Journal of Algebra and Related Topics
Vol. 6, No 1, (2018), pp 45-54

TIGHT CLOSURE OF A GRADED IDEAL RELATIVE TO A GRADED MODULE

F. DOROSTKAR * AND R. KHOSRAVI

Abstract. In this paper we will study the tight closure of a graded ideal relative to a graded module.

1. introduction

Throughout this paper G is an arbitrary abelian group with identity e and R will denote a commutative ring with identity and with prime characteristic p. Also R° will denote the subset of R consisting of all elements which are not contained in any minimal prime ideal of R. Further $\mathbf{N}$ and $\mathbf{Z}$ will denote respectively the set of natural numbers and the set of integer numbers.

The main idea of tight closure of an ideal in a commutative Noetherian ring (with prime characteristic) was introduce by Hochster and Huneke in [7].

Let R be a Noetherian ring and I be an ideal of R. We recall that an element x of R is said to be in tight closure, I^*, of I, if there exists an element $c \in R^{\circ}$ such that for all sufficiently large e, $cx^{p^e} \in (a^{p^e} : a \in I)$. The ideal $(a^{p^e} : a \in I)$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of I. In particular if $I = (a_1, a_2, ..., a_n)$, then $I^{[p^e]} = (a_1^{p^e}, a_2^{p^e}, ..., a_n^{p^e})$. The reader is referred to [12] for the tight closure of an ideal.

MSC(2010): 13A35
Keywords: Graded ring, graded ideal, graded module, tight closure relative to a module, tightly closed relative to a module.
Received: 13 February 2018, Accepted: 23 September 2018.
∗Corresponding author .

In the remainder of this paper, to simplify notation, we will write q to stand for a power p^e of p. For any ideals I and J, $I^{[q]}+J^{[q]}=(I+J)^{[q]}$, $I^{[q]}J^{[q]}=(IJ)^{[q]}$.

In [2], the dual notion of tight closure of ideals relative to modules was introduced and some properties of this concept which reflect results of tight closure in the classical situation were obtained. It is appropriate for us to begin by briefly summarizing some of main aspects.

Again let R be a Noetherian ring. Let I and J be ideals of R and let M be an $R-$module. I is an $F-$reduction of the ideal J relative to M, if $I \subseteq J$ and there exists a $c \in R^{\circ}$ such that

$$(0:_M I^{[q]}) \subseteq (0:_M cJ^{[q]}) \text{ } for \text{ } all \text{ } q \gg 0.$$

It is straightforward to see that the set of ideals of R which have I as an $F-$reduction relative to M has a unique maximal member, denoted by $I^{*[M]}$ and called the tight closure of I relative to M. This is in fact the largest ideal which has I as $F-$reduction relative to M (see [2]).

An element x of R is said to be tight dependent on I relative to M, if there exists an element $c \in R^{\circ}$ such that

$$(0:_M I^{[q]}) \subseteq (0:_M cx^q) \text{ } for \text{ } all \text{ } q \gg 0.$$

Moreover in [2], it was shown that

$$I^{*[M]} = \{x \in R : x \text{ } is \text{ } tight \text{ } dependent \text{ } on \text{ } I \text{ } relative \text{ } to \text{ } M\}.$$

In this paper we will introduce the tight closure of a graded ideal relative to a graded module and we will prove some properties for it.

2. Auxiliary results

Let G be a group. A ring R is said to be a $G-$graded or a graded ring of type G if there exists a family $\{R_\sigma : \sigma \in g\}$ of additive subgroups of R such that $R = \bigoplus_{\sigma\in G} R_\sigma$ and $R_\sigma R_\delta \subseteq R_{\sigma\delta}$ for every $\sigma, \delta \in G$. Every element of $h(R) = \bigcup_{\sigma\in G} R_\sigma$ is called a homogeneous element. Further any nonzero homogeneous element $r_\sigma \in R_\sigma$ is called a homogeneous element of degree σ and we will write $deg(r_\sigma) = \sigma$. It is well known that R_e is a subring of R and $1 \in R_e$.

If R is a graded ring, then every nonzero element $r \in R$ has a unique expression $r = r_{\sigma_1} + r_{\sigma_2} + ... + r_{\sigma_n}$ as a finite sum of nonzero homogeneous elements. The elements $r_{\sigma_1}, r_{\sigma_2}, ..., r_{\sigma_n}$ are called the homogeneous components of r. Also if I is an ideal of the graded ring R, then I^{gr} denotes the ideal of R generated by the elements $h(I) = I \cap h(R)$. Clearly $h(I)$ is the set of the homogeneous elements in I.

Let R be a graded ring of type G. An $R-$module M is said to be a graded left $R-$module if there exists a family $\{M_\sigma : \sigma \in g\}$ of additive subgroups of M such that $M = \bigoplus_{\sigma\in G} M_\sigma$ and $R_\sigma M_\delta \subseteq M_{\sigma\delta}$ for every $\sigma, \delta \in G$. Every element of $h(M) = \bigcup_{\sigma\in G} M_\sigma$ is called a homogeneous element. Further any nonzero homogeneous element $m_\sigma \in M_\sigma$ is called a homogeneous element of degree σ and we will write $deg(m_\sigma) = \sigma$.

Every nonzero element $m \in M$ has a unique expression $m = m_{\sigma_1} + m_{\sigma_2} + ... + m_{\sigma_n}$ as a finite sum of nonzero homogeneous elements. The elements $m_{\sigma_1}, m_{\sigma_2}, ..., m_{\sigma_n}$ are called the homogeneous components of m. An submodule N of M is said to be a graded submodule if for every $n \in N$ the homogeneous components of n are in N. A submodule N of graded $R-$module M is a graded submodule if and only if N is generated by some homogeneous elements. An ideal I of graded ring R is called a graded ideal if it is a graded submodule of $R-$module R (see [8]).

Let R be a graded ring of type G and let M be a graded $R-$module. The injective hull of M in the category of graded $R-$modules is denoted by $E^g(M)$. It follows from [6, 1.1], that $E^g(M)$ is a submodule of the ordinary injective hull $E(M)$. A graded injective $R-$module E is a graded injective cogenerator if it is a cogenerator in the category of graded $R-$modules.

Let R be a $G-$graded ring and S be a multiplicatively closed set of G-homogeneous elements not containing 0. Then R_S is a graded ring where for every $g \in G$

$$(R_S)_g := \{\frac{r}{s} : r \in R_h, s \in R_{hg^{-1}}, h \in G\}.$$

Similarly, if M is a graded $R-$module, then the graded R_S-module M_S is defined. If P is a $G-$graded prime ideal of R, then $S = h(R) - P$ is a multiplicatively closed set of G-homogeneous elements. Also the $G-$graded ring R_S and R_S-module M_S is denoted by $R_{(P)}$ and $M_{(P)}$. We know the ring $R_{(P)}$ is a graded local ring with graded maximal ideal $PR_{(P)}$. For every $x \in h(R) - P$ multiplication by x induces an automorphism of $E^{gr}(R/P)$ and so $E^{gr}(R/P)$ is an $R_{(P)}-$module.

Let R be a $G-$graded ring. A graded $R-$module M is called graded Noetherian or $gr-$Noetherian if M satisfies the ascending chain condition for graded $R-$submodules of M. We know that, a graded $R-$module M is graded Noetherian if and only if each graded submodule of M is finitely generated or if and only if each non-empty family of graded submodules of M has a maximal element (see [8]). A commutative $G-$graded ring R is called graded Noetherian or $gr-$Noetherian if R is graded Noetherian as an $R-$module. If R is a $Z-$graded Noetherian ring then R is a Noetherian ring (see [6, 2.1]).

In the remainder of the paper, we assume that R is a commutative graded Noetherian ring of type G (or when we declare, it is a $Z-$graded Noetherian) with characteristic p.

By using a method similar that they used in [3, 1.5], one can obtain the next proposition.

Proposition 2.1. *Let R be $\mathbf{Z}-$graded ring and F be a graded injective cogenerator $R-$module. Let I and J be graded ideals of R. Then $I \subseteq J$ if and only if $(0 :_F J) \subseteq (0 :_F I)$.*

Corollary 2.2. *Let R be $\mathbf{Z}-$graded ring and I, J be graded ideals. Let P be a graded prime ideal of R and $E = E^{gr}(R/P)$. Then the following conditions are equivalent:*

(a) $(0 :_E J) \subseteq (0 :_E I)$;
(b) $IR_{(P)} \subseteq JR_{(P)}$.

Proof. This is well knowon from [10, 2.1(D)] that $E^{gr}(R_{(P)}/PR_{(P)})$ is a graded injective cogenerator. We know from [6, 4.5],

$$E^{gr}(R_{(P)}/PR_{(P)}) \simeq (E^{gr}(R/P))_{(P)} \simeq E^{gr}(R/P).$$

Now since $(0 :_{E^{gr}(R/P)} J) \subseteq (0 :_{E^{gr}(R/P)} I)$ if and only if $(0 :_{E^{gr}(R_{(P)}/PR_{(P)})} J) \subseteq (0 :_{E^{gr}(R_{(P)}/PR_{(P)})} I)$ the proof is clear from 2.1. $\square$

It is well-known (cf. [11]) that, if I is a graded ideal of R then I^* is also a graded ideal.

Definition 2.3. Let I and J be graded ideals of R and M be an graded $R-$module. We say that I is a graded $F-$reduction of the ideal J relative to M, if $I \subseteq J$ and there exists a $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cJ^{[q]}) \ for\ all\ q \gg 0.$$

If the graded ideal I is a $F-$reduction of the the graded ideals J and J' relative to M then I is a $F-$reduction of the graded ideal $J+J'$ relative to M. Since R is a graded Notherian ring, the set of graded ideals R which have the graded ideal I as a graded $F-$reduction relative to M has a unique maximal member, denoted by $I^{*gr[M]}$ and is called the gr-tight closure of I relative to M. This is in fact the largest graded ideal which has I as $F-$reduction relative to M.

Definition 2.4. Let I be a graded ideal of R and M be a graded $R-$module. A homogenous element $a \in h(R)$ is called graded tight dependent on I relative to M, if there exists a $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M ca^q) \ for\ all\ q \gg 0.$$

Also an element $x \in R$ is called graded tight dependent on I relative to M, if every homogenous component of x is graded tight dependent on I relative to M.

Remark 2.5. Let I be a graded ideal of R and M be a graded $R-$module. Since $I^{[q]}$ is a graded ideal of R, $(0 :_M I^{[q]})$ is a graded submodule of M (see [9, Chap. 2, Sec. 11, Prop. 31]).

Remark 2.6. Let I be a graded ideal of R and let M be a graded $R-$module and $x \in R$. Also assume that, x has the homogeneous components $x_{\sigma_1}, ..., x_{\sigma_n} \in h(R)$. Then x is graded tight dependent on I relative to M if and only if I is a graded $F-$reduction of the graded ideal $I + Rx_{\sigma_1} + ... + Rx_{\sigma_n}$ relative to M.

Remark 2.7. Let I and J be graded ideals of R and let M be a graded $R-$module. Then the following conditions hold.

(a) $I \subseteq I^* \subseteq I^{*_{gr}[M]}$.
(b) If $I \subseteq J$ then $I^{*_{gr}[M]} \subseteq J^{*_{gr}[M]}$.
(c) $(I^{*_{gr}[M]})^{*_{gr}[M]} = I^{*_{gr}[M]}$.
(d) $I^{*_{gr}[M]} J^{*_{gr}[M]} \subseteq (IJ)^{*_{gr}[M]}$.

Theorem 2.8. *Let I be a graded ideal of R and let M be a graded $R-$module. Then*

$$I^{*_{gr}[M]} = \{x \in R : x \text{ is graded tight dependent on } I \text{ relative to } M\}.$$

Proof. Let x be graded tight dependent on I relative to M. Let x have the homogeneous components $x_{\sigma_1}, ..., x_{\sigma_n} \in h(R)$. Then for every $1 \leq i \leq n$ there exists a $c_i \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c_i x_{\sigma_i}^q) \text{ for all } q \gg 0.$$

Let $c = c_1 c_2 ... c_n$. Then $c \in R^\circ$ and for all $q \gg 0$ we have

$$(0 :_M I^{[q]}) \subseteq \bigcap_{i=1}^{n} (0 :_M cRx_{\sigma_i}^q) \subseteq (0 :_M c(\sum_{i=1}^{n} Rx_{\sigma_i})^{[q]}).$$

This shows that I is a graded $F-$reduction of the graded ideal $I + \sum_{i=1}^{n} Rx_{\sigma_i}$ relative to M. Thus $I + \sum_{i=1}^{n} Rx_{\sigma_i} \subseteq I^{*_{gr}[M]}$ and so $x = x_{\sigma_1} + ... + x_{\sigma_n} \in I^{*_{gr}[M]}$.

For converse inclusion, let $y \in I^{*_{gr}[M]}$. Since I is a graded $F-$reduction of the graded ideal $I^{*_{gr}[M]}$ relative to M, there exists a $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c(I^{*_{gr}[M]})^{[q]}) \text{ for all } q \gg 0.$$

Let $y = y_{\sigma_1} + ... + y_{\sigma_k}$ where $y_{\sigma_1}, ..., y_{\sigma_k} \in h(R)$. Since $I^{*[M]}$ is a graded ideal, $y_{\sigma_1}, ..., y_{\sigma_k} \in I^{*_{gr}[M]}$. This implies that for all $1 \leq i \leq k$ and $q \gg 0$ and so we have

$$(0 :_M I^{[q]}) \subseteq (0 :_M c(I^{*_{gr}[M]})^{[q]}) \subseteq (0 :_M cy^q_{\sigma_1}).$$

Hence $y = y_{\sigma_1} + ... + y_{\sigma_k}$ is tightly dependent on I relative to M and this completes the proof. □

Proposition 2.9. *Let I be a graded ideal of R and M be a graded $R-$module. Further assume that S is a multiplicatively closed subset of R. Then*

$$S^{-1}(I^{*_{gr}[M]}) \subseteq (S^{-1}I)^{*_{gr}[S^{-1}M]}.$$

Proof. Let $\frac{x}{1} \in S^{-1}(I^{*_{gr}[M]})$. Let $x = x_{\sigma_1} + x_{\sigma_2} + ... + x_{\sigma_n}$ where $x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n} \in h(R)$. By 2.8, x_{σ_i} is graded tight dependent on I relative to M for every $1 \leq i \leq n$. So for each $1 \leq i \leq n$, there exists a $c_i \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c_i x^q_{\sigma_i}) \; for \; all \; q \gg 0.$$

Let $c = c_1 c_2 ... c_n$. It is straightforward to see that $\frac{c}{1} \in (S^{-1}R)^\circ$ and for every $1 \leq i \leq n$ we have

$$(0 :_{S^{-1}M} S^{-1}I^{[q]}) \subseteq (0 :_M \frac{c}{1}\frac{x^q_{\sigma_i}}{1}) \; for \; all \; q \gg 0.$$

This follows that $\frac{x}{1} = \frac{x_{\sigma_1}}{1} + \frac{x_{\sigma_2}}{1} + ... + \frac{x_{\sigma_n}}{1}$ is graded tight dependent on $S^{-1}I$ relative to $S^{-1}M$ and so $\frac{x}{1} \in (S^{-1}I)^{*_{gr}[S^{-1}M]}$ by 2.8. □

3. Main results

Theorem 3.1. *Let R be a Noetherian $G-$graded ring and M be a graded $R-$ module. Then for every graded ideal I of R we have*

$$I^{*_{gr}[M]} = (I^{*[M]})^{gr}.$$

Proof. Since R is a Noetherian ring, $I^{*[M]}$ can be defined. But I is a graded $F-$reduction of $I^{*_{gr}[M]}$ relative to M then $I^{*_{gr}[M]} \subseteq I^{*[M]}$ and so $I^{*_{gr}[M]} \subseteq (I^{*[M]})^{gr}$. Now let $x \in (I^{*[M]})^{gr}$ and $x = \sum_{i=1}^{n} x_{\sigma_i}$ where x_{σ_i} is a homogeneous element of degree σ_i for every $1 \leq i \leq n$. Since $(I^{*[M]})^{gr}$ is a graded ideal, we have $x_{\sigma_1}, x_{\sigma_2}, ..., x_{\sigma_n} \in (I^{*[M]})^{gr} \subseteq I^{*[M]}$. This follows that for every $1 \leq i \leq n$, there exists $c_i \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c_i {x_{\sigma_i}}^q) \; for \; all \; q \gg 0.$$

Then $x = \sum_{i=1}^{n} x_{\sigma_i} \in I^{*_{gr}[M]}$. So $(I^{*[M]})^{gr} \subseteq I^{*_{gr}[M]}$ and this completes the proof. □

Theorem 3.2. *Let R be a $\mathbf{Z}-$graded Noetherian ring and M be a graded $R-module$. Then $I^{*[M]}$ is a graded ideal.*

Proof. Since R is a $\mathbf{Z}-$graded Noetherian ring, R is a Noetherian ring. Then the integral closure of the ideal I relative to $R-$module M namely $I^{*[M]}$ can be defined. Let $x \in I^{*[M]}$ and let $x = x_{\sigma_1} + ... + x_{\sigma_n}$ where $x_{\sigma_1}, ..., x_{\sigma_n} \in h(R)$ and $deg(x_{\sigma_j}) = \sigma_j$ for all $1 \leq j \leq n$. Since $x \in I^{*[M]}$ there exists a $c \in R^{\circ}$ such that for all $q \gg 0$ we have

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q) \subseteq (0 :_M c(x^q_{\sigma_1} + ... + x^q_{\sigma_n})).$$

Assume that $c = c_{t_1} + c_{t_2} + ... + c_{t_r}$, where $c_{t_1}, c_{t_2}, ..., c_{t_r} \in h(R)$ and $deg(c_{t_i}) = t_i$ for all $1 \leq i \leq r$. By 2.5, $(0 :_M I^{[q]})$ is a graded submodule of M. If m_λ is a homogeneous element of $(0 :_M I^{[q]})$ then $\sum_{i=1}^{r}\sum_{j=1}^{n} c_{t_i}x^q_{\sigma_j}m_\lambda = 0$. Let

$$q > \{\frac{t_{i_1} - t_{i_2}}{\sigma_{j_2} - \sigma_{j_1}} : 1 \leq i_1, i_2 \leq r, 1 \leq j_1, j_2 \leq n\}.$$

If $i_1 \neq i_2$ or $j_1 \neq j_2$ then $deg(c_{t_{i_1}}x^q_{\sigma_{j_1}}m_\lambda) \neq deg(c_{t_{i_2}}x^q_{\sigma_{j_2}}m_\lambda)$. This follows that for all $q \gg 0$, $c_{t_i}x^q_{\sigma_j}m_\lambda = 0$ and so $cx^q_{\sigma_i}m_\lambda = 0$. Thus

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q_{\sigma_i}) \text{ } for \text{ } all \text{ } q \gg 0$$

for every $1 \leq i \leq n$. Then $x_{\sigma_i} \in I^{*[M]}$ and so $I^{*[M]}$ is a graded ideal. □

Corollary 3.3. *Let R be a $\mathbf{Z}-$graded Noetherian ring and M be a graded $R-module$. Then for every graded ideal I of R we have*

$$I^{*_{gr}[M]} = I^{*[M]}.$$

Proof. This immediately follows from 3.1 and 3.2. □

Lemma 3.4. *Let R be a $\mathbf{Z}-$graded Noetherian integral domain and let I be a graded ideal of R. Let $P \in Ass(R)$. Then*

$$I^{*_{gr}[E^{gr}(R/P)]} = I^{*_{gr}[R/P]}.$$

Proof. By 3.3, $I^{*_{gr}[E^{gr}(R/P)]} = I^{*[E^{gr}(R/P)]}$ and $I^{*_{gr}[R/P]} = I^{*[R/P]}$. Since $R/P \leq E^{gr}(R/P)$, $I^{*_{gr}[E^{gr}(R/P)]} \subseteq I^{*_{gr}[R/P]}$. Now let $x \in I^{*_{gr}[R/P]}$. Then there exists $c \in R^{\circ}$ such that

$$(0 :_{R/P} I^{[q]}) \subseteq (0 :_{R/P} cx^q) \text{ } for \text{ } all \text{ } q \gg 0.$$

We will show that $x \in I^{*[E^{gr}(R/P)]}$. Let $y \in (0 :_{E^{gr}(R/P)} I^{[q]})$. By using a method similar that they used in [1, 3.6], one can see that there exists $t \in R \setminus P$ such that $ty \in R/P$. Since $ty \in (0 :_{R/P} I^{[q]})$, $cx^qty = 0$.

Since the multiplication by t provide an automorphism on $E^{gr}(R/P)$, we have that $y \in (0 :_{E^{gr}(R/P)} cx^q)$. Thus we have

$$(0 :_{E^{gr}(R/P)} I^{[q]}) \subseteq (0 :_{E^{gr}(R/P)} cx^q) \ for \ all \ q \gg 0$$

Therefore $x \in I^{*_{gr}[E^{gr}(R/P)]}$. So the proof is complete. □

Proposition 3.5. *Let R be a graded Noetherian integral domain of type $\mathbf{Z}$. Let I be a graded ideal of R and M be a graded $R-$module. Then any homogenous element of degree lees than the degree of generators of the generators of the graded ideal I can not be in $I^{*_{gr}[E^{gr}(M)]}$ unless it is nilpotent.*

Proof. Assume that the graded ideal I can be generated by the homogeneous elements $a_1, a_2, ..., a_n$ all of degree at least δ. Let x be a nonzero homogeneous element of R such that $deg(x) < \delta$. Further assume that x is not nilpotent and M is a graded $R-$module. Let $P \in Ass(M)$. By [4, 1.5.6], P is a prime graded ideal. Since $E^{gr}(R/P) \leq E^{gr}(M)$, $I^{*_{gr}[E^{gr}(M)]} \subseteq I^{*_{gr}[E^{gr}(R/P)]}$. If we show $x \notin I^{*_{gr}[E^{gr}(R/P)]}$ then x can not be in $I^{*_{gr}[E^{gr}(M)]}$.

So let $x \in I^{*_{gr}[E^{gr}(R/P)]}$. Then there exists a $c \in R^{\circ}$ such that

$$(0 :_{E^{gr}(R/P)} I^{[q]}) \subseteq (0 :_{E^{gr}(R/P)} cx^q) \ for \ all \ q \gg 0.$$

Then by 2.2, we have $\frac{cx^q}{1} \in I^{[q]}R_{(P)}$ for all $q \gg 0$. Let $c = c_1+c_2+...+c_k$ where $c_1, c_2, ..., c_k \in h(R)$. This follows that $\frac{cx^q}{1}$ has the expression $\frac{cx^q}{1} = \frac{c_1x^q}{1} + \frac{c_2x^q}{1} + ... + \frac{c_kx^q}{1}$ as a finite sum of homogeneous elements. Since x is not nilpotent and R is a graded Noetherian integral domain of type $\mathbf{Z}$ we can see $\frac{c_ix^q}{1} \neq 0$ for every $1 \leq i \leq k$ and so $\frac{cx^q}{1} \neq 0$. But for every $1 \leq i \leq k$,

$$deg(\frac{c_ix^q}{1}) = deg(c_ix^q) = deg(c_i) + qdeg(x) \ll q\delta \quad for \quad all \quad q \gg 0.$$

Then $\frac{c_ix^q}{1} \notin I^{[q]}R_{(P)}$ for every $1 \leq i \leq k$. Since $I^{[q]}R_{(P)}$ is a graded ideal and $\frac{cx^q}{1} \in I^{[q]}R_{(P)}$, $\frac{c_ix^q}{1} \in I^{[q]}R_{(P)}$ for every $1 \leq i \leq k$. This contradiction shows that $x \notin {}_{gr}I^{*_{gr}[E^{gr}(R/P)]}$. □

Corollary 3.6. *(See [11, 2.1].) Let R be a $\mathbf{Z}-$graded Noetherian integral domain. Let I be a graded ideal of R and M be a graded $R-$module. Then any homogenous element of degree lees than the degree of generators of the generators of the graded ideal I can not be in I^* unless it is nilpotent.*

Proof. This is clear by 2.7(a) and 3.5. □

Theorem 3.7. *Let I be a graded ideal of R and M be a graded Noetherian $R-$module. Further assume that S is a multiplicatively closed set of homogeneous elements of R. Then*

$$S^{-1}(I^{*_{gr}[M]}) = (S^{-1}I)^{*_{gr}[S^{-1}M]}.$$

Proof. Let $\frac{x}{1} \in {}_{gr}(S^{-1}I)^{*_{gr}[S^{-1}M]}$. Let $\frac{x}{1}$ be a finite sum of non-zero homogeneous elements as $\frac{x}{1} = \frac{x_{\sigma_1}}{r_1} + \frac{x_{\sigma_2}}{r_2} + ... + \frac{x_{\sigma_n}}{r_n}$. If $\overline{r}_i = r_1 r_2 ... r_{i-1} r_{i+1} ... r_n$ for each $1 \leq i \leq n$ and $r = r_1 r_2 ... r_n$ then $\frac{x_{\sigma_i}}{r_i} = \frac{\overline{r}_i x_{\sigma_i}}{r}$ and there exists $\frac{c_i}{1} \in (S^{-1}R)^{\circ}$ such that

$$(0 :_{S^{-1}M} S^{-1}I^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c_i}{1}(\frac{\overline{r}_i x_{\sigma_i}}{r})^q) \ for\ all\ q \gg 0$$

for each $1 \leq i \leq n$. By [5, 2.2], we can assume that $c_i \in R^{\circ}$. If $c = c_1 c_2 ... c_n$ then $c \in R^{\circ}$ and

$$(0 :_{S^{-1}M} S^{-1}I^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c}{1}(\frac{\overline{r}_i x_{\sigma_i}}{r})^q) \ for\ all\ q \gg 0$$

for each $1 \leq i \leq n$. For every $k \in \mathbf{N}$, let $q_k = p^k$. Since M is graded Noetherian $R-$module there exists $n \in \mathbf{N}$ such that

$$(0 :_M I^{[q_n]}) = (0 :_M I^{[q_k]}) \quad \forall k \geq n.$$

Now we can choose a $t \in S$ such that

$$(0 :_M I^{[q_n]}) \subseteq (0 :_M c(t\overline{r}_i x_{\sigma_i})^{q_n})$$

for every $1 \leq i \leq n$. This implies that

$$(0 :_M I^{[q]}) \subseteq (0 :_M c(t\overline{r}_i x_{\sigma_i})^q) \ for\ all\ q \gg 0$$

for every $1 \leq i \leq n$. So the homogeneous elements $t\overline{r}_i x_{\sigma_i}$ is graded tight dependent on I relative to M and so

$$\frac{x}{1} = \frac{t\overline{r}_1 x_{\sigma_i} + t\overline{r}_2 x_{\sigma_2} + ... + t\overline{r}_n x_{\sigma_n}}{tr} \in S^{-1}I^{*_{gr}[M]}.$$

Hence $(S^{-1}I)^{*_{gr}[S^{-1}M]} \subseteq S^{-1}(I^{*_{gr}[M]})$. The inverse inclusion follows from 2.9 and so the proof completes. ∎

Acknowledgments

The authors would like to thank the referee for careful reading.

References

1. H. Ansari-Toroghy and F. Dorostkar, *On the integral closure of ideals*, Honam Math. J., (4) **29** (2007), 653-666.
2. H. Ansari-Toroghy and F. Dorostkar, *Tight closure of ideals relative to modules*, Honam Math. J., (4) **32** (2010), 675-687.
3. H. Ansari-Toroghy and R. Y. Sharp, *Integral closure of ideals relative to injective modules over commutative Noetherian rings*, Quart. J. Math. Oxford, (2) **42** (1991), 393-402.
4. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, 1993.
5. F. Dorostkar and R. Khosravi, *F-Regularity Relative To Modules*, Journal of Algebra and Related Topics, (1) **3** (2015), 41-50.
6. J. R. Fossum and H.B. Foxby, *The Category of Graded Modules*, Math. Scandinavica, (2) **35** (1974), 288-300.
7. M. Hochster and C. Huneke, *Tight closure, invariant theory, and Briançon-Skoda theorem*, J. of the Amer. Math. Soc., **3** (1990), 31-116.
8. C. Năstăsescu and F. Van Oystaeyen, *Methods of graded rings*, Lecture Notes in Mathematics, 1836, Springer-Verlag, Berlin, 2004.
9. D. G. Northcott, *Lessons on Rings, modules and multiplicities*, Cambridge University Press, London, 1968.
10. F. Rohrer, *Coarsenings Injectives And HOM Functors*, Rev. Roumaine Math. Pures Appl., (3) **57** (2012), 275-287.
11. K. E. Smith, *Tight closure in graded rings*, J. Math. Kyoto Univ. **37** (1997), 35-53.
12. I. Swanson and C. Huneke, *Integral closure of ideals, rings, and modules*, Cambridge Univ. Press, New York, 2006.

F. Dorostkar

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.

Email: `dorostkar@guilan.ac.ir`

R. Khosravi

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.

Email: `Khosravi@phd.guilan.ac.ir`