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THE INVERSE MONOID ASSOCIATED TO A GROUP AND THE SEMIDIRECT PRODUCT OF GROUPS

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Abstract. In this paper, we construct an inverse monoid $M(G)$ associated to a given group G by using the notion of the join of subgroups and then, by applying the left action of monoid M on a semigroup S, we form a semigroup $S\omega M$ on the set $S \times M$. The finally result is to build the semi direct product of groups associated to the group action on an another group.

1. Intoduction and Preliminaries

Semigroups and monoids are convenient algebraic systems for stating theorems on groups and playing an important role in algebra and in many other branches of science. In semigroup theory, by using the actions of semigroups, we can introduce new algebraic structures which may employ in other area like computer science. The same is true for the wreath product as a specialized product of two groups. It helps to construct interesting examples of groups and can be applicable in semigroups as well. For example, it is used to prove the theorem on the decomposition of every finite semigroup automation into a step wise combination of flip-flop and simple group automata. With the help of these notions, we introduce new algebraic structures.

In this Section, we recall some requisite definitions and then, in Section 2, we construct an inverse monoid $M(G)$ associated to the group

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G. Finally, we draw our conclusions in Section 3. Throughout the article, our notations are based on [3, 5, 7] and [11].

Let S be a non-empty set. A binary operation on S is a mapping of $S \times S$ into S denoted by a dot " $\cdot$ " such that the image of the ordered pares (a, b) in $S \times S$ is $a \cdot b$. For the sake of simplicity, we shall omit the dot and write ab. A *semigroup* S is a non-empty set equipped with a binary operation as above such that for all x, y and z of S, $(xy)\, z = x\,(yz)$. A familiar example of semigroup is the set of functions on a non-empty set X under the operation of composition. A semigroup S with the identity element is called *monoid*. By given an arbitrary semigroup S, we define S^1 to be S if S is a monoid and to be $S \cup \{1\}$ if it is not a monoid where $1x = x1 = x$, for all x in S. In this way, S^1 is a monoid. An element e in S is called an *idempotent* if $e^2 = ee = e$. The set of idempotents of S is denoted by $E\,(S)$. An element a in S is called *regular*, if and only if a in aSa, i.e, $a = axa$ for some x in S. A semigroup S is called *regular* if every element of S is regular. A semigroup S is said to be a *inverse semigroup* if for every element a in S, there is a unique element a^{-1} in S in the sense that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. The element a^{-1} is usually called an *inverse* of a in S. We recall that the following conditions on a regular semigroup S are equivalent: (i) idempotents commute and (ii) inverses are unique. We also note that to each a in S there corresponds a pair of idempotents e and f such that

$$aa^{-1} = e, a^{-1}a = f, ea = a, af = a$$

The idempotents e and f are called respectively the left and right *units* of a. Moreover, for any two elements a, b in $S; (ab)^{-1} = b^{-1}a^{-1}$. Alternatively, inverse semigroup are precisely regular semigroups whose idempotents commute.

A subset T of S is an inverse subsemigroup of S if T is closed under the operations of S, that is, for all t_1 and t_2 of T, t_1t_2 in T and t_1^{-1} in T. If e in $E\,(S)$ then we denote by S_e, the inverse subsemigroup eSe (See [8, 13, 1]).

A partial permutation on a set X is a bijection map from a subset of X to a subset of X. The set of partial one-to-one transformations on a non-empty set X under the operation of composition is an important example of inverse semigroup. This semigroup is called the *symetric inverse semigroup* on X and denoted by $I\,(X)$. By a theorem of Vagner [15] and Preston [14] every inverse semigroup S is isomorphic to an inverse subsemigroup of $I\,(X)$ for a suitable set X.

Let S is an inverse semigroup. For each s in S take τ_s in $I(S)$ where $\tau_s(x) = sx$, x in Ss^{-1}. Then the mapping $\tau : S \longrightarrow I(S)$ defined by $\tau(s) = \tau_s$ is an embedding of S into $I(S)$.

Let S be a semigroup and M be a monoid with 1 as an identity. To simplify notation, we will write S additively, without assuming that S is commutative. A *left action* of M on S is a mapping of $M \times S$ into S defined by $(m, s) \longmapsto ms$ and satisfying for all s, s_1 and s_2 in S, m, m_1 and m_2 in M:

(i) $m(s_1 + s_2) = ms_1 + ms_2$,
(ii) $m_1(m_2 s) = (m_1 m_2) s$,
(iii) $1s = s$.

Of course, this just amounts to giving a morphism from M to the monoid of endomorphisms acting on the left of S. This action is used to form a semigroup $S\omega M$ on the set $S \times M$ with the multiplication defined by $(s, m)(s, m') = (s + ms', mm')$. Note that $S\omega M$ is called a *semidirect product* of S and M [4]. Moreover, if the elements of $S \times M$ are represented by matrices of the form $\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix}$ where $s \in S$ and $m \in M$ then the previous formula can be written as

$$\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s' & m' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s + ms' & mm' \end{pmatrix}.$$

If the elements of $S \times M$ are denoted by (s, m), then one can define the product $M\omega S$ by $(m, s)(m', s') = (mm', sm' + s')$. If the elements of $M \times S$ are represented by matrices of the form $\begin{pmatrix} m & 0 \\ s & 1 \end{pmatrix}$ then the formula can be written as

$$\begin{pmatrix} m & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} m' & 0 \\ s' & 1 \end{pmatrix} = \begin{pmatrix} mm' & 0 \\ sm' + s' & 1 \end{pmatrix}.$$

Recall that, if S is a semilattice and let G is a group, then $S\omega M$ is an inverse semigroup for any left action of G on S. (See [13, 2, 9, 12])

A *group* G is an ordered pair $(G, \cdot)$ consisting of a non void set G equipped with a binary operation " $\cdot$ " which satisfies the following properties:

(i) For all elements x, y and z of G, $(xy)z = x(yz)$,
(ii) There exists an element 1_G in G such that for all $x \in G$, $1_G x = x 1_G = x$,
(iii) For each x in G, there exists an element $x^{-1} \in G$ such that $x^{-1}x = xx^{-1} = 1_G$.

A non empty subset H of a group G is called a *subgroup* of G, denoting by $H \leqslant G$, if and only if for all $x, y \in H$, $xy \in H$ and x^{-1} in H. We may use $Sub(G)$ to stand for the set of all subgroups of the group G.

Obviously, if $H \leqslant G$ and $a \in G$, then $aHa^{-1} \subseteq Sub(G)$. For subgroups H and K of group G, the set $HK = \{hk : h \in H \text{ and } k \in K\}$ may not be a subgroup of G. In fact, $HK \leqslant G$ if and only if $HK = KH$ (See [6, 10, 1]). By $H \vee K$ we mean the joint subgroup of subgroups H and K of group G.

2. Main Results

In this section, we construct an inverse semigroup $M(G)$ associated to group G.

Theorem 2.1. *Let G be a group and let the inclusion $\subseteq$ be totally ordered relation on $Sub(G)$. Consider the set*

$$M(G) = \{Ha : H \leqslant G, a \in G\},$$

and define an operation "$*$" on $M(G)$ by $Ha * Kb = (H \vee aKa^{-1})\, ab$. Then the followings hold:

(i) *The set $(H \vee aKa^{-1})\, ab$ is the smallest element of $M(G)$ containing the product $HaKb$,*
(ii) *$(M(G), *)$ is an inverse monoid.*
(iii) *$Sub(G)$ is the set of idempotents.*

Proof. Since the inclusion relation $\subseteq$ is totally ordered on $Sub(G)$ then we have $(H \vee aKa^{-1}) = H$ or $(H \vee aKa^{-1}) = aKa^{-1}$. Moreover, for all $K \in Sub(G)$ and $a \in G$, $aKa^{-1} \in Sub(G)$ because the set $M(G)$ is closed under the operation $*$. Let $x = hakb \in HaKb$ arbitrarily where h in H and k in K. Since h is an element of $H \subseteq H \vee aKa^{-1}$ and $aka^{-1} \in aKa^{-1} \subseteq H \vee aKa^{-1}$ so $x = (haka^{-1})\, ab$ in $(H \vee aKa^{-1})\, ab$ which yields that $HaKb \subseteq (H \vee aKa^{-1})\, ab$. Let $Lc \in M(G)$ contains $HaKb$. Since $HaKb \subseteq Lc$, $1_G \in H$ and $1_G \in K$ so $1_G a 1_G b = ab \in Lc$. Moreover $HaKb \subseteq Lc$ and $1_G \in K$, implies $Hab \subseteq Lc$. On the other hand, $HaKb = (HaKa^{-1})\, ab \subseteq Lc$ and $1_G \in H$ implies $(aKa^{-1})\, ab \subseteq Lc$. Therefore we get $(H \vee aKa^{-1})\, ab \subseteq Lc$ which completes the proof of (i). By considering elements Ha, Kb and Lc of $M(G)$ we have

$$\begin{aligned}
(Ha * Kb) * Lc &= \left(\left(H \vee aKa^{-1}\right) ab\right) * Lc, \\
&= \left(\left(H \vee aKa^{-1}\right) \vee (ab)\, L\, (ab)^{-1}\right) (ab)\, c, \\
&= \left(H \vee aKa^{-1} \vee abLb^{-1}a^{-1}\right) abc.
\end{aligned}$$

Also, we get

$$\begin{aligned}
Ha * (Kb * Lc) &= Ha * \left(K \vee bLb^{-1}\right) bc, \\
&= \left(H \vee a\left(K \vee bLb^{-1}\right) a^{-1}\right) abc, \\
&= \left(H \vee aKa^{-1} \vee abLb^{-1}a^{-1}\right) abc.
\end{aligned}$$

This shows that the operation $*$ is associative and so $M(G)$ is a semigroup. If $Ha \in M(G)$ then

$$Ha * \{1_G\} = \{1_G\} * Ha = Ha,$$

where $\{1_G\} = \{1_G\} 1_G \in Sub(G) \cap M(G)$. So, the identity element for $(M(G), *)$ is $\{1_G\}$. For the elements Ha and $(a^{-1}Ha) a^{-1}$ of $M(G)$ we have

$$\begin{aligned}\left(Ha * \left(a^{-1}Ha\right) a^{-1}\right) * Ha &= \left(H \vee a \left(a^{-1}Ha\right) a^{-1}\right) aa^{-1} * Ha,\\ &= (H \vee H) 1_G * Ha = H * Ha = Ha.\end{aligned}$$

And

$$\left(a^{-1}Ha\right) a^{-1} * Ha * \left(a^{-1}Ha\right) a^{-1} = \left(a^{-1}Ha\right) a^{-1},$$

which show that $(a^{-1}Ha) a^{-1}$ is an inverse of Ha in monoid $(M(G), *)$. So the proof of (ii) is completed. For (iii) suppose that Ha is idempotent:

$$Ha = Ha * Ha = \left(H \vee aHa^{-1}\right) a^2,$$

Then, in particular, $a^2 = 1_G a^2 \in Ha$, i.e. $a^2 = ha$ for some $h \in H$. Hence $a = h \in H$ so $Ha = H$. This means that the idempotents of $(M(G), *)$ are precisely the subgroups of G. Since for any two subgroups H, K of G, $H * K = K * H = H \vee K$ thus the idempotents commute and so $(M(G), *)$ is an inverse monoid. □

Remark 2.2. For each $Ha \in M(G)$, the map $\tau_{Ha} : M(G) \to M(G)$ defined by $\tau_{Ha}(Kb) = (H \vee aKa^{-1}) ab$ where $Kb \in M(G) * (a^{-1}Ha) a^{-1}$ is an element of $I(M(G))$. Moreover, the map $\tau : M(G) \to I(M(G))$ defined by $\tau(Ha) = \tau_{Ha}$ is an embedding of $M(G)$ into $I(M(G))$.

Proof. $(M(G), *)$ is an inverse monoid and by using the inverse of $Ha \in M(G)$, i.e. $(Ha)^{-1} = (a^{-1}Ha) a^{-1}$ the proof is complete. □

Remark 2.3. (i) For all $H \in Sub(G)$, $H * M(G) * H$ is the inverse subsemigroup of $(M(G), *)$. (ii) For all $Ha \in M(G)$,

$$Ha * \left(a^{-1}Ha\right) a^{-1} * M(G) * Ha * \left(a^{-1}Ha\right) a^{-1},$$

and

$$\left(a^{-1}Ha\right) a^{-1} * Ha * M(G) * \left(a^{-1}Ha\right) a^{-1} * Ha,$$

are the inverse subsemigroups of $(M(G), *)$.

Proof. For (i) we use the fact that $E_{M(G)} = Sub(G)$. (ii) Since for all $Ha \in M(G)$, we have $(Ha)^{-1} = (a^{-1}Ha) a^{-1}$,

$$Ha * \left(a^{-1}Ha\right) a^{-1} * Ha * \left(a^{-1}Ha\right) a^{-1} \in E_{M(G)},$$

and

$$\left(a^{-1}Ha\right)a^{-1} * Ha * \left(a^{-1}Ha\right)a^{-1} * Ha \in E_{M(G)},$$

so the proof is complete. $\square$

Theorem 2.4. *Let G be a group such that for all $H, K \in Sub\,(G)\,, HK = KH$. Define an operation "$\Delta$" on $M\,(G)$ by $Ha\Delta Kb = (HK)\,ab$. Then the followings hold:*

(i) $(M\,(G)\,, \Delta)$ is an inverse monoid.
(ii) $Sub\,(G)$ is the set of idempotents.

Proof. (i) Obviously $M(G)$ is closed with respect to Δ since for all $H, K \in Sub\,(G)$, $HK \in Sub(G)$. If $Ha, Kb, Lc \in M\,(G)$ then

$$(Ha\Delta Kb)\,\Delta Lc = ((HK)\,ab)\,\Delta Lc = ((HK)\,L)\,(ab)\,c = (HKL)\,abc,$$

And

$$Ha\Delta\,(Kb\Delta Lc) = Ha\Delta\,(KL)\,bc = (H\,(KL))\,abc = (HKL)\,abc,$$

so the Δ is an associative operation.

The identity element in $M\,(G)$ is $\{1_G\}$. Indeed, if $Ha \in M\,(G)$ then we have $Ha\Delta\,\{1_G\} = \{1_G\}\,\Delta Ha = Ha$. Since

$$Ha\Delta Ha^{-1}\Delta Ha = HHaa^{-1}\Delta Ha = H1_G\Delta Ha = H\Delta Ha = Ha$$

and $Ha^{-1}\Delta Ha\Delta Ha^{-1} = Ha^{-1}$, so Ha^{-1} is an inverse of Ha in the monoid $(M\,(G)\,, \Delta)$. Now, suppose that Ha is idempotent, i.e. $Ha = Ha\Delta Ha = Ha^2$. Then in particular, $a^2 = 1_G a^2 \in Ha$, i.e, $a^2 = ha$ for some $h \in H$. Hence $a = h \in H$ and so $Ha = H$. In fact the idempotents of $(M\,(G)\,, \Delta)$ are precisely the subgroups of G. For any two subgroups H, K of G $H\Delta K = K\Delta H = HK$. Thus idempotents commute and so $(M\,(G)\,, \Delta)$ is an inverse monoid. $\square$

Remark 2.5. For each $Ha \in M\,(G)$, let $\tau_{Ha} \in I\,(M\,(G))$ defined by $\tau_{Ha}\,(Kb) = (HK)\,ab$, for all $Kb \in M\,(G)\,\Delta Ha^{-1}$. Then the mapping $\tau : M\,(G) \longrightarrow I\,(M\,(G))$ defined by $\tau\,(Ha) = \tau_{Ha}$ is an embedding of $M\,(G)$ into $I\,(M\,(G))$.

Proof. The proof yields from the fact that $(M\,(G)\,, \Delta)$ is an inverse monoid and for $Ha \in M\,(G)$, we have $(Ha)^{-1} = Ha^{-1}$. $\square$

Remark 2.6. (i) For all $H \in Sub\,(G)$, $H\Delta M\,(G)\,\Delta H$ is the inverse subsemigroup of $(M\,(G)\,, \Delta)$. (ii) For all $Ha \in M\,(G)$,

$$Ha\Delta Ha^{-1}\Delta M\,(G)\,Ha\Delta Ha^{-1},$$

and

$$Ha^{-1}\Delta Ha\Delta M\,(G)\,\Delta Ha^{-1}\Delta Ha,$$

are inverse subsemigroups of $(M\,(G)\,, \Delta)$.

Proof. The part (i) is a consequence of $E_{M(G)} = Sub\,(G)$. For (ii), it is enough to note that for all $Ha \in M\,(G)$, we have $(Ha)^{-1} = Ha^{-1}$ and $Ha\Delta Ha^{-1} \in E_{M(G)}$, $Ha^{-1}\Delta Ha \in E_{M(G)}$. □

Theorem 2.7. *Let $(S, +, 0)$ and $(M, \cdot, 1)$ be two monoids and consider the left action of M on S:*

$$\begin{aligned} M \times S &\longrightarrow S \\ (m, s) &\longmapsto ms, \end{aligned}$$

such that it satisfies

- $m\,(s_1 + s_2) = ms_1 + ms_2$,
- $m_1\,(m_2 s) = (m_1 m_2)\,s$,
- $1s = s$,
- $m0 = 0$.

for all $s, s_1, s_2 \in S$ and $m, m_1, m_2 \in M$. Then the followings hold

(i) *For all $m \in M$; the map $\theta_m : S \longrightarrow S, s \longmapsto ms$ is an element of $End\,(S)$,*
(ii) *The map $\theta : (M, \cdot, 1) \longrightarrow (End\,(S), \circ, id_S), m \longmapsto \theta_m$ is a morphism of monoids.*
(iii) *The set $S \times M$ with the multiplication $(s, m)(s', m') = (s + ms', mm')$ is a monoid.*
(iv) *If $K = \{\left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right), s \in S, m \in M\}$, then $(K, \times)$ is a monoid.*
(v) *The map $h : (S \times M, .) \longrightarrow (K, \times), (s, m) \longmapsto \left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right)$ is an isomorphism of monoids.*

Proof. For all $s, s' \in S$, $\theta\,(s + s') = m\,(s + s') = ms + ms' = \theta\,(s) + \theta\,(s')$ and $\theta\,(0) = m0 = 0$ so $\theta_m \in End\,(S)$ and this proved (i). Since for all $s \in S$, $\theta_{mm'}\,(s) = (mm')\,s$ and $(\theta_m \circ \theta_{m'})\,(s) = \theta_m\,(m's) = m\,(m's)$ so $\theta\,(mm') = \theta\,(m) \circ \theta\,(m)$ where $m, m' \in M$ which left the part (ii) proved. For (iii), the closure property follows from the definition of multiplication as follows. Take elements $(s, m), (s', m')$ and (s'', m'') of $S \times M$. Then

$$\begin{aligned} ((s, m)(s', m'))(s'', m'') &= (s + ms', mm')(s'', m''), \\ &= (s + ms' + (mm')\,s'', (mm')\,m''). \end{aligned}$$

Also, we have in the same manner that,

$$\begin{aligned} (s, m)((s', m')(s'', m'')) &= (s, m)(s' + m's'', m'm''), \\ &= (s + m\,(s' + m's''), m\,(m'm'')), \\ &= (s + ms' + (mm')\,s'', m\,(m'm'')). \end{aligned}$$

Hence the multiplication is associative. Moreover, for all $(s,m) \in S \times M$

$$(s,m)(0,1) = (0,1)(s,m) = (s,m).$$

This implies that identity element exists so $S \times M$ is a monoid. For elements $\left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ s' & m' \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ s'' & m'' \end{smallmatrix}\right)$ of K we have

$$\begin{aligned}\left[\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ s' & m' \end{pmatrix}\right] \times \begin{pmatrix} 1 & 0 \\ s'' & m'' \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ s+ms' & mm' \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ s'' & m'' \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ s+ms'+mm's'' & (mm')m'' \end{pmatrix}.\end{aligned}$$

And

$$\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix} \times \left[\begin{pmatrix} 1 & 0 \\ s' & m' \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ s'' & m'' \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ s+ms'+mm's'' & m(m'm'') \end{pmatrix}.$$

Obviously, the identity element of $(K, \times)$ is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. So $(K, \times)$ is a monoid. For (v), it is easy to see that h is a bijective. Also for all $(s,m), (s',m') \in S \times M$:

$$\begin{aligned} h\left[(s,m)(s',m')\right] = h(s+ms', mm') &= \begin{pmatrix} 1 & 0 \\ s+ms' & mm' \end{pmatrix}, \\ &= \begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ s' & m' \end{pmatrix}, \\ &= h(s,m) \times h(s',m'). \end{aligned}$$

□

Theorem 2.8. *Let $(S,+,0)$ and $(M,\cdot,1)$ be two groups. Let the left action of M on S $M \times S \longrightarrow S, (m,s) \longmapsto ms$ which satisfies the following conditions for all $s, s_1, s_2 \in S$ and $m, m_1, m_2 \in M$*

- $m(s_1+s_2) = ms_1 + ms_2$,
- $m_1(m_2 s) = (m_1 m_2) s$,
- $1s = s$,
- $m0 = 0$,
- $m^{-1}(s) = m(-s) = -ms$.

Then we have

(i) *For all $m \in M, \theta_m : S \longrightarrow S, s \longmapsto ms$, $\theta_m \in End(S)$,*
(ii) *The mapping $\theta : (M,\cdot,1) \longrightarrow (End(S), \circ, id_S), m \longmapsto \theta_m$ is a morphism of groups,*
(iii) *The set $S \times M$ with multiplication $(s,m)(s',m') = (s+ms', mm')$ is a group,*
(iv) *If $K = \{\left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right), s \in S, m \in M\}$, then $(K, \times)$ is a group,*

(v) *The mapping* $h : (S \times M, .) \longrightarrow (K, \times), (s, m) \longmapsto \left(\begin{smallmatrix} 1 & 0 \\ s & m \end{smallmatrix}\right)$ *is an isomorphism of groups.*

Proof. It is suffices to show that, for each $(s, m) \in S \times M$, there exists $(s', m') \in S \times M$ such that $(s, m)(s', m') = (s', m')(s, m) = (0, 1)$. We have

$$(s, m)\left(-m^{-1}s, m^{-1}\right) = \left(s + m\left(-m^{-1}s\right), mm^{-1}\right) = (0, 1),$$

and

$$\left(-m^{-1}s, m^{-1}\right)(s, m) = \left(-m^{-1}s + m^{-1}s, m^{-1}m\right) = (0, 1).$$

We also have

$$\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -m^{-1}s & m^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -m^{-1}s & m^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ s & m \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

3. Conclusion

In this paper, we present some notes on the inverse monoid $M(G)$ associated to a group G. Also we construct the semidirect product of groups associated to the group action on another group.

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