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Lie Symmetry Analysis method to first-order M-fractional differential equations

Mousa Ilie [a,*], Ali Khoshkenar [a]

[a] Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran

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ABSTRACT

Finding analytical or numerical solutions of fractional differential equations is one of the bothersome and challenging issues among mathematicians and engineers, specifically in recent years. The objective of this paper is to solve linear and nonlinear fractional differential equations for instance first order linear fractional differential equation, Bernoulli, and Riccati fractional differential equations by using Lie Symmetry method, in accordance with M-fractional derivative. For each equation, some numerical examples are presented to illustrate the proposed approach.

1. Introduction

Fractional calculus is as old as the usual calculus. In the past several years, many of researchers have been trying to generalize the concept of the usual derivatives. Nowadays there are many definitions for the fractional derivative. Two the earliest of definitions are as follows [1-8].

Definition 1. ***(Riemann-Liouville definition)*** If n is a positive integer and $\alpha \in [n-1\,,n)$ the αth derivative of f is given by

$$D_a^{\alpha}(f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt. \tag{1}$$

Definition 2. (***Caputo definition)***: If n is a positive integer for $\alpha \in [n-1\,,n)$ the αth derivative of f is

$$D_a^{\alpha}(f)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^n(t)}{(x-t)^{\alpha-n+1}}dt. \tag{2}$$

* Corresponding author.
E-mail addresses: ilie@iaurasht.ac.ir, mousailie52@gmail.com (M.Ilie)

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The presented definitions are attempted to satisfy the usual properties of the standard derivative [1]. The only property inherited by all definitions of fractional derivative is the linearity property, but there are some disadvantages that caused their application confront with difficulty [1].

In 2014, Khalil et al. proposed the so-called conformable fractional derivative of order integer α to generalize the classical properties of calculus [1]. One of the definition that have been presented recently is conformable fractional derivative that removed some of drawbacks the presented definitions. More recently, in 2014, Katugampola has also proposed an alternative fractional derivative with classical properties, which refers to the Leibniz and Newton calculus, similar to the conformable fractional derivative [9]. In 2017, Sousa and et al., introduced an M-fractional derivative involving a Mittag-Leffler function with one parameter that also satisfies the properties of integer-order calculus [10,11]. In this sense, Sousa and Oliveira introduced a truncated M-fractional derivative type that unifies four existing fractional derivative types mentioned above and which also satisfied the classical properties of integer-order calculus [12].

Definition 3. ***(Truncated Mittag-Leffler function)*** With $\beta > 0$, and $z \in \mathbb{C}$, the truncated Mittag-Leffler function of one parameter is defined by[10,12]

$$ {}_i\mathbb{E}_\beta(z) = \sum_{k=0}^{i} \frac{z^k}{\Gamma(k\beta+1)}. \tag{3}$$

Definition 4. ***(Truncated M-fractional derivative)*** Given a function $f: [0,\infty) \to \mathbb{R}$. Then the truncated M-fractional derivative of f of order α is defined by

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(x) = \lim_{\varepsilon\to 0} \frac{f\left(x\ {}_i\mathbb{E}_\beta(\varepsilon x^{-\alpha}\right) - f(x)}{\varepsilon} \tag{4}$$

for all $x > 0, \alpha \in (0,1)$, where ${}_i\mathbb{E}_\beta(.)$, $\beta > 0$ is the Mittag-Leffler function with one parameter as defined by in Eq. (3) [12].

Note that if f is α- differentiable in some $(0,\ a), a > 0$, and $\lim_{x\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(x)$ exists, then one can define[12]

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(0) = \lim_{x\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(x).$$

If the M-fractional derivative of f of order α exists, then we simply say that f is α- differentiable [12].

One can easily show that truncated M-fractional derivative satisfies all the following properties [12]. Let $\alpha \in (0,1)$ and f, g be functions α-differentiable at a point $x > 0$, Then

A. (Linearity Rule)For $a, b \epsilon \mathbb{R}$ ${}_i\mathcal{D}_M^{\alpha,\beta}(af + bg) = a\left({}_i\mathcal{D}_M^{\alpha,\beta} f\right) + b\left({}_i\mathcal{D}_M^{\alpha,\beta} g\right)$,

B. For all $p \epsilon \mathbb{R}$, ${}_i\mathcal{D}_M^{\alpha,\beta} x^p = \frac{p}{\Gamma(\beta+1)} x^{p-\alpha}$,

C. For all constant functions $f(x) = \lambda$, ${}_i\mathcal{D}_M^{\alpha,\beta} \lambda = 0$,

D. (Product Rule) ${}_i\mathcal{D}_M^{\alpha,\beta}(f.g) = g.\left({}_i\mathcal{D}_M^{\alpha,\beta} f\right) + f.\left({}_i\mathcal{D}_M^{\alpha,\beta} g\right)$,

E. (Quotient Rule) ${}_i\mathcal{D}_M^{\alpha,\beta}\left(\frac{f}{g}\right) = \frac{g.\left({}_i\mathcal{D}_M^{\alpha,\beta} f\right) - f.\left({}_i\mathcal{D}_M^{\alpha,\beta} g\right)}{g^2}$,

F. (Chain Rule) If function f, ordinary differentiable at $g(x)$, then

${}_i\mathcal{D}_M^{\alpha,\beta}(fog) = f'(g(x).\left({}_i\mathcal{D}_M^{\alpha,\beta} g\right)$,

G. ${}_i\mathcal{D}_M^{\alpha,\beta} f(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \frac{df}{dx}$.

Definition 5. Let $\beta > 0$, $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}$, and f, n, times differentiable (in the classical of sense) for $x > 0$. Then the local M-derivative of order n, of function f is defined by

$$ {}_{i}\mathcal{D}_{M}^{\alpha,\beta;n} f(x) = \lim_{\varepsilon\to 0} \frac{f^{(n)}\left(x \ {}_{i}\mathbb{E}_{\beta}(\varepsilon x^{-\alpha})\right) - f^{(n)}(x)}{\varepsilon}, \tag{5} $$

if and only if the limit exists [12].

The study of fractional differential equations has demonstrated very valuable over time. Solving fractional differential equations is very important, due to this fact, finding an exact solution and an approximate solution of fractional differential equations is clearly an important task. The authors of this article suggest dear researchers to refer to the articles cited to see some useful methods for solving fractional differential equations [14-36]. The purpose of this paper is solving fractional differential equations by Lie Symmetry method, on the basis of truncated M-fractional derivative.

The organization of this paper is as follows: In Section 2, Lie invariance condition will be described. In subsection 3.1, 3.2, and 3.3 respectively the method will be used to solve first order linear M-fractional differential equations, Bernoulli M-fractional differential equation, and Riccati M-fractional differential equation. For each equation there are some examples, as well. Finally, in section 4. discussion will be given.

2. Lie Symmetry method

Let consider the invariance of

$$ \frac{dy}{dx} = F(x,y), \tag{6} $$

under the infinitesimal transformation

$$ \bar{x} = x + X(x,y)\varepsilon + O(\varepsilon^2), \ \ \bar{y} = y + Y(x,y)\varepsilon + O(\varepsilon^2). \tag{7} $$

The derivatives transform under the infinitesimal transformations (7) is as follows [36-39]

$$ \frac{d\bar{y}}{d\bar{x}} = \frac{dy}{dx} + \left(\frac{\partial Y}{\partial x} + \left[\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right]\frac{dy}{dx} - \frac{\partial X}{\partial y}\left(\frac{dy}{dx}\right)^2\right)\varepsilon + O(\varepsilon^2). \tag{8} $$

Consider the following ODE

$$ \frac{d\bar{y}}{d\bar{x}} = F(\bar{x},\bar{y}). \tag{9} $$

Substituting the infinitesimal transformations (7) and first-order derivative transformation (8) into (9) yields

$$ \frac{dy}{dx} + \left(\frac{\partial Y}{\partial x} + \left[\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right]\frac{dy}{dx} - \frac{\partial X}{\partial y}\left(\frac{dy}{dx}\right)^2\right)\varepsilon + O(\varepsilon^2) = F\left(x + X(x,y)\varepsilon + O(\varepsilon^2), y + Y(x,y)\varepsilon + O(\varepsilon^2)\right). $$

Expanding to order $O(\varepsilon^2)$ gives

$$ \frac{dy}{dx} + \left(\frac{\partial Y}{\partial x} + \left[\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right]\frac{dy}{dx} - \frac{\partial X}{\partial y}\left(\frac{dy}{dx}\right)^2\right)\varepsilon + O(\varepsilon^2) = F(x,y) + \left(X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y}\right)\varepsilon + O(\varepsilon^2), \tag{10} $$

by using (6), Eq. (10) is satisfied to $O(\varepsilon^2)$ if

$$ \frac{\partial Y}{\partial x} + \left[\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right]F - \frac{\partial X}{\partial y}F^2 = X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y}. \tag{11} $$

This is known as Lie's Invariance Condition. For a given $F(x,y)$, any functions $X(x,y)$ and $Y(x,y)$ that solve equation (11) are the infinitesimals [36-39].

A transformation can be constructed that would lead to a separable equation involving r, and s. Consider

$$r = r(x,y), \quad s = s(x,y) \tag{12}$$

and require that (12) be invariant, that is,

$$\bar{r} = r(\bar{x},\bar{y}), \quad \bar{s} = s(\bar{x},\bar{y}). \tag{13}$$

The separable equation

$$\frac{ds}{dr} = G(r),$$

is invariant under

$$\bar{r} = r, \quad \bar{s} = s + \varepsilon. \tag{14}$$

Differentiating (13) with respect to ε and setting $\varepsilon = 0$, and using Eqs. (14), gives

$$X(x,y)\frac{\partial r}{\partial x} + Y(x,y)\frac{\partial r}{\partial y} = 0, \quad X(x,y)\frac{\partial s}{\partial x} + Y(x,y)\frac{\partial s}{\partial y} = 1. \tag{15}$$

Thus, if the infinitesimals X and Y are had, solving (15) would give rise to the transformation that will separate the given ordinary differential equation (6) [36-39].

3. Some applications of lie symmetry method for solving M-fractional equations

In this section by using lie symmetry method, presented a general solution for linear and nonlinear first order M-fractional differential equations.

3.1. First order linear M-fractional differential equations

General form of a first order linear M-fractional differential equations is as follow,

$${}_i\mathcal{D}_M^{\alpha,\beta} y(x) + p(x)y(x) = q(x), \tag{16}$$

where $p(x), q(x)$ are α −differentiable functions, and $y(x)$ is an unknown function [13,22,36].

By using the property (G), Eq. (16) can be written as the following form

$$\frac{x^{1-\alpha}}{\Gamma(\beta+1)} y'(x) + p(x)y(x) = q(x),$$

so,

$$y'(x) + P(x)y(x) = Q(x), \tag{17}$$

where $P(x) = \Gamma(\beta+1)x^{\alpha-1}p(x)$ and $Q(x) = \Gamma(\beta+1)x^{\alpha-1}q(x)$. The Eq. (17) is a first order linear ordinary differential equation [13].

This is invariant under the Lie group

$$\bar{x} = x \quad , \quad \bar{y} = y + \varepsilon e^{-\int P(x)dx}, \tag{18}$$

gives

$$\frac{dy}{dx} - \varepsilon P(x) e^{-\int P(x)dx} + P(x)\left(y + \varepsilon e^{-\int P(x)dx}\right) = Q(x). \tag{19}$$

Expanding (19) gives

$$\frac{dy}{dx} + P(x)y(x) = Q(x).$$

From the lie group (18), we obtain the infinitesimals $X = 0$ and $Y = e^{-\int P(x)dx}$, [36].

The change of variables are obtained by solving (15), as follows

$$e^{-\int P(x)dx} \frac{\partial r}{\partial y} = 0 \,, \; e^{-\int P(x)dx} \frac{\partial s}{\partial y} = 1.$$

Thus

$$r = R(x) \;, \; s = y\, e^{\int P(x)dx} + S(x),$$

where $R(x)$ and $S(x)$ are arbitrary functions. Choosing $R(x) = x$ and $S(x) = 0$ results in

$$x = r \,, \; y = s e^{-\int P(r)dr}. \tag{20}$$

By calculating $\frac{dy}{dx}$, we obtain

$$\frac{dy}{dx} = \frac{ds}{dr}\, e^{-\int P(r)dr} - s\, P(r) e^{-\int P(r)dr}.$$

and substituting into (17) and simplifying, the following result will be given.

$$\frac{ds}{dr} = Q(r) e^{\int P(r)dr}. \tag{21}$$

which (21) is a separable equation [36-39].

Example 1. Consider the linear M-fractional differential equation

$${}_i\mathcal{D}_M^{0.5,\beta} y + \sqrt{x}\, y = x\sqrt{x}\,. \tag{22}$$

By using property (G), Eq. (22), can be rewritten as follows

$$y' + \Gamma(\beta + 1)y = \Gamma(\beta + 1)x, \tag{23}$$

that $P(x) = \Gamma(\beta + 1)$, and $Q(x) = \Gamma(\beta + 1)x$, thereupon by Lie Symmetry method

$$x = r \;\;, \; y = s\, e^{-\Gamma(\beta+1)r}. \tag{24}$$

$$\frac{ds}{dr} = \Gamma(\beta + 1) r e^{\Gamma(\beta+1)r},$$

which is the separable equation having a general solution is as follows

$$s = r e^{\Gamma(\beta+1)r} - \frac{e^{\Gamma(\beta+1)r}}{\Gamma(\beta+1)} + C. \tag{25}$$

By substitute (25) in (24) the general solution to M-fractional equation (22) can be presented as the following form

$$y = C\,\mathbb{E}_1(-\Gamma(\beta+1)x) + x - \frac{1}{\Gamma(\beta+1)},$$

wherever C is constant among and arbitrary. In figure 1. The analytical solution of the M-fractional differential Eq. (22) for $C = -50$, and values different β, are plotted.

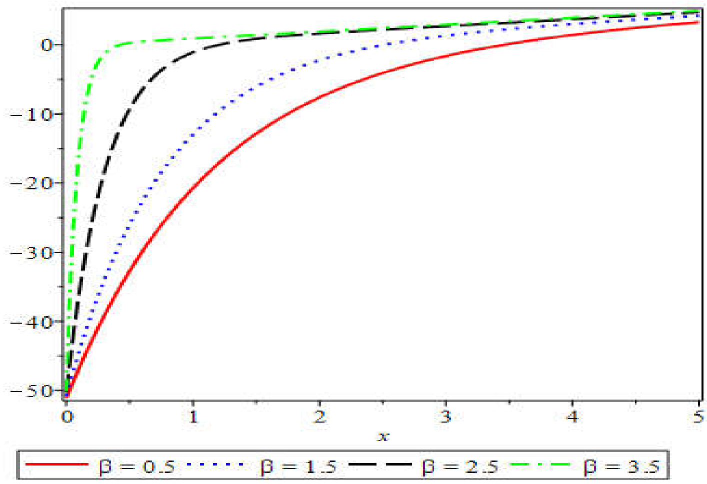

Figure.1. The exact solution of example 1. for $\beta = 0.5, 1.5, 2.5, 3.5, C = -50$.

3.2. The Bernoulli M-fractional differential equation

The Bernoulli M-fractional differential equations have the following general form,

$${}_i\mathcal{D}_M^{\alpha,\beta} y + p(x)y = q(x)y^n, \qquad n \neq 0,1 \tag{26}$$

where α −differentiable functions, $y(x)$ is an unknown function [13,22,36].

By using property (G) Eq. (26) leads to

$$\frac{x^{1-\alpha}}{\Gamma(\beta+1)} y' + p(x)y = q(x)y^n,$$

so,

$$y'(x) + P(x)y = Q(x)y^n, \tag{27}$$

where $P(x) = \Gamma(\beta+1)x^{\alpha-1}p(x)$ and $Q(x) = \Gamma(\beta+1)x^{\alpha-1}q(x)$, equation (27) is the Bernoulli equation [13].

Assuming $X = 0$, Lie's invariance condition becomes,

$$\frac{\partial Y}{\partial x} + \frac{\partial Y}{\partial y}\left(Q(x)y^n - P(x)y\right) = Y\left(nQ(x)y^{n-1} - P(x)\right). \tag{28}$$

That

$$Y = e^{(n-1)\int P(x)dx}\, y^n,$$

satisfies (28). To obtain a change of variables, it is necessary to solve [36-39]

$$e^{(n-1)\int P(x)dx}\, y^n \frac{\partial r}{\partial y} = 0\,,\ e^{(n-1)\int P(x)dx}\, y^n\, \frac{\partial s}{\partial y} = 1.$$

Example 2. Consider the Bernoulli M-fractional differential equation

$${}_i\mathcal{D}_M^{\frac{2}{3},\beta} y = 2\sqrt[3]{x}\, y + x\sqrt[3]{x}\, y^2. \tag{29}$$

By using property (G), in equation (29), leads to

$$y' - 2\Gamma(\beta+1)y = \Gamma(\beta+1)xy^2. \tag{30}$$

here $P(x) = -2\Gamma(\beta+1)$, $n = 2$, Choosing $R(x) = x$, and $S(x) = 0$, leads to

$$r = x \ \ ,\ \ s = \frac{-1}{ye^{-2\Gamma(\beta+1)x}}. \tag{31}$$

Under this change of variables, the Bernoulli differential equation (30) becomes

$$\frac{ds}{dr} = re^{2r}.$$

The general solution of M-fractional differential equation (29) is as the following form

$$y = \frac{4\Gamma(\beta+1)}{C\Gamma(\beta+1)\mathbb{E}_1(-2\Gamma(\beta+1)x)-2\Gamma(\beta+1)x+1},$$

wherever C is constant among and arbitrary. In figure 2. The analytical solution of the M-fractional differential Eq. (29) for $C = -30$, and values different β, are plotted.

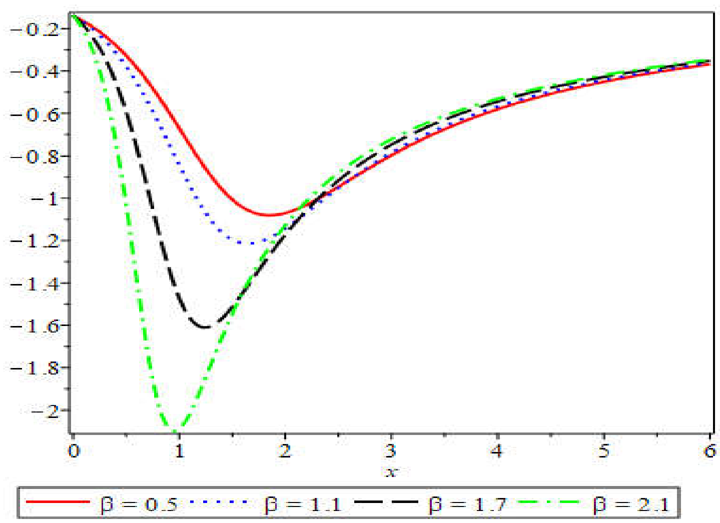

Figure. 2. The exact solution of example 1. for $\beta = 0.5, 1.1, 1.7, 2.1, C = -30$.

3.3. The Riccati M-fractional differential equation

The general form of a Riccati M-fractional differential equation is as the following

$${}_i\mathcal{D}_M^{\alpha,\beta} y = p(x)y^2 + q(x)y + r(x), \tag{32}$$

where α −differentiable functions and $y(x)$ is an unknown function [13,22,36].

By using property (G), we obtain

$$\frac{x^{1-\alpha}}{\Gamma(\beta+1)} y' = p(x)y^2 + q(x)y + r(x),$$

so,

$$y' = P(x)y^2 + Q(x)y + R(x), \tag{33}$$

where $P(x) = \Gamma(\beta+1)x^{\alpha-1}p(x)$, $Q(x) = \Gamma(\beta+1)x^{\alpha-1}q(x)$ and $R(x) = \Gamma(\beta+1)x^{\alpha-1}r(x)$. The equation (33) is the Riccati equation [13]. We will assume $X = 0$, giving Lie's invariance condition as [36]

$$\frac{\partial Y}{\partial x} + \left(P(x)y^2 + Q(x)y + R(x)\right)\frac{\partial Y}{\partial y} = \left(2P(x)y + Q(x)\right)Y. \tag{34}$$

One solution of (34) is

$$Y = (y - y_1)^2\, F(x),$$

where y_1 is one solution to (32) and F satisfies

$$F' + (2Py_1 + Q)F = 0\,. \tag{35}$$

Variables r and s, it is necessary to solve

$$(y - y_1)^2\, F(x)\frac{\partial r}{\partial y} = 0, \quad (y - y_1)^2\, F(x)\frac{\partial s}{\partial y} = 1,$$

from which we obtain

$$r = K(x)\ ,\ s = S(x) - \frac{1}{(y - y_1)F}\,,$$

where $K(x)$ and $S(x)$ are arbitrary functions. Setting $K(x) = x$ and $S(x) = 0$, yields to

$$x = r\ ,\ y = y_1 - \frac{1}{s\,F(r)}\,, \tag{36}$$

thereby transforming the original Riccati equation (33) to

$$\frac{ds}{dr} = \frac{a(r)}{F(r)}.$$

It is interesting that the usual linearizing transformation is recovered using Lie Symmetry method [36-39].

Example 3. Consider Riccati fractional equation as follows

$${}_i\mathcal{D}_M^{0.5,\beta}u(x) + 2x^2\sqrt{x} = \frac{1}{\Gamma(\beta+1)\sqrt{x}}\ u(x) + 2\sqrt{x}\,(u(x))^2\,, \tag{37}$$

that has a solution such as $y_1 = x$, [22].

Clearly by using property (G), this equation changing to

$$y' = 2\Gamma(\beta+1)y^2 + \frac{1}{x}\,y - 2\Gamma(\beta+1)x^2. \tag{38}$$

From (35) F is as the following form

$$F(x) = \frac{1}{x}e^{-4\Gamma(\beta+1)x},$$

Thus, under the change of variables given in (36), namely

$$x = r\ ,\ y = r - \frac{r}{se^{-4\Gamma(\beta+1)r}}\,, \tag{39}$$

the original ordinary differential equation becomes

$$\frac{ds}{dr} = re^{4\Gamma(\beta+1)r}\,.$$

Which is a separable equation having a general solution is as follows

$$s = \frac{r}{4\Gamma(\beta+1)} e^{4\Gamma(\beta+1)r} - \frac{1}{16\Gamma(\beta+1)} e^{4\Gamma(\beta+1)r} + C,$$

and the general solution of nonlinear M-fractional equation (37), can be presented as the following form

$$u(x) = \frac{x[C+\mathbb{E}_1(2\Gamma(\beta+1)x^2)]}{C-\mathbb{E}_1(2\Gamma(\beta+1)x^2)},$$

wherever C is constant among and arbitrary. In figure 3. The analytical solution of the M-fractional differential Eq. (37) for $C = -4$, and values different β, are plotted.

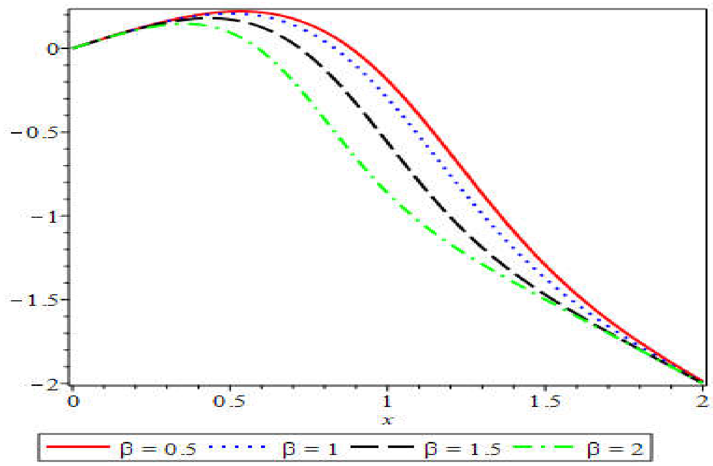

Figure. 3. The exact solution of example 1. for $\beta = 0.5, 1, 1.5, 2, C = -4$.

4. Conclusion

In this paper, Lie Symmetry Analysis method have been applied for solving fractional differential equations, appropriate to truncated M-fractional derivative. First order linear M-fractional differential equations, Bernoulli and Riccati M-fractional differential equations, have been solved by the presented method. For each cases some examples are given for more description and illumination. The results demonstrated that the offered method is easily applicable for this kind of M-fractional differential equations.

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