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On the Conditions of Similar Analytical Solutions of Homotopy Perturbation, Taylor Series Expansion and Differential Transformation Methods

M. G. Sobamowo [a,b*]

[a]Department of Mechanical Engineering, University of Lagos, Nigeria.

[b]Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria

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ABSTRACT

The developments of approximate analytical solutions to nonlinear differential equations have been achieved through the use of various approximate analytical and semi-analytical methods. These methods provide different analytical expressions which give difference values for the same input data and variables. However, under some certain conditions, the methods provide similar analytical expressions, thereby give the same values for the same input data and variables. Therefore, in this work, the conditions of similar analytical solutions by homotopy perturbation, differential transformation and Taylor series methods for linear and nonlinear differential equations are investigated. From the analysis, it is established that if some specific values or functions are assigned to the auxiliary parameters in the homotopy perturbation method, the approximate analytical solutions provided by homotopy perturbation method is entirely similar to the approximate analytical solutions given by differential transformation and Taylor series methods. Also, it is found that the results of Taylor series method when expansion is at the center, is exactly the same to the results of homotopy perturbation and differential transformation methods. It is hoped that this work will great assist and enhance the understanding of mathematical solutions providers and enthusiasts as it provides better insight into finding analytical solutions to linear and nonlinear differential equations.

1. Introduction

Mathematical modeling of real-life problems results in nonlinear differential and integral equations in which their exact analytical solutions are very difficult to develop. However, the developments of approximate analytical solutions to the nonlinear differential and integral equations have been achieved through the use of various approximate analytical and semi-analytical methods [1-19]. In these pool of the approximate analytical and semi-analytical methods, one major concern is the

* Corresponding author.
E-mail address: gsobamowo@unilag.edu.ng (M. G. Sobamowo)

approximate analytical solutions given by homotopy perturbation, differential transformation, and Taylor series expansion methods. As with the other approximate analytical and semi-analytical methods, these three methods of interests solve linear and nonlinear differential equations without requiring linearization, discretization, restrictive assumptions, or small perturbation parameter.

Homotopy perturbation method (HPM) is an approximate analytical method which is relative simple and provides acceptable analytical results with convenient convergence and stability. Although, it is a type of perturbation methods, it does not require small perturbation parameter like the traditional perturbation methods (regular and singular perturbation methods) to provide approximate analytical solutions to the nonlinear problems. It should be stated that when it comes to finding solutions to boundary-value and initial-boundary value problems, homotopy perturbation method is a total analytical method (non-semi-analytical method or non-analytic-numeric method) as it does not necessarily require any numerical scheme to find the value(s) or function(s) that satisfy the terminal boundary condition(s). The versatility of this method has made it to be widely applied to various science and engineering problems (19-34].

Taylor series expansion method (TSM) is one of the earliest total analytic methods for finding approximate analytical solutions to differential equations. This method provides an advantage of making is a differentiable approximate solution to be obtained, which can be replaced into differential equations and the initial or boundary conditions. However, Taylor series expansion method is not frequently applied because it requires more function evaluations than well-known classical algorithms and the overelaborate tasks of calculations of the higher-order derivatives involve in finding approximate solutions of differential equations. Some of the applications of the method can be found in [35-50].

Differential transform method (DTM) is a type of non-perturbation semi-analytical method that is based on the Taylor series expansion method. It is taken as an extended Taylor series expansion method as it allows easy generalization of the Taylor series expansion method to various derivation procedures. As with the other methods, DTM can also be directly applied to differential equations without requiring linearization, discretization, restrictive assumptions or perturbation [51-68]. It is capable of greatly reducing the size of computational work and time while still accurately providing the series solution with fast convergence rate. The method minimizes the computational difficulties of the Taylor series in that the components are easily determined. The method solves nonlinear problems in a like manner as linear problems, thus overcoming the deficiency of linearization or perturbation. With the aid of differential transformation method, solving differential equations is sufficiently done with simple calculations whereas the Taylor series method suffers from certain computational difficulties.

The HPM, TSEM and DTM provide different analytical expressions which give difference results for the same input data and variables. However, under some certain conditions, the methods provide similar analytical expressions, thereby give the same values for the same input data and variables. To the best of the author's knowledge, these conditions have not been well established in literature. Therefore, in this work, the conditions of similar analytical solutions for the three approximate analytical methods for linear and nonlinear differential equations are investigated and presented. Various examples are provided for the approximate analytical solutions to linear and nonlinear ordinary and partial differential equations to establish the conditions.

2. Principles of the Approximate Analytical Methods of Solutions

In this section, the basic definitions and the principles of the homotopy perturbation, Taylor series expansion and differential transformation methods are presented.

2.1 The basic idea of homotopy perturbation method

In order to establish the basic idea behind homotopy perturbation method, consider a system of nonlinear differential equations given as

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{1}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, \qquad r \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator which is a function of the dependent variable and its derivatives, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω

The operator A can be divided into two parts, which are L and N, where L is a linear operator, N is a non-linear operator. Eq. (1) can be therefore rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, a homotopy $U(r, p): \Omega \times [0,1] \to R$ can be constructed, which satisfies

$$H(U, p) = (1-p)\left[L(U) - L(U_o)\right] + p\left[A(U) - f(r)\right] = 0, \quad p \in [0,1], \tag{4}$$

or

$$H(U, p) = L(U) - L(U_o) + pL(U_o) + p\left[N(U) - f(r)\right] = 0. \tag{5}$$

In the above Eqs. (4) and (5), $p \in [0,1]$ is an embedding parameter, u_o is an initial approximation of equation of Eq.(1), which satisfies the boundary conditions.

Also, from Eqs. (4) and Eq. (5), we will have

$$H(U,0) = L(U) - L(U_o) = 0, \tag{6}$$

or

$$H(U,0) = A(U) - f(r) = 0. \tag{7}$$

The changing process of p from zero to unity is just that of $U(r, p)$ from $u_o(r)$ to $u(r)$. This is referred to homotopy in topology. Using the embedding parameter p as a small parameter, the solution of Eqs. (4) and Eq. (5) can be assumed to be written as a power series in p as given in Eq. (8)

$$U = U_o + pU_1 + p^2U_2 + ... \tag{8}$$

It should be pointed out that of all the values of p between 0 and 1, $p=1$ produces the best result. Therefore, setting $p = 1$, results in the approximation solution of Eq. (9)

$$u = \lim_{p \to 1} U = U_o + U_1 + U_2 + ... \tag{9}$$

Therefore

$$u = U_o + U_1 + U_2 + ... \quad (10)$$

The basic idea expressed above is a combination of homotopy and perturbation method. Hence, the method is called homotopy perturbation method (HPM), which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantages of the traditional perturbation techniques.

2.2 The basic principle of Taylor series method

The basic principle of Taylor series method for solving differential equation is as follows:
Given a differential equation

$$f(x, y, y', y'', ...y^n) = 0. \quad (11)$$

From the n^{th}–order Taylor series of a smooth function about the point $x = x_o$, the series solution of the differential equation is given by

$$y(x) = y(x_o) + y'(x_o)(x - x_o) + \frac{1}{2!}y''(x_o)(x - x_o)^2 + \frac{1}{3!}y'''(x_o)(x - x_o)^3 + ... + \frac{1}{n!}y^n(x_o)(x - x_o)^n \quad (12)$$

using the above series in Equ. (12), each term in the differential equation as presented in Eq. (11) can be found. The various developed series expressions are introduced into the given differential equation to evaluate the coefficients $y(x_o),\ y'(x_o),\ y''(x_o),\ y'''(x_o), ..., y^n(x_o)$. The values or analytical expressions of $y(x_o),\ y'(x_o),\ y''(x_o),\ y'''(x_o), ..., y^n(x_o)$ are substituted into the series solution in Eq. (12) to establish the Taylor series expansion solution of the differential equation.

2.3 The basic definitions and operational properties of differential transformation method

The basic definitions of the method are stated that if $u(t)$ is analytic in the domain *T*, then it will be differentiated continuously with respect to time *t*.

$$\frac{d^k u(t)}{dt^k} = \varphi(t, k) \qquad \text{for} \qquad \text{all } t \in T \quad (13)$$

for $t = t_i$, then $\varphi(t, k) = \varphi(t_i, k)$, where *k* belongs to the set of non-negative integers, denoted as the *k*-domain. Therefore Eq. (13) can be rewritten as

$$U(k) = \varphi(t_i, k) = \left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i} \quad (14)$$

Where U_k is called the spectrum of $u(t)$ at $t = t_i$

If $u(t)$ can be expressed by Taylor's series, the $u(t)$ can be represented as

$$u(t) = \sum_{k}^{\infty}\left[\frac{(t - t_i)^k}{k!}\right] U(k) \quad (15)$$

where Equ. (15) is called the inverse of $U(k)$ using the symbol 'D' denoting the differential transformation process and combining (14) and (15), it is obtained that

$$u(t)=\sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]U(k)=D^{-1}U(k) \tag{16}$$

2.2.1 Operational properties of differential transformation method

If $u(t)$ *and* $v(t)$ are two independent functions with time (t) where $U(k)$ and $V(k)$ are the transformed function corresponding to $u(t)$ and $v(t)$, then it can be proved from the fundamental mathematics operations performed by differential transformation that

i. If $z(t)=u(t)\pm v(t),$ then $Z(k)=U(k)\pm V(k)$

ii. If $z(t)=\alpha u(t),$ then $Z(k)=\alpha U(k)$

iii. If $z(t)=\dfrac{du(t)}{dt},$ then $Z(k)=(k-1)\,U(k+1)$

iv. If $z(t)=u(t)v(t),$ then $Z(t)=\sum_{i=0}^{K}V(l)U(k-l)$

v. If $z(t)=u^m(t)$, then $Z(t)=\sum_{I=0}^{K}U^{m-1}(l)U(k-l)$

vi. If $z(t)=u^n(t)v^n(t),$ then $Z(t)=\sum_{l-0}^{k}\left[\sum_{j=0}^{l}[V(j)U(l-j)]\;\sum_{j=0}^{k-l}[V(j)U(k-l-j)]\right]$

vii. If $z(t)=u(t)v(t),$ then $Z(k)=\sum_{l=0}^{k}(l+1)V(l+1)U(k-l)$

viii. If $z(t)=t^m$, then $Z(t)=\begin{cases}1 & k=m\\ 0 & k\neq m\end{cases}$

3. Presentation of approximate analytical solutions to different differential equations

It has been stated in the previous sections that HPM, TSM and DTM provide different analytical expressions for a given differential equation. However, under some certain conditions, the methods give similar analytical expressions for the given differential equation. In order to establish these conditions, the three series methods are applied separately to some set of linear and nonlinear ordinary differential equations as presented under this section.

3.1 Solutions of linear differential equations

Example 1: Apply homotopy perturbation method, Taylor series and differential transformation methods to solve the following second-order differential equation

$$y'' - 2xy' + 2\alpha y = 0, \quad y(0) = a_o, \; y'(0) = a_1 \tag{17}$$

Using homotopy perturbation method, we have the following

$$L(y) = y'', \quad N(y) = -2xy' + 2\alpha y, \quad f(x) = 0, \tag{18}$$

From the definition of HPM in Eq. (5), we have

$$H(y,p) = [y''] - [y_0''] + p[y_0''] + p[-2xy' + 2\alpha y] = 0. \tag{19}$$

Since $y_0' = a_1 \quad \Rightarrow y_0'' = 0$, therefore, Eq. (19) becomes,

$$H(y,p) = [y''] + p[-2xy' + 2\alpha y] = 0. \tag{20}$$

The solution of Eq. (17) can be assumed to be written as a power series in p as given in Eq. (20)

$$y = y_0 + py_1 + p^2 y_2 + p^3 y_3 + ... \tag{21}$$

Substitute the assume solution in Eq. (21) into Eq. (20), we have

$$H(y,p) = \left[y_0'' + py_1'' + p^2 y_2'' + p^3 y_3'' + ...\right] + p\begin{bmatrix} -2x\left(y_0' + py_1' + p^2 y_2' + p^3 y_3' + ...\right) \\ +2\alpha\left(y_0 + py_1 + p^2 y_2 + p^3 y_3 ...\right) \end{bmatrix} = 0. \tag{22}$$

$$\begin{aligned} H(y,p) = {} & y_0'' + py_1'' + p^2 y_2'' + p^3 y_3'' + ... - 2xpy_0' - 2xp^2 y_1' - 2xp^3 y_2' - 2xp^4 y_3' - ... \\ & + 2\alpha py_0 + 2\alpha p^2 y_1 + 2\alpha p^3 y_2 + 2\alpha p^4 y_3 + ... = 0. \end{aligned} \tag{23}$$

Also, for the initial conditions, we have

$$y_0(0) + py_1(0) + p^2 y_2(0) + p^3 y_3(0) + ... = a_o, \qquad y_0'(0) + py_1'(0) + p^2 y_2'(0) + p^3 y_3'(0) + ... = a_1 \tag{24}$$

Arrange the equation and the initial conditions according to the power of the embedding parameter p, we have

$$p^0: \quad y_0'' = 0, \quad y_0(0) = a_o, \quad y_0'(0) = a_1 \tag{25}$$

$$p^1: \quad y_1'' - 2xy_0' + 2\alpha y_0 = 0, \quad y_1(0) = 0, \; y_1'(0) = 0 \tag{26}$$

$$p^2: \quad y_2'' - 2xy_1' + 2\alpha y_1 = 0, \quad y_2(0) = 0, \; y_2'(0) = 0 \tag{27}$$

$$p^3: \quad y_3'' - 2xy_2' + 2\alpha y_2 = 0, \quad y_3(0) = 0, \; y_3'(0) = 0 \tag{28}$$

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Solving the above Eqs. (25), (26), (27) and (28), we have

$$y_0 = a_o + a_1 x, \tag{29}$$

$$y_1 = -\alpha a_0 x^2 - \frac{(\alpha-1)}{3} a_1 x^3, \tag{30}$$

$$y_2 = \frac{\alpha(\alpha-2)}{6} a_0 x^4 + \frac{(\alpha-1)(\alpha-3)}{30} a_1 x^5, \tag{31}$$

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After substitution of solutions in Eqs. (29), (30) and (31) into the assumed solution in Eq. (21), we have,

$$y = a_o + a_1 x + \left(-\alpha a_0 x^2 - \frac{(\alpha-1)}{3} a_1 x^3 \right) p + \left(\frac{\alpha(\alpha-2)}{6} a_0 x^4 + \frac{(\alpha-1)(\alpha-3)}{30} a_1 x^5 \right) p^2 + \dots \tag{32}$$

From the definition of HPM,

$$y = \lim_{p\to 1} \left(a_o + a_1 x + \left(-\alpha a_0 x^2 - \frac{(\alpha-1)}{3} a_1 x^3 \right) p + \left(\frac{\alpha(\alpha-2)}{6} a_0 x^4 + \frac{(\alpha-1)(\alpha-3)}{30} a_1 x^5 \right) p^2 + \dots \right) \tag{33}$$

Therefore

$$y = a_o + a_1 x - \alpha a_0 x^2 - \frac{(\alpha-1)}{3} a_1 x^3 + \frac{\alpha(\alpha-2)}{6} a_0 x^4 + \frac{(\alpha-1)(\alpha-3)}{30} a_1 x^5 + \dots \tag{34}$$

Applying Taylor series method to Example 1 i.e. Eq. (17), we have the Taylor series solution about the center point $x = 0$ of the linear differential equation in Eq. (17) as

$$y(x) = y(0) + xy'(0) + \frac{1}{2!} x^2 y''(0) + \frac{1}{3!} x^3 y'''(0) + \frac{1}{4!} x^4 y^{iv}(0) + \frac{1}{5!} x^5 y^{v}(0) + \dots \tag{35}$$

Therefore,

$$y'(x) = y'(0) + xy''(0) + \frac{1}{2} x^2 y'''(0) + \frac{1}{6} x^3 y^{iv}(0) + \frac{1}{24} x^4 y^{v}(0) + \dots \tag{36}$$

and

$$y''(x) = y''(0) + xy'''(0) + \frac{1}{2} x^2 y^{iv}(0) + \frac{1}{6} x^3 y^{v}(0) + \dots \tag{37}$$

After substitution of the initial conditions $y(0) = a_o$, $y'(0) = a_1$ into Eq. (91) and (92), we have

$$y(x) = a_0 + a_1 x + \frac{1}{2!} x^2 y''(0) + \frac{1}{3!} x^3 y'''(0) + \frac{1}{4!} x^4 y^{iv}(0) + \frac{1}{5!} x^5 y^{v}(0) + \dots \tag{38}$$

$$y'(x) = a_1 + xy''(0) + \frac{1}{2} x^2 y'''(0) + \frac{1}{6} x^3 y^{iv}(0) + \frac{1}{24} x^4 y^{v}(0) + \dots \tag{39}$$

On substituting Eqs. (37), (38) and (39) into the given differential equation, one arrives at

$$\begin{aligned}&\left[y''(0)+xy'''(0)+\frac{1}{2}x^2y^{iv}(0)+\frac{1}{6}x^3y^{v}(0)+...\right]-2x\left[a_1+xy''(0)+\frac{1}{2}x^2y'''(0)+\frac{1}{6}x^3y^{iv}(0)+\frac{1}{24}x^4y^{v}(0)+...\right]\\&+2\alpha\left[a_0+a_1x+\frac{1}{2!}x^2y''(0)+\frac{1}{3!}x^3y'''(0)+\frac{1}{4!}x^4y^{iv}(0)+\frac{1}{5!}x^5y^{v}(0)+...\right]=0,\end{aligned} \tag{40}$$

Collection of like terms in Eq. (40) provides,

$$\begin{aligned}&\left(y''(0)+2\alpha a_0\right)+\left[2\alpha a_1+y'''(0)-2a_1\right]x+\left[\frac{2}{2!}\alpha y''(0)-2y''(0)+\frac{1}{2}y^{iv}(0)\right]x^2\\&+\left[\frac{1}{3!}y^{v}(0)-\frac{2}{2!}y'''(0)+\frac{1}{3!}y'''(0)\right]x^3+...\end{aligned} \tag{41}$$

Which provides

$$\begin{aligned}&\left(y''(0)+2\alpha a_0\right)=0\\&\left[2\alpha a_1+y'''(0)-2a_1\right]=0\\&\left[\frac{2}{2!}\alpha y''(0)-2y''(0)+\frac{1}{2!}y^{iv}(0)\right]=0\\&\left[\frac{1}{3!}y^{v}(0)-\frac{2}{2!}y'''(0)+\frac{2\alpha}{3!}y'''(0)\right]=0\end{aligned} \tag{42}$$

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From Eq. (42), one can see that

$$\begin{aligned}&y''(0)=-2\alpha a_0\\&y'''(0)=-2a_1\left(\alpha-1\right)\\&y^{iv}(0)=4\alpha a_0\left(\alpha-2\right)\\&y^{v}(0)=4a_1\left(\alpha-3\right)\left(\alpha-1\right)\end{aligned} \tag{43}$$

Substitute the solution in Eq. (38) into Eq. (44), we have

$$y(x)=a_0+a_1x-\alpha a_0x^2-\frac{\left(\alpha-1\right)}{3}a_1x^3+\frac{\alpha\left(\alpha-2\right)}{6}a_0x^4+\frac{\left(\alpha-3\right)\left(\alpha-1\right)}{30}a_1x^5+... \tag{44}$$

Now, we apply differential transformation method. The differential transform of equation is given as

$$\left(k+1\right)\left(k+2\right)Y\left(k+2\right)-2\left(k\right)Y\left(k\right)+2\alpha Y\left(k\right)=0 \tag{45}$$

From the given initial condition, one can write that

$$Y(0)=a_0 \qquad Y(1)=a_1 \tag{46}$$

From Eq. (101), for *k=0, 1, 2, 3,…,* it can be stated that

$$Y(2)=-\alpha a_0,\;\; Y(3)=\frac{-(\alpha-1)a_1}{3},\;\; Y(4)=\frac{\alpha(\alpha-2)a_1}{6},\;\; Y(5)=\frac{(\alpha-1)(\alpha-3)a_1}{30}, \tag{47}$$

Following the definition of differential transformation method, we have

$$y=Y(0)+xY(1)+x^2Y(2)+x^3Y(3)+x^4Y(4)+x^5Y(5)+... \tag{48}$$

Substituting the solutions in Eq. Eq. (47) into (48), we have

$$y=a_o+a_1x-\alpha a_0x^2-\frac{(\alpha-1)}{3}a_1x^3+\frac{\alpha(\alpha-2)}{6}a_0x^4+\frac{(\alpha-1)(\alpha-3)}{30}a_1x^5+... \tag{49}$$

Again, one arrives at the same analytical solution using the three methods. However, in the homotopy perturbation method, if the auxiliary parameters are chosen such that

$L(y)=y''-2xy'+2\alpha y,\quad N(y)=0,\;\; f(x)=0$, or $L(y)=y''+2\alpha y,\quad N(y)=-2xy',\;\; f(x)=0$

The analytical solution of the homotopy perturbation method will not the same as the similar analytical solutions of Taylor series and differential transformation methods given in Eqs. (34) and (49).

Example 2: Apply homotopy perturbation method, Taylor series expansion and differential transformation methods to solve the following partial differential heat equation

$$u_t=u_{xx} \qquad u(x,0)=4x^2, \tag{50}$$

As previously presented, we apply homotopy perturbation method, we have the following

$$L(u)=u_t,\;\; N(u)=-u_{xx},\;\; f(x,t)=0,\;\; u_0=4x^2\rightarrow L(u_0)=\frac{\partial}{\partial t}\left(4x^2\right)=0, \tag{51}$$

Therefore, from the definition of HPM, we have

$$H(u,p)=u_t(x,t,p)+p\left(-u_{xx}(x,t,p)\right)=0,\;\; u(x,0,p)=4x^2, \tag{52}$$

As done previously, the solution of Eq. (106) can be assumed to be written as a power series in *p* as

$$u(x,t,p)=u_0(x,t,p)+pu_1(x,t,p)+p^2u_2(x,t,p)+p^3u_3(x,t,p)+... \tag{53}$$

Substitute the assume solution in Eq. (53) into Eq. (52), we have

$$H(u,p)=\left(\frac{\partial u_0}{\partial t}+p\frac{\partial u_1}{\partial t}+p^2\frac{\partial u_2}{\partial t}+p^3\frac{\partial u_3}{\partial t}+...\right)+p\left(-\left(\frac{\partial^2 u_0}{\partial x^2}+p\frac{\partial^2 u_1}{\partial x^2}+p^2\frac{\partial^2 u_2}{\partial x^2}+p^3\frac{\partial^2 u_3}{\partial x^2}+...\right)\right)=0 \tag{54}$$

Also, for the initial condition

$$u(x,0,p)=u_0(x,0,p)+pu_1(x,0,p)+p^2u_2(x,0,p)+p^3u_3(x,0,p)+...=4x^2 \quad (55)$$

Arrange the equation and the initial conditions according to the power of the embedding parameter p, we have

$$p^0:\ \frac{\partial u_0}{\partial t}=0,\ u(x,0)=4x^2, \quad (56)$$

$$p^1:\ \frac{\partial u_1}{\partial t}-\frac{\partial^2 u_0}{\partial x^2}=0,\ u_1(x,0)=0, \quad (57)$$

$$p^2:\ \frac{\partial u_2}{\partial t}-\frac{\partial^2 u_1}{\partial x^2}=0,\ u_2(x,0)=0, \quad (58)$$

$$p^3:\ \frac{\partial u_3}{\partial t}-\frac{\partial^2 u_2}{\partial x^2}=0,\ u_3(x,0)=0, \quad (59)$$

$$p^4:\ \frac{\partial u_4}{\partial t}-\frac{\partial^2 u_3}{\partial x^2}=0,\ u_4(x,0)=0, \quad (60)$$

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Solving the above Eqs. (56), (57), (58), (59) and (60), we have

$$u_0=4x^2,\quad u_1=8t,\quad u_2=0,\quad u_3=0,\quad u_4=0,\ ...,\ u_n=0, \quad (61)$$

Substitute Eq. (53) into Eqs. (61), gives,

$$u(x,t)=4x^2+8tp, \quad (62)$$

From the definition of HPM,

$$u(x,t)=\lim_{p\to 1}\left(4x^2+8tp\right) \quad (63)$$

Therefore

$$u(x,t)=4x^2+8t \quad (64)$$

In order to apply Taylor series method to find the analytical solution to Example 2 i.e. Eq. (50), it is stated that the Taylor series expansion for two independent variables is given as

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)\right]+... \quad (64)$$

Therefore, if we write the Taylor's series expansion for $u(x,t)$ about the point $x=0$ and $t=0$, we have

$$\begin{aligned}u(x,t) &= u(0,0)+xu_x(0,0)+tu_t(0,0)+\frac{1}{2!}\left[x^2u_{xx}(0,0)+2xtu_{xt}(0,0)+t^2u_{tt}(0,0)\right]\\ &+\frac{1}{3!}\left[x^3u_{xxx}(0,0)+3x^2tu_{xxt}(0,0)+3xt^2u_{xtt}(0,0)+t^3u_{ttt}(0,0)\right]+\\ &+\frac{1}{4!}\left[x^4u_{xxxx}(0,0)+4x^3tu_{xxxt}(0,0)+6x^2t^2u_{xxtt}(0,0)+4xt^3u_{xttt}(0,0)+t^4u_{tttt}(0,0)\right]+\\ &+\frac{1}{5!}\left[x^5u_{xxxxx}(0,0)+5x^4tu_{xxxxt}(0,0)+10x^3t^2u_{xxxtt}(0,0)+10x^2t^3u_{xxttt}(0,0)+5xt^4u_{xtttt}(0,0)+t^5u_{ttttt}(0,0)\right]+\ldots\end{aligned} \tag{65}$$

From the given equation,

$$u_t(x,t)=u_{xx}(x,t) \tag{66}$$

From Eq. (66), it be stated that

$$u_t(0,0)=u_{xx}(0,0) \tag{67}$$

Given that

$$u(x,0)=4x^2 \rightarrow u(0,0)=0,\ u_x(0,0)=0,\quad u_{xx}(0,0)=8 \tag{68}$$

On substituting Eq. (68) into Eq. (67), we have

$$u_t(0,0)=8 \tag{69}$$

Also, differentiating Eq. (66) with respect to "*t*", we have

$$u_{tt}(x,t)=u_{xxt}(x,t) \tag{70}$$

Then,

$$u_{tt}(0,0)=u_{xxt}(0,0) \tag{71}$$

One can say that

$$u(0,0)=0,\rightarrow u_x(0,0)=0,\quad u_{xx}(0,0)=0,\quad u_{xxt}(0,0)=0,\ \ u_t(0,0)=0 \tag{72}$$

On substituting Eq. (72) into Eq. (71), we have

$$u_{tt}(0,0)=0 \tag{73}$$

Again, on differentiating Eq. (66) with respect to "*x*", one has

$$u_{xt}(x,t)=u_{xxx}(x,t) \tag{74}$$

Then,

$$u_{xt}(0,0)=u_{xxx}(0,0) \tag{75}$$

It can be stated that

$$u(0,0)=0, \rightarrow u_x(0,0)=2,\ u_{xx}(0,0)=8,\ u_{xxx}(0,0)=0\ u_{xxt}(0,0)=0, \tag{76}$$

On substituting Eq. (76) into Eq. (75), we have

$$u_{xt}(0,0)=0 \tag{77}$$

Similarly

$u_{xxt}(0,0)=0$, $u_{xtt}(0,0)=0$, $u_{xttt}(0,0)=0,\ \ u_{xtttt}(0,0)=0$. In the same way, the other coefficients in the Eq. (65) are also found.

Therefore, we have

$$\begin{aligned} &u(0,0)=0,\ \ u_x(0,0)=0,\ \ u_t(0,0)=8,\ \ u_{xx}(0,0)=8,\ \ u_{xt}(0,0)=0,\\ &u_{tt}(0,0)=0,\ \ u_{xxx}(0,0)=0,\ \ u_{xxt}(0,0)=0,\ \ u_{xtt}(0,0)=0,\ \ u_{ttt}(0,0)=0\\ &u_{xxxx}(0,0)=0,\ \ u_{xxxt}(0,0)=0,\ \ u_{xxtt}(0,0)=0,\ \ u_{xttt}(0,0)=0,\ \ u_{tttt}(0,0)=0\\ &u_{xxxxx}(0,0)=0,\ \ u_{xxxxt}(0,0)=0,\ \ u_{xxxtt}(0,0)=0\ \ u_{xxttt}(0,0)=0,\ \ u_{xtttt}(0,0)=0,\ \ u_{ttttt}(0,0)=0 \end{aligned} \tag{78}$$

Substitute the coefficients in Eq. (78) into Eqs. (65), gives,

$$u(x,t)=4x^2+8t \tag{79}$$

Using differential transformation method, the differential transform of equation is given as

$$(h+1)U(k,h+1)=(k+1)(k+2)U(k+2,h) \tag{80}$$

Which gives

$$U(k,h+1)=\frac{(k+1)(k+2)U(k+2,h)}{(h+1)} \tag{81}$$

The initial condition is transformed as

$$U(k,0)=4\begin{cases}1 & k=2,\\ 0 & k\neq 2\end{cases}$$

$$U(2,0)=4,\ U(0,0)=U(1,0)=U(3,0)=U(4,0)=U(5,0)=...=U(N,0)=0 \tag{82}$$

From Eq. (81),

$$\begin{aligned} &U(0,1)=8,\ U(0,1)=0,\ \ =U(0,2)=0,\ \ U(0,3)=0,\ \ U(0,4)=0...=U(0,N)=0\\ &U(1,1)=0,\ \ =U(1,2)=0,\ \ U(1,3)=0,\ \ U(1,4)=0...=U(1,N)=0 \end{aligned} \tag{83}$$

The other coefficients are zero.

From the definition of differential transformation method, one can write that

$$\begin{aligned} u(x,y)=&U(0,0)+xU(1,0)+tU(0,1)+xtU(1,1)+x^2U(2,0)\\ &+t^2U(0,2)+x^2tU(2,1)+xt^2U(1,2)+x^2t^2U(2,2)+... \end{aligned} \tag{84}$$

Substitute the coefficients in Eq. (83) into Eq. (84), gives,

$$u(x,t) = 4x^2 + 8t \tag{85}$$

It can also be seen that the same analytical solution is given using the three methods. However, in the homotopy perturbation method, if the auxiliary parameters are chosen such that

$$L(u) = u_t - u_{xx}, \quad N(u) = 0, \;\; f(x,t) = 0,,$$

The analytical solution of the homotopy perturbation method will not the same as the similar analytical solutions of Taylor series and differential transformation methods given in Eqs. (79) and (85).

3.2 Solutions of Nonlinear differential equations

Example 3: Solve the following nonlinear differential equation using homotopy perturbation, Taylor series expansion and differential transformation methods.

$$y' + y^2 = 0, \quad y(0) = 1 \tag{80}$$

Using homotopy perturbation method, we have the following

$$L(y) = y', \quad N(y) = y^2, \;\; f(x) = 0, \tag{81}$$

By the homotopy technique, one can construct homotopy which satisfies

$$H(y,p) = (1-p)\left[L(y) - L(y_o)\right] + p\left[L(y) + N(y) - f(x)\right] = 0. \tag{82}$$

After substituting of Eq. (81) into Eq. (82) and, one arrives at

$$H(y,p) = (1-p)\left[y' - y_0'\right] + p\left[y' + y^2\right] = 0. \tag{83}$$

Since $y_0 = 1 \;\; \Rightarrow y_0' = 0$, therefore, Eq. (83) becomes,

$$H(y,p) = y' + p\left[y^2\right] = 0. \tag{84}$$

The solution of Eq. (80) can be assumed to be written as a power series in *p* as given in Eq. (85)

$$y = y_0 + py_1 + p^2 y_2 + p^3 y_3 ... \tag{85}$$

Substitute the assume solution in Eq. (86) into Eq. (84), we have

$$H(y,p) = y_0' + py_1' + p^2 y_2' + p^3 y_3' + ... + p\left(y_0 + py_1 + p^2 y_2 + p^3 y_3 + ...\right)^2 = 0. \tag{86}$$

Arrange the equation according to the power of the embedding parameter p, we have

$$p^0: \quad y_0' = 0, \quad y_0(0) = 1 \tag{87}$$

$$p^1: \quad y_1' + y_0^2 = 0, \quad y_1(0) = 0 \tag{88}$$

$$p^2: \quad y_2' + 2y_0 y_1 = 0, \quad y_2(0) = 0 \tag{89}$$

$$p^3: \quad y_3' + 2y_0 y_2 + y_1 y_1 = 0, \quad y_3(0) = 0 \tag{90}$$

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The solutions the above Eqs. (87), (88), (89) and (90) are

$$y_0 = 1, \quad y_1 = -x, \quad y_2 = x^2, \quad y_3 = -x^3, \ldots . \tag{91}$$

After substitution of solutions in Eq. (91) into Eq. (85), the result is

$$y = 1 - px + p^2 x^2 - p^3 x^3 + \ldots \tag{92}$$

From the definition of homotopy perturbation method, we have

$$y = \lim_{p \to 1} \left(1 - px + p^2 x^2 - p^3 x^3 + \ldots\right) = 1 - x + x^2 - x^3 + \ldots \tag{93}$$

Therefore

$$y = 1 - x + x^2 - x^3 + \ldots \tag{94}$$

The above solution can be extended to a generalization as

$$y = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots = \frac{1}{1+x}, \quad |x| < 1 \tag{95}$$

Using Taylor series expansion method, suppose the series solution about the point $x = 0$ of the linear differential equation in Eq. (85) is given as

$$y(x) = y(0) + xy'(0) + \frac{1}{2!} x^2 y''(0) + \frac{1}{3!} x^3 y'''(0) + \ldots \tag{96}$$

It is given that

$$y(0) = 1 \tag{97}$$

Then from the given differential equation, one gets

$$y'(0) + y^2(0) = 0, \tag{98}$$

Given that

$$y(0) = 1 \to y'(0) = -1, \tag{99}$$

Differentiating the given differential equation, we have

$$y'' + 2yy' = 0, \tag{100}$$

Which gives

$$y''(0) + 2y(0)y'(0) = 0, \tag{101}$$

Then, the result is

$$y''(0) = -2y(0)y'(0) = -2(1)(-1) = 2, \tag{102}$$

Differentiation of Eq. (100) with respect to *x*, gives

$$y''' + 2yy'' + 2(y')^2 = 0, \tag{103}$$

Which provides

$$y'''(0) = -2(1)(2) - 2((-1))^2 = -6, \tag{104}$$

Therefore

$$y(0) = 1, \;\; y'(0) = -1, \;\; y''(0) = 2, \;\; y'''(0) = -6, \tag{105}$$

On substituting the solution of Eq. (105) into Eq. (96), one arrives as

$$y = 1 - x + x^2 - x^3 + ... \tag{106}$$

The above solution can be extended to a generalization as

$$y = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + ... = \frac{1}{1+x}, \quad |x| < 1 \tag{107}$$

Application of differential transformation method, the differential transform of equation is given as

$$(k+1)Y(k+1) + \sum_{l=0}^{k} Y(l)Y(k-l) = 0 \tag{108}$$

Which can be written as

$$Y(k+1) = \frac{-\sum_{l=0}^{k} Y(l)Y(k-l)}{(k+1)} \tag{109}$$

From the given initial condition, we have

$$Y(0) = 1 \tag{110}$$

From Eq. (174), for *k=0, 1, 2, 3,…,* one has

$$Y(1) = -1, \;\; Y(2) = 1, \;\; Y(3) = -1, \tag{111}$$

From the definition of differential transformation method, one can write that

$$y = Y(0) + xY(1) + x^2Y(2) + x^3Y(3) + ... \tag{112}$$

On substituting the solutions in Eq. (110) and (111) into Eq. (172), one arrives as

$$y = 1 - x + x^2 - x^3 + ... \tag{113}$$

The above solution can be extended to a generalization as

$$y = 1 - x + x^2 - x^3 + ... + (-1)^n x^n + ... = \frac{1}{1+x}, \quad |x| < 1 \tag{114}$$

Example 4: Apply homotopy perturbation method, Taylor series expansion and differential transformation methods to solve the following partial differential Burger's equation

$$u_t + uu_x = u_{xx} \qquad u(x,0) = 2x, \tag{115}$$

Using homotopy perturbation method, we have the following

$$L(u) = u_t, \;\; N(u) = \left(uu_x - u_{xx}\right), \;\; f(x,t) = 0, \;\; u_0 = 2x \to L(u_0) = \frac{\partial}{\partial t}\left(2x\right) = 0\,, \tag{116}$$

Therefore, from the definition of HPM, we have

$$H\left(u, p\right) = u_t(x,t,p) + p\left(u(x,t,p)u_x(x,t,p) - u_{xx}(x,t,p)\right) = 0, \;\; u(x,0,p) = 2x, \tag{117}$$

As done previously, the solution of Eq. (115) can be assumed to be written as a power series in *p* as

$$u(x,t,p) = u_0(x,t,p) + pu_1(x,t,p) + p^2u_2(x,t,p) + p^3u_3(x,t,p) + ... \tag{118}$$

Substitute the assumed solution in Eq. (118) into Eq. (117), we have

$$H\left(u,p\right) = \left(\frac{\partial u_0}{\partial t} + p\frac{\partial u_1}{\partial t} + p^2\frac{\partial u_2}{\partial t} + p^3\frac{\partial u_3}{\partial t} + ...\right) + p\left(\begin{array}{l} \left(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...\right)\left(\dfrac{\partial u_0}{\partial x} + p\dfrac{\partial u_1}{\partial x} + p^2\dfrac{\partial u_2}{\partial x} + p^3\dfrac{\partial u_3}{\partial x} + ...\right) \\ -\left(\dfrac{\partial^2 u_0}{\partial x^2} + p\dfrac{\partial^2 u_1}{\partial x^2} + p^2\dfrac{\partial^2 u_2}{\partial x^2} + p^3\dfrac{\partial^2 u_3}{\partial x^2} + ...\right) \end{array}\right) = 0 \tag{119}$$

Also, for the initial condition

$$u(x,0,p) = u_0(x,0,p) + pu_1(x,0,p) + p^2u_2(x,0,p) + p^3u_3(x,0,p) + ... = 2x \tag{120}$$

Arrange the equation and the initial condition according to the power of the embedding parameter p, we have

$$p^0: \;\; \frac{\partial u_0}{\partial t} = 0, \;\; u(x,0) = 2x, \tag{121}$$

$$p^1: \;\; \frac{\partial u_1}{\partial t} + u_0\frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} = 0, \;\; u_1(x,0) = 0, \tag{122}$$

$$p^2: \ \frac{\partial u_2}{\partial t} + u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_0}{\partial x} - \frac{\partial^2 u_1}{\partial x^2} = 0, \ u_2(x,0) = 0, \tag{123}$$

$$p^3: \ \frac{\partial u_3}{\partial t} + u_0 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} - \frac{\partial^2 u_2}{\partial x^2} = 0, \ u_3(x,0) = 0, \tag{124}$$

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Solving the above Eqs. (121), (122), (123) and (124), we have

$$u_0 = 2x, \quad u_1 = -4xt, \quad u_2 = 8xt^2, \quad u_3 = -16xt^3, \quad u_4 = 32xt^4, \ \ldots, \ u_n = (-1)^n \, 2^{n+1} x t^n, \tag{125}$$

Substitute the solutions in Eqs. (125) into Eqs. (118), gives,

$$u(x,t) = 2x - 4xtp + 8xt^2 p^2 - 16xt^3 p^3 + 32xt^4 p^4 + \ldots + (-1)^n \, 2^{n+1} x t^n p^n, \tag{126}$$

From the definition of HPM,

$$u(x,t) = \lim_{p \to 1} \left(2x - 4xtp + 8xt^2 p^2 - 16xt^3 p^3 + 32xt^4 p^4 + \ldots + (-1)^n \, 2^{n+1} x t^n p^n + \ldots \right) \tag{127}$$

Therefore

$$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 + \ldots + (-1)^n \, 2^{n+1} x t^n \tag{128}$$

The above series is an expansion of $\dfrac{2x}{1+2t}$

$$u(x,t) = 2x - 4xt + 8xt^2 - 16xt^3 + 32xt^4 + \ldots + (-1)^n \, 2^{n+1} x t^n = \frac{2x}{1+2t} \tag{129}$$

Therefore,

$$u(x,t) = \frac{2x}{1+2t} \tag{130}$$

In order to apply Taylor series to solution to Example 4 i.e. Eq. (115), we know the Taylor series expansion for two independent variables is given as

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b) \right] + \ldots \tag{121}$$

Therefore, if we write the Taylor's series expansion for $u(x,t)$ about the point $x = 0$ and $t=0$, we have

$$\begin{aligned} u(x,t) &= u(0,0) + xu_x(0,0) + tu_t(0,0) + \frac{1}{2!}\left[x^2u_{xx}(0,0) + 2xtu_{xt}(0,0) + t^2u_{tt}(0,0)\right] \\ &+ \frac{1}{3!}\left[x^3u_{xxx}(0,0) + 3x^2tu_{xxt}(0,0) + 3xt^2u_{xtt}(0,0) + t^3u_{ttt}(0,0)\right] + \\ &+ \frac{1}{4!}\left[x^4u_{xxxx}(0,0) + 4x^3tu_{xxxt}(0,0) + 6x^2t^2u_{xxtt}(0,0) + 4xt^3u_{xttt}(0,0) + t^4u_{tttt}(0,0)\right] + \\ &+ \frac{1}{5!}\left[x^5u_{xxxxx}(0,0) + 5x^4tu_{xxxxt}(0,0) + 10x^3t^2u_{xxxtt}(0,0) + 10x^2t^3u_{xxttt}(0,0) + 5xt^4u_{xtttt}(0,0) + t^5u_{ttttt}(0,0)\right] + ... \end{aligned} \quad (122)$$

From the given equation,

$$u_t(x,t) + u(x,t)u_x(x,t) = u_{xx}(x,t) \quad (123)$$

From Eq. (123), it be stated that

$$u_t(0,0) + u(0,0)u_x(0,0) = u_{xx}(0,0) \quad (124)$$

Given that

$$u(x,0) = 2x$$

Then

$$u(0,0) = 0, \rightarrow u_x(0,0) = 2, \quad u_{xx}(0,0) = 0 \quad (125)$$

On substituting Eq. (125) into Eq. (124), we have

$$u_t(0,0) = 0 \quad (126)$$

Also, differentiating Eq. (123) with respect to "*t*", we have

$$u_{tt}(x,t) + u_t(x,t)u_x(x,t) + u(x,t)u_{xt}(x,t) = u_{xxt}(x,t) \quad (127)$$

Then,

$$u_{tt}(0,0) + u_t(0,0)u_x(0,0) + u(0,0)u_{xt}(x,t) = u_{xxt}(0,0) \quad (128)$$

One can say that

$$u(0,0) = 0, \rightarrow u_x(0,0) = 2, \quad u_{xx}(0,0) = 0, \quad u_{xxt}(0,0) = 0, \quad u_t(0,0) = 0 \quad (129)$$

On substituting Eq. (129) into Eq. (128), we have

$$u_{tt}(0,0) = 0 \quad (130)$$

Again, differentiating Eq. (123) with respect to "*x*", we have

$$u_{xt}(x,t) + u_x(x,t)u_x(x,t) + u(x,t)u_{xx}(x,t) = u_{xxx}(x,t) \quad (213)$$

Then,

$$u_{xt}(0,0) + u_x(0,0)u_x(0,0) + u(0,0)u_{xx}(0,0) = u_{xxx}(0,0) \quad (124)$$

It can be stated that

$$u(0,0)=0, \rightarrow u_x(0,0)=2,\; u_{xx}(0,0)=0,\; u_{xxx}(0,0)=0\;\; u_{xxt}(0,0)=0, \tag{125}$$

On substituting Eq. (125) into Eq. (124), we have

$$u_{xt}(0,0)=-4 \tag{126}$$

Now, differentiating Eq. (123) with respect to "*x*", we have

$$u_{xxt}(x,t)+2u_{xx}(x,t)u_x(x,t)+u_x(x,t)u_{xx}(x,t)+u(x,t)u_{xxx}(x,t)=u_{xxxx}(x,t) \tag{127}$$

Then

$$u_{xxt}(0,0)+2u_{xx}(0,0)u_x(0,0)+u_x(0,0)u_{xx}(0,0)+u(0,0)u_{xxx}(0,0)=u_{xxxx}(0,0) \tag{128}$$

It was found that

$$u(0,0)=0, \rightarrow u_x(0,0)=2,\; u_{xx}(0,0)=0,\; u_{xxx}(0,0)=0\;\; u_{xxt}(0,0)=0,\; u_{xxxx}(0,0)=0 \tag{129}$$

On substituting Eq. (129) into Eq. (128), we have

$$\rightarrow u_{xxt}(0,0)=0 \tag{130}$$

Now, differentiating Eq. (123) with respect to "*x*", we have

$$u_{xtt}(x,t)+2u_x(x,t)u_{xt}(x,t)+u_t(x,t)u_{xx}(x,t)+u(x,t)u_{xxt}(x,t)=u_{xxxt}(x,t) \tag{131}$$

Then

$$u_{xtt}(0,0)+2u_{xt}(0,0)u_x(0,0)+u_t(0,0)u_{xx}(x,t)+u(0,0)u_{xxt}(0,0)=u_{xxxt}(0,0) \tag{132}$$

It was found that

$$u(0,0)=0, \rightarrow u_x(0,0)=2,\; u_{xt}(0,0)=-4,\; u_{xx}(0,0)=0\;\; u_{xxt}(0,0)=0,\; u_{xxxt}(0,0)=0,\; u_t(0,0)=0 \tag{133}$$

On substituting the solutions in Eq. (133) into Eq. (132), we have

$$\rightarrow u_{xtt}(0,0)=16 \tag{134}$$

Similarly

$u_{xttt}(0,0)=-96,\;\; u_{xtttt}(0,0)=768$. In the same way, the other coefficients in the Eq. (122) are also found.

Therefore, we have

$$\begin{gathered} u(0,0)=0,\;\; u_x(0,0)=0,\;\; u_t(0,0)=2,\;\; u_{xx}(0,0)=0,\;\; u_{xt}(0,0)=-4, \\ u_{tt}(0,0)=0,\;\; u_{xxx}(0,0)=0,\;\; u_{xxt}(0,0)=0,\;\; u_{xtt}(0,0)=16,\;\; u_{ttt}(0,0)=0 \\ u_{xxxx}(0,0)=0,\;\; u_{xxxt}(0,0)=0,\;\; u_{xxtt}(0,0)=0,\;\; u_{xttt}(0,0)=-96,\;\; u_{tttt}(0,0)=0 \\ u_{xxxxx}(0,0)=0,\;\; u_{xxxxt}(0,0)=0,\;\; u_{xxxtt}(0,0)=0\;\; u_{xxttt}(0,0)=0,\;\; u_{xtttt}(0,0)=768,\;\; u_{ttttt}(0,0)=0 \end{gathered} \tag{135}$$

Substitute the coefficients in Eq. (135) into Eqs. (122), gives,

$$u(x,t)=2x-4xt+8xt^2-16xt^3+32xt^4+...+\left(-1\right)^n 2^{n+1}xt^n \tag{136}$$

As done previously,

$$u(x,t)=2x-4xt+8xt^2-16xt^3+32xt^4+...+(-1)^n 2^{n+1}xt^n=\frac{2x}{1+2t} \tag{137}$$

Therefore,

$$u(x,t)=\frac{2x}{1+2t} \tag{138}$$

Using differential transformation method, the differential transform of equation is given as

$$(h+1)U(k,h+1)+\sum_{l=0}^{k}\sum_{p=0}^{h}U(l,h-p)(k+1-l)U(k-l+1,p)=(k+1)(k+2)U(k+2,h) \tag{139}$$

Which gives

$$U(k,h+1)=\frac{(k+1)(k+2)U(k+2,h)-\sum_{l=0}^{k}\sum_{p=0}^{h}U(l,h-p)(k+1-l)U(k-l+1,p)}{(h+1)} \tag{140}$$

The initial condition is transformed as

$$U(k,0)=2\begin{cases}1 & k=1,\\ 0 & k\neq 1\end{cases} \tag{141}$$

$$U(1,0)=2,\ U(0,0)=U(2,0)=U(3,0)=U(4,0)=U(5,0)=...=U(N,0)=0 \tag{142}$$

From Eq. (230),

$$U(1,0)=2,\ U(1,1)=-4,\ =U(1,2)=-8,\ U(1,3)=16,\ U(1,4)=32...=U(1,N)=(-1)^N 2^{N+1} \tag{143}$$

The other coefficients are zero.

From the definition of differential transformation method, one can write that

$$\begin{aligned}u(x,y)=&U(0,0)+xU(1,0)+tU(0,1)+xtU(1,1)+x^2U(2,0)\\&+t^2U(0,2)+x^2tU(2,1)+xt^2U(1,2)+x^2t^2U(2,2)+...\end{aligned} \tag{144}$$

Substitute the coefficients in Eq. (143) into Eqs. (144), gives,

$$u(x,t)=2x-4xt+8xt^2-16xt^3+32xt^4+...+(-1)^n 2^{n+1}xt^n=\frac{2x}{1+2t} \tag{145}$$

Therefore,

$$u(x,t)=\frac{2x}{1+2t} \tag{146}$$

The three methods give the same analytical solutions. However, in homotopy perturbation method, if the auxiliary parameters are chosen such that $L(u)=u_t-u_{xx},\ N(u)=(uu_x),\ f(x,t)=0,$ the analytical solution is not $\frac{2x}{1+2t}$.

3.3 Solutions of Nonlinear Differential Equations in Practical Situations

Example 5: Kinetic model of a biochemical reaction: The dimensionless forms of Michaelis-Menten biochemical reaction model are

$$\frac{dx}{dt}=-x+(\beta-\alpha)y+xy \tag{147}$$

$$\frac{dy}{dt}=\frac{1}{\varepsilon}(x-\beta y-xy) \tag{148}$$

where *x* is the dimensionless form of the concentration of substrate while *y* is the dimensionless form of the concentration intermediate complex between enzyme and substrate. $\alpha,\ \varepsilon$ and β are dimensionless parameters.

The initial conditions are given as

$$x(0)=1,\ y(0)=0 \tag{149}$$

Given $\alpha=\frac{3}{8},\ \varepsilon=\frac{1}{10},$ and $\beta=1$ and using homotopy perturbation method, Taylor series expansion and differential transformation methods to solve the dimensionless forms of Michaelis-Menten biochemical reaction model, we have the same approximate analytical solutions

$$x(t)=1-t+\frac{69}{8}t^2-\frac{757}{12}t^3+\frac{47767}{128}t^4-\frac{3800401}{1920}t^5+\frac{156000923}{15360}t^6+\ldots \tag{150}$$

$$y(t)=10t-105t^2+\frac{9145}{12}t^3-\frac{17785}{4}t^4+\frac{4440661}{192}t^5-\frac{44551057}{384}t^6+\ldots \tag{151}$$

For the above solutions, the choice of the auxiliary parameters for the homotopy perturbation method is,

$$L(x)=\frac{dx}{dt},\quad N(x,y)=x-(\beta-\alpha)y-xy,\ \ f(t)=0,\ \ x_0=1, \tag{152}$$

$$L(y)=\frac{dy}{dt},\quad N(x,y)=-\frac{1}{\varepsilon}(x-\beta y-xy),\ \ f(t)=0,\ \ y_0=0, \tag{153}$$

The Taylor series expansion solutions are the expansions about $t = 0$.

Example 6: Nonlinear thermal model of a stationary fin with constant thermal conductivity: The dimensionless form of one-dimensional heat transfer model in longitudinal rectangular fin is given as

$$\frac{d^2\theta}{dx^2} - M^2\theta^{n+1} = 0 \qquad 0 < x < 1 \tag{154}$$

where *x* is the dimensionless length of the fin and θ is the dimensionless temperature.

The boundary conditions are given as

$$x = 0, \quad \frac{d\theta}{dx} = 0,$$

$$x = 1, \quad \theta = 1, \tag{155}$$

using homotopy perturbation method, Taylor series expansion and differential transformation methods to solve the above equation, we have the same approximate analytical solution as

$$\theta(x) = \theta_e + \frac{M^2\theta_e^{n+1}}{2}x^2 + \frac{M^4\theta_e^{2n+1}(n+1)}{24}x^4 + \frac{M^6\theta_e^{3n+1}(n+1)(4n+1)}{720}x^6 + \frac{M^8\theta_e^{4n+1}(n+1)(34n^2+5n+1)}{40320}x^8 + \frac{M^{10}\theta_e^{5n+1}(n+1)(496n^3-66n^2+69n+1)}{3628800}x^{10} + \ldots \tag{156}$$

For the above solutions, the choice of the auxiliary parameters for the homotopy perturbation method is,

$$L(\theta) = \frac{d^2\theta}{dx^2}, \quad N(\theta) = -M^2\theta^{n+1}, \quad f(x) = 0, \quad \theta_0(x) = \theta_e \tag{157}$$

The Taylor series expansion solutions are the expansions about $x = 0$.

The complete solution is obtained once the constant θ_e is determined by imposing the second boundary conditions given by $\theta(1) = 1$. Note that the value of θ_e lies in the interval (0,1).

Example 7: Nonlinear thermal model of a stationary fin with variable thermal conductivity: The dimensionless form of one-dimensional heat transfer model in longitudinal rectangular fin is given as

$$\frac{d}{dx}\left[(1+\beta\theta)\frac{d\theta}{dx}\right] = M^2\theta^{n+1} \qquad 0 < x < 1 \tag{158}$$

where *x* is the dimensionless length of the fin and θ is the dimensionless temperature.

The boundary conditions are given as

$$x = 0, \quad \frac{d\theta}{dx} = 0,$$

$$x = 1, \quad \theta = 1, \tag{159}$$

using homotopy perturbation method, Taylor series expansion and differential transformation methods to solve the above equation, we also have the same approximate analytical solution as

$$\theta(x)=\theta_e+\frac{M^2\theta_e^{n+1}}{2!(1+\beta\theta_e)}X^2+\frac{M^4\theta_e^{2n+1}\left[(n+1)+(n-2)\beta\theta_e\right]}{4!(1+\beta\theta_e)^3}X^4+\frac{M^6\theta_e^{3n+1}\begin{bmatrix}(n+1)(4n+1)+8(n+1)(n-2)\beta\theta_e\\+(4n^2-13n+8)\beta^2\theta_e^2\end{bmatrix}}{6!(1+\beta\theta_e)^5}X^6$$
$$+\frac{M^8\theta_e^{4n+1}\begin{bmatrix}(n+1)(34n^2+5n+1)+3(n+1)(34n^2+5n+1)\beta\theta_e\\+3(n+1)(34n^2-135n+200)\beta^2\theta_e^2\\+(34n^3+171n^2-474n-896)\beta^3\theta_e^3\end{bmatrix}}{8!(1+\beta\theta_e)^7}X^8$$
$$+\frac{M^{10}\theta_e^{5n+1}\begin{bmatrix}(n+1)(496n^3-66n^2+69n+1)\\+2(n+1)(992n^3-2094n^2-687n-166)\beta\theta_e\\+12(n+1)(248n^3-1014n^2+1129n+651)\beta^2\theta_e^2\\+2(n+1)(992n^3-6018n^2+15747n-19948)\beta^3\theta_e^3\\+(496n^4-3494n^3+12513n^2-31538n+51184)\beta^4\theta_e^4\end{bmatrix}}{10!(1+\beta\theta_e)^9}X^{10}+\ldots \tag{160}$$

For the above solutions, the choice of the auxiliary parameters for the homotopy perturbation method is,

$$L(\theta)=\frac{d^2\theta}{dx^2},\quad N(\theta)=\beta\theta\frac{d^2\theta}{dx^2}+\beta\left(\frac{d\theta}{dx}\right)^2-M^2\theta^{n+1},\ f(x)=0,\ \theta_0(x)=\theta_e \tag{161}$$

The Taylor series expansion solutions are the expansions about $x = 0$.

The complete solution is obtained once the constant θ_e is determined by imposing the second boundary conditions given by $\theta(1)=1$. Note that the value of θ_e lies in the interval (0,1).

4. Conclusions

In this work, the conditions of similar analytical solutions of homotopy perturbation, differential transformation and Taylor series expansion methods for linear and nonlinear differential equations have been investigated and established. From the different examples presented, it was established that if some specific values or functions are assigned to the auxiliary parameters in the homotopy perturbation method, the approximate analytical solutions provided by homotopy perturbation method is entirely similar to the approximate analytical solutions given by differential transformation and traditional Taylor series expansion methods. This means that the choice of the auxiliary parameters determines the possibility that homotopy perturbation will give similar approximate analytical solutions like Taylor series expansion and differential transformation methods. The results of TSM at the center exactly correspond to the results of HPM and DTM. These methods give the solutions in the form of a convergent series with easily computable components. Also, additional results were established from the studies which are stated as follows:

i. While the DTM and TSM can be taken as semi-analytical methods for boundary-value problems, HPM is a total approximate analytical method as it does not need any numerical scheme as practice in DTM and TSM when the value(s) that will satisfy the terminal boundary condition(s) which is/are to be found when solving boundary-value problems.

ii. The DTM can be taken as an extended Taylor series expansion method as it allows an easy generalization of the Taylor series expansion method to various derivation procedures. However, Taylor series method suffers from certain computational difficulties. Also, DTM produces fast convergence to the available analytical solution. DTM requires less computations than the traditional Taylor series expansion method because DTM does not need to calculate derivatives or partial derivatives as done in Taylor series expansion method.

iii. The study also showed that the differential transformation method is simple and easy to use. It produces reliable results with relatively low computational works and algorithmic nature. Also, DTM minimizes the computational difficulties of the TSM in that the components are easily determined. Further, HPM is capable of reducing the volume of the computational work as compared to the TSM.

It is believed that this work will greatly assist and enhance the understanding of mathematical solutions providers and enthusiasts as it provides better physical insight into the development of analytical solutions of linear and nonlinear differential equations.

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