forall a \in I, ra = 0\}.$$

Let A be an $m \times n$ matrix over R and consider the following descending chain

$$R = I_0(A) \supseteq I_1(A) \supseteq \cdots \supseteq I_{\min\{m,n\}}(A),$$

where $I_i(A)$ is the ideal of R generated by the $i \times i$ minors of A for $1 \leq i \leq \min\{m, n\}$. We have

$$\{0\} = \mathrm{Ann}_R(I_0(A)) \subseteq \mathrm{Ann}_R(I_1(A)) \subseteq \cdots \subseteq \mathrm{Ann}_R(I_{\min\{m,n\}}(A)).$$

The *rank* of A, denoted $\mathrm{rk}(A)$, is the largest integer i such that

$$\mathrm{Ann}_R(I_i(A)) = \{0\}.$$

The following statements are well-known and important to this paper.

(1) If R is a field, then $\mathrm{rk}(A)$ is equal to the usual rank of A.
(2) If R is a finite local ring with the unique maximal ideal M, then $\mathrm{rk}(A)$ is equal to $\mathrm{rk}(\pi(A))$ over its residue field $\pi(R) = R/M$, where π is the usual canonical map [1].

The structure of this paper: We define and study a generalized orthogonal graph in Section 2. This includes the computation on number of vertices and degree. We also analyze arc transitivity and find the diameter of the graphs. In Section 3, the first subconstituent of this generalized orthogonal graph is studied. We use similar technique as in Section 2 to determine all parameters of this subconstituent.

2. Generalized orthogonal graphs

Let R be a finite local ring of characteristic a power of 2 with the unique maximal ideal M. We denote $R^{\times} = R \setminus M$ the units group of R. An $n \times n$ matrix over $K = (k_{ij})$ over R is said to be *alternate* if $K^T = K$ and $k_{ii} = 0$ for all $i = 1, 2, \ldots, n$. Here, K^T is the transpose of K.

For a positive integer $\nu \geq 2$ and $\delta = 0, 1$ or 2, let $R^{(2\nu+\delta)} = R \oplus R \oplus \cdots \oplus R$ be the free R-module of rank $2\nu + \delta$. Let $G_{2\nu+\delta}$ be the $(2\nu + \delta) \times (2\nu + \delta)$ matrix over R given by

$$G_{2\nu+\delta} = \begin{pmatrix} 0 & I_\nu & \\ & 0 & \\ & & \Delta \end{pmatrix},$$

where

$$\Delta = \begin{cases} \emptyset(\text{disappear}) & \text{if } \delta = 0, \\ (1) & \text{if } \delta = 1, \\ \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix} & \text{if } \delta = 2, \end{cases}$$

and z is a fixed element in $R \setminus N$ where $N = \{x^2 + x \mid x \in R\}$. For convenience, we sometimes write G for $G_{2\nu+\delta}$. A free submodule X of $R^{(2\nu+\delta)}$ is called a *totally singular submodule* if XGX^T is alternate. For more details of these terminologies over finite fields, the reader is referred to [19].

Let X be a free submodule of $R^{(2\nu+\delta)}$ of rank $m \geq 2$ with basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$. Then $X = R\vec{x}_1 \oplus R\vec{x}_2 \oplus \cdots \oplus R\vec{x}_m$ and by abuse of notation, we use X to denote an $m \times (2\nu + \delta)$ matrix containing $\vec{x}_i$ in its ith row, i.e.

$$X = \begin{pmatrix} \vec{x}_1 \\ \vec{x}_2 \\ \vdots \\ \vec{x}_m \end{pmatrix}.$$

If Y is another free submodule of $R^{(2\nu+\delta)}$ of rank m, we let $\binom{X}{Y}$ denote a $2m \times (2\nu + \delta)$ matrix whose rows are from the matrices X and Y, respectively.

Huo and Zhang [11] recently introduced the orthogonal graph of type $(m, m-1, 0)$ with respect to $G_{2\nu+\delta}$ over a finite field of characteristic 2. Its vertex set is the set of all m-dimensional totally singular subspaces and any two distinct vertices X and Y are adjacent if and only if

$$\text{rk}(X(G + G^T)Y^T) = 0 \text{ and } \dim(X \cap Y) = m - 1.$$

Note that $\dim(X \cap Y) = m - 1$ is equivalent to $\mathrm{rk}\binom{X}{Y} = m + 1$. Now, let's turn our attention to ring case. In general, the intersection of two submodules X and Y over a finite local ring may not be free, even both X and Y are free. Thus, the dimension may be invalid. However, the rank $\mathrm{rk}\binom{X}{Y}$ is always valid. Therefore, we can generalize this orthogonal graph to the case of finite local ring using McCoy rank in the following way.

Let m be a positive integer such that $\nu > m$. A *generalized orthogonal graph of* $R^{(2\nu+\delta)}$, denoted $\mathcal{O}_m^{2\nu+\delta}(R)$, is the graph whose vertex set is the set of totally singular free submodules of $R^{(2\nu+\delta)}$ of rank m and its adjacency condition is given by

$$X \text{ is adjacent to } Y \iff \mathrm{rk}(X(G+G^T)Y^T) = 0 \text{ and } \mathrm{rk}\begin{pmatrix} X \\ Y \end{pmatrix} = m+1.$$

For convenience, we sometimes use $\mathcal{V}(\mathcal{O}_m^{2\nu+\delta}(R))$ for the vertex set of this graph. Note that if R is a finite field, then $\mathcal{O}_m^{2\nu+\delta}(R)$ is an orthogonal graph of type $(m, m-1, 0)$ in [11]. To extend the results from this reference, we consider the following preparation about a local ring.

Let $k = R/M$ be the residue field of R. Note that $u + m \in R^\times$ for all $u \in R^\times$ and $m \in M$. In this paper, we mainly use the canonical map $\pi : R \to k$ defined by $\pi(r) = r + M =: \overline{r}$ for all $r \in R$ to investigate relations between a generalized orthogonal graph over R and its induced graph over k.

Proposition 2.1. *Let $r \in R$. Then $r \in \{x^2 + x \mid x \in R\}$ if and only if $\overline{r} \in \{\bar{y}^2 + \bar{y} \mid \bar{y} \in k\}$.*

Proof. Note that for $x, y \in R$, if $x^2+x = y^2+y$, then $(x-y)(x+y+1) = 0$. Since $x - y$ or $x + y + 1$ is a unit, it follows that either $x = y$ or $x = -y - 1$. Conversely, if $y = -x - 1$, then $y^2 + y = x^2 + x$. It is obvious that $x \neq -x - 1$, otherwise $2x = -1$, which contradicts the fact $\mathrm{char}(R)$ is a power of 2. Thus, for $x, y \in R, y^2 + y = x^2 + x$ if and only if $y = x$ or $y = -x - 1$. Therefore, the map $x \mapsto x^2 + x$ defined on R, and likewise on k, is two-to-one. Let

$$S = \{x^2 + x \mid x \in R\} \text{ and } \overline{S} = \{\bar{y}^2 + \bar{y} \mid \bar{y} \in k\}.$$

It is clear that $S \subseteq \pi^{-1}(\overline{S})$. However,

$$|\pi^{-1}(\overline{S})| = |M||\overline{S}| = \frac{|M||k|}{2} = \frac{|M||R|}{2|M|} = \frac{|R|}{2} = |S|.$$

Therefore, $S = \pi^{-1}(\overline{S})$ which proves the proposition. $\square$

Now, we consider each vertex of the graph $\mathcal{O}_m^{2\nu+\delta}(R)$. Let X be a vertex in $\mathcal{O}_m^{2\nu+\delta}(R)$. Then

$$X = R\vec{x}_1 \oplus R\vec{x}_2 \oplus \cdots \oplus R\vec{x}_m$$

where $\vec{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,2\nu+\delta}), i = 1, \ldots, m$, for some $x_{i,1}, \ldots, x_{i,2\nu+\delta} \in R$. Then $\vec{x}_i$'s must be linearly independent and so, $x_{i,j} \in R^\times$ for some $j \in \{1, 2, \ldots, 2\nu + \delta\}$. In fact, we have the following lemma.

Lemma 2.2. *Under the above setting, there exists $j \in \{1, 2, \ldots, 2\nu\}$ such that $x_{i,j} \in R^\times$.*

Proof. The result is immediate when $\delta = 0$. If $\delta = 1$ and $x_{i,1}, \ldots, x_{i,2\nu} \in M$, then, since $\vec{x}_i G \vec{x}_i^T = 0$,

$$x_{i,1}x_{i,\nu+1} + x_{i,2}x_{i,\nu+2} + \cdots + x_{i,2\nu}x_{i,2\nu} + x_{i,2\nu+1}^2 = 0$$

implies $x_{i,2\nu+1}$ is also an element in M which is a contradiction. Now assume that $\delta = 2$ and $x_{i,1}, x_{i,2}, \ldots, x_{i,2\nu} \in M$. Then

$$x_{i,1}x_{i,\nu+1}+x_{i,2}x_{i,\nu+2}+\cdots+x_{i,2\nu}x_{i,2\nu}+zx_{i,2\nu+1}^2+x_{i,2\nu+1}x_{i,2\nu+2}+zx_{i,2\nu+2}^2 = 0$$

implies $zx_{i,2\nu+1}^2 + x_{i,2\nu+1}x_{i,2\nu+2} + zx_{i,2\nu+2}^2 \in M$ and so

$$\bar{z}\bar{x}_{i,2\nu+1}^2 + \bar{x}_{i,2\nu+1}\bar{x}_{i,2\nu+2} + \bar{z}\bar{x}_{i,2\nu+2}^2 = \bar{0}$$

in the residue field k. Without loss of generality, we may assume that $\bar{x}_{i,2\nu+1} = \bar{1}$. Then $\bar{z}^2(1 + \bar{x}_{i,2\nu+2})^2 = \bar{z}\bar{x}_{i,2\nu+2}$ and so,

$$\bar{z}^2(1 + \bar{x}_{i,2\nu+2})^2 + \bar{z}(1 + \bar{x}_{i,2\nu+2}) + \bar{z} = \bar{0}$$

which contradicts the condition of the element z and Proposition 2.1. ☐

Remark. The argument in the proof of this Lemma also applies in [18]. Therefore, one can see the results there hold for any finite local ring of characteristic a power of 2 as well.

Since $k = R/M$ is the residue field of R, the matrix G over R induces the matrix $\overline{G}$ over k in an obvious manner via the canonical map $\pi : R \to k$ by Proposition 2.1. Moreover, we have

$$\pi(\vec{x})(\overline{G} + \overline{G}^T)\pi(\vec{y}^T) = \pi(\vec{x}(G + G^T)\vec{y}^T)$$

for all $\vec{x}, \vec{y} \in R^{(2\nu+\delta)}$. Here, we write

$$\pi(\vec{x}) = (\pi(x_1), \pi(x_2), \ldots, \pi(x_{2\nu+\delta}))$$

for all $\vec{x} = (x_1, x_2, \ldots, x_{2\nu+\delta}) \in R^{(2\nu+\delta)}$. Moreover, for any vertex $X = R\vec{x}_1 \oplus R\vec{x}_2 \oplus \cdots \oplus R\vec{x}_m$, we have $\pi(X) = k\pi(\vec{x}_1) \oplus k\pi(\vec{x}_2) \oplus \cdots \oplus k\pi(\vec{x}_m)$.

The main purpose of this work is to extend the results from [11]. So, from now on, we only consider $\delta = 0$ or 2 likewise.

The following lemma provides the relation between the graph $\mathcal{O}_m^{2\nu+\delta}(R)$ and the graph $\mathcal{O}_m^{2\nu+\delta}(k)$. One can see that, in some sense, the first graph is a lift of the latter one with a restriction for all vertices being totally sigular.

Lemma 2.3. *(Lifting Lemma) By the above setting, we have the following statements.*

(1) If X is a vertex of $\mathcal{O}_m^{2\nu+\delta}(R)$, then there are $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}$ many vertices which are lifts of the vertex $\pi(X)$ of $\mathcal{O}_m^{2\nu+\delta}(k)$, i.e.,

$$|\{Y \in \mathcal{V}(\mathcal{O}_m^{2\nu+\delta}(R)) \mid \pi(X) = \pi(Y)\}| = |M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}.$$

(2) Two vertices X and Y are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(R)$ if and only if the vertices $\pi(X)$ and $\pi(Y)$ are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(k)$.

Proof. Using Lemma 2.2, the proof can be done analogously to the proofs of Theorem 3.3 and Theorem 3.4 of [17] by replacing s by m. We recall some important details here.

Suppose that a vertex X of $\mathcal{O}_m^{2\nu+\delta}(R)$ is of the form

$$X = R\vec{x}_1 \oplus R\vec{x}_2 \oplus \cdots \oplus R\vec{x}_m.$$

Then a lift of X is of the form

$$R(\vec{x}_1 + \vec{m}_1) \oplus R(\vec{x}_2 + \vec{m}_2) \oplus \cdots \oplus R(\vec{x}_m + \vec{m}_m)$$

where $\vec{m}_i \in M^{2\nu+\delta}$ and it must be totally singular. Thus, we have $|M|^{(2\nu+\delta)m} - \binom{m+1}{2}$ choices of $\vec{m}_i$'s. However, some of choices give the same lifts. Hence, the number is reduced to $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}$. This proves the first statement.

For (2), we note that X and Y are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(R)$ if and only if

$$\operatorname{rk}(X(G+G^T)Y^T) = 0 \text{ and } \operatorname{rk}\begin{pmatrix} X \\ Y \end{pmatrix} = m+1$$

if and only if

$$\operatorname{rk}(\pi(X)(\bar{G}+\bar{G}^T)\pi(Y)^T) = 0 \text{ and } \operatorname{rk}\begin{pmatrix} \pi(X) \\ \pi(Y) \end{pmatrix} = m+1$$

if and only if $\pi(X)$ and $\pi(Y)$ are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(k)$. □

According to Huo and Zhang (Proposition 2.3 and Corollary 3.2 of [11]), the graph $\mathcal{O}_m^{2\nu+\delta}(k)$ is a regular graph with

$$n_k(2\nu+\delta, m) = \frac{\prod_{i=\nu-m+1}^{\nu}(|k|^i - 1)(|k|^{i+\delta-1} + 1)}{\prod_{i=1}^{m}(|k|^i - 1)}$$

many vertices and the valency of each vertex is

$$d_k(2\nu+\delta,m)=\frac{(|k|^{2\nu-2m+\delta}-|k|)(|k|^m-1)}{(|k|-1)^2}.$$

In the following theorem, we find the number of vertices and the valency of each vertex of the graph $\mathcal{O}_m^{2\nu+\delta}(R)$ by using the lifting lemma.

Theorem 2.4. *The graph $\mathcal{O}_m^{2\nu+\delta}(R)$ is $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}d_k(2\nu+\delta,m)$-regular on*

$$|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}n_k(2\nu+\delta,m)$$

many vertices.

Proof. By Lemma 2.3 (1), each vertex of $\mathcal{O}_m^{2\nu+\delta}(k)$ can be lifted to

$$|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}$$

vertices in $\mathcal{O}_m^{2\nu+\delta}(R)$. Thus the number of vertices of $\mathcal{O}_m^{2\nu+\delta}(R)$ is

$$|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}n_k(2\nu+\delta,m).$$

Since the graph $\mathcal{O}_m^{2\nu+\delta}(k)$ is $d_k(2\nu+\delta,m)$-regular, Lemma 2.3 (2) implies that $\mathcal{O}_m^{2\nu+\delta}(R)$ is regular of degree $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}d_k(2\nu+\delta,m)$ □

A graph G is *vertex transitive* if its automorphism group acts transitively on the vertex set. That is, for any two vertices of G, there is an automorphism carrying one to the other. An *arc* in G is an ordered pair of adjacent vertices, and G is *arc transitive* if its automorphism group acts transitively on its arcs.

Lemma 2.5. [11] *The orthogonal graph $\mathcal{O}_m^{2\nu+\delta}(k)$ is vertex transitive and arc transitive.*

For any set A, we denote the set of all permutations on A by $\mathrm{Sym}(|A|)$. For any vertex $X=R\vec{x}_1\oplus R\vec{x}_2\oplus\cdots\oplus R\vec{x}_m$ in $\mathcal{O}_m^{2\nu+\delta}(R)$, let

$$X^M:=\{R(\vec{x}_1+\vec{m}_1)\oplus\cdots\oplus R(\vec{x}_m+\vec{m}_m)\mid \vec{m}_i\in M^{(2\nu+\delta)}\ \text{ and }\ R(\vec{x}_1+\vec{m}_1)\oplus\cdots\oplus R(\vec{x}_m+\vec{m}_m)\text{ is totally singular}\}.$$

This is the set of all vertex in $\mathcal{O}_m^{2\nu+\delta}(R)$ which are the lifts of the vertex $k(\pi(\vec{x_1}))\oplus\cdots\oplus k(\pi(\vec{x_m}))$ in $\mathcal{O}_m^{2\nu+\delta}(k)$. Note that each permutation in $\mathrm{Sym}(|X^M|)$ can be regard as an automorphism of this orthogonal graph. Then we have:

Theorem 2.6. *The orthogonal graph $\mathcal{O}_m^{2\nu+\delta}(R)$ is vertex transitive and arc transitive.*

Proof. It suffices to show that $\mathcal{O}_m^{2\nu+\delta}(R)$ is arc transitive. Let X_1, X_2, Y_1 and Y_2 be vertices of $\mathcal{O}_m^{2\nu+\delta}(R)$ such that X_1 is adjacent to Y_1 and X_2 is adjacent to Y_2. By applying four suitable permutations in $\mathrm{Sym}(|X_1^M|)$, $\mathrm{Sym}(|X_2^M|)$, $\mathrm{Sym}(|Y_1^M|)$ and $\mathrm{Sym}(|Y_2^M|)$, respectively, X_1, X_2, Y_1 and Y_2 can be regarded as vertices of $\mathcal{O}_m^{2\nu+\delta}(k)$. By Lemma 2.5, the result follows directly. □

The *distance* between two vertices x and y in a graph G, denoted $d(x, y)$, is the length of the shortest path from x to y. The *diameter* of the graph is the maximum distance between two distinct vertices. The orthogonal graph $\mathcal{O}_m^{2\nu+\delta}(k)$ has the diameter $m + 1$ [11]. By Lemma 2.3 (2), we have:

Theorem 2.7. *The diameter of $\mathcal{O}_m^{2\nu+\delta}(R)$ is $m + 1$.*

3. The first subconstituent

In this section, the first subconstituent of $\mathcal{O}_m^{2\nu+\delta}(R)$ is studied. The author believes that the other subconstituents are very complicated even for finite fields case. Some results of subconstituents of $\mathcal{O}_m^{2\nu+\delta}(k)$ are illustrated in [11]. Here we study the analogous results of the first subconstituent. First, we introduce some notations. Let $E = R\vec{e}_1 \oplus R\vec{e}_2 \oplus \cdots \oplus R\vec{e}_m$ where $\vec{e}_i$ is a vector with 1 on the ith position and 0 elsewhere. Then E is a vertex in $\mathcal{O}_m^{2\nu+\delta}(R)$. From Theorem 2.7, the distance $d(X, E) \le m+1$ if $X \ne E$. The *subconstituent* $\mathcal{O}_m^{2\nu+\delta}(R, i)$, $i = 1, 2, \ldots, m + 1$, is defined to be the induced subgraphs of $\mathcal{O}_m^{2\nu+\delta}(R)$ on the sets of all vertices X satisfying $d(X, E) = i$. We observe that it is possible to define other subconstituents associated with other vertices. However, our graph $\mathcal{O}_m^{2\nu+\delta}(R)$ is vertex and arc transitive by Theorem 2.6. Therefore, it suffices to consider only the one associated with the vertex E. From the definition, the first subconstituent $\mathcal{O}_m^{2\nu+\delta}(R, 1)$ is a graph whose vertices are adjacent to the vertex E.

Lemma 3.1. *By the above setting, we have the following statements.*

1. *If X is a vertex of $\mathcal{O}_m^{2\nu+\delta}(R, 1)$, then there are $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}$ many vertices which are lifts of vertex $\pi(X)$ of $\mathcal{O}_m^{2\nu+\delta}(k, 1)$.*
2. *Two vertices X and Y are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(R, 1)$ if and only if the vertices $\pi(X)$ and $\pi(Y)$ are adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(k, 1)$.*

Proof. The proof is analogous to the proof of Lemma 2.3 since X is adjacent to E if and only if $\pi(X)$ is adjacent to $\pi(E)$. □

A regular graph of degree l on n vertices is a *generalized strongly regular graph* with the parameters $(n, l, \lambda_1, \lambda_2, \ldots, \lambda_m; \mu_1, \mu_2, \ldots, \mu_n)$

if any two adjacent vertices have λ_i common neighbors and any two non-adjacent vertices have μ_i common neighbors for some i.

By [11], the first subconstituent $\mathcal{O}_m^{2\nu+\delta}(k,1)$ is a generalized strongly graph with the parameters $(n', l', \lambda'_1, \lambda'_2, \lambda'_3; \mu'_1, \mu'_2, \mu'_3)$ where

$$\begin{aligned}
n' &= \frac{(|k|^{2\nu-2m+\delta})(|k|^m-1)}{(|k|-1)^2},\\
l' &= \frac{|k|^{2\nu-2m+\delta-1}+|k|^{m+1}-|k|^2-|k|}{|k|-1},\\
\lambda'_1 &= \frac{|k|^{m+1}-3|k|+2}{|k|-1},\\
\lambda'_2 &= \frac{|k|^{2\nu-2m+\delta-2}-|k|}{|k|-1},\\
\lambda'_3 &= \frac{|k|^{2\nu-2m+\delta-1}+|k|^{m+1}-|k|^2-|k|}{|k|-1},\\
\mu'_1 &= \frac{|k|^{2\nu-2m+\delta-2}-|k|}{|k|-1},\\
\mu'_2 &= 2|k|,\\
\mu'_3 &= 0.
\end{aligned}$$

Theorem 3.2. *The subconstituent $\mathcal{O}_m^{2\nu+\delta}(R,1)$ is a generalized strongly regular graph with parameters $(n, l, \lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3)$ where*

$$\begin{aligned}
n &= |M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2} n',\\
l &= |M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2} l',\\
\lambda_i &= |M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2} \lambda'_i,\\
\mu_i &= |M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2} \mu'_i,
\end{aligned}$$

for all $i = 1, 2$ or 3.

Proof. For the parameters $n, l, \lambda_1, \lambda_2$ and λ_3, they follow directly from Lemma 3.1. Now, assume that X and Y are non-adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(R,1)$. If $\pi(X) \neq \pi(Y)$, then $\pi(X)$ and $\pi(Y)$ are non-adjacent vertices in $\mathcal{O}_m^{2\nu+\delta}(k,1)$, so the number of common neighbors of X and Y is the product of common neighbors of $\pi(X)$ and $\pi(Y)$ and $|M|^{(2\nu+\delta)m-\binom{m+1}{2}-m^2}$ by Lemma 2.3. If $\pi(X) = \pi(Y)$, the number of common neighbors is the degree of regularity of $\mathcal{O}_m^{2\nu+\delta}(R,1)$ which is equal to λ_3. □

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