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NON-REDUCED RINGS OF SMALL ORDER AND THEIR MAXIMAL GRAPH

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ABSTRACT. Let R be a commutative ring with nonzero identity. Let $\Gamma(R)$ denotes the maximal graph corresponding to the non-unit elements of R, that is, $\Gamma(R)$ is a graph with vertices the non-unit elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. In this paper, we investigate that for a given positive integer n, is there a non-reduced ring R with n non-units? For $n \leq 100$, a complete list of non-reduced decomposable rings $R = \prod_{i=1}^{k} R_i$ (up to cardinalities of constituent local rings R_i's) with n non-units is given. We also show that for which n, $(1 \leq n \leq 7500)$, $|Center(\Gamma(R))|$ attains the bounds in the inequality $1 \leq |Center(\Gamma(R))| \leq n$ and for which n, $(2 \leq n \leq 100)$, $|Center(\Gamma(R))|$ attains the value between the bounds.

1. INTRODUCTION

The maximal graph $G(R)$ associated to R was introduced by the authors [3] in 2013. The authors considered $G(R)$ as a simple graph whose vertices are elements of R, and two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. In [4], the authors defined $\Gamma(R)$ as the restriction of $G(R)$ to the non-unit elements of R, that is, $\Gamma(R)$ is a simple graph whose vertices are the non-unit elements of R such that two distinct vertices a and b are adjacent if and only if $a, b \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. $\Gamma(R)$

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was also named as maximal graph of R as the units in R are just the isolated vertices in $G(R)$.

This paper is inspired by a simple question: Given any positive integer n, is there a commutative ring with nonzero identity having n non-units? One can easily verify that a ring R has a finite number $n \geq 2$ of non-units only if R is finite. So, to answer this question, we need to consider finite rings only.

Of course, the question is somewhat trivial if one removes the requirement that the ring must have an identity. Letting A_k denote the additive group $\mathbb{Z}_k$ with the trivial multiplication ($xy = 0$ for all $x, y \in A_k$), then A_k has k non-units. Thus, for this paper, all rings considered will be finite with nonzero identity. We use $\mathbb{F}_k$ to denote the finite field with k elements.

Restricting the question to local rings (rings which have a unique maximal ideal, including fields) can give examples only for certain values of n. For a finite local ring R with $\mathfrak{m}$ its maximal ideal, $|R| = p^{k\alpha}$ and $|\mathfrak{m}| = p^{(k-1)\alpha}$ for some prime p and some positive integer k. Hence, one must look beyond local rings to answer this question in general.

For finite commutative rings with nonzero identity, every non-unit is zero-divisor. In [6], it was shown that there is no commutative ring with nonzero identity and 1210 non-units. Moreover, for $1 \leq n \leq 7500$, $n = 1210$, $n = 3342$, and $n = 5466$ are the only positive integers for which there is no commutative ring R with nonzero identity and n non-units [6]. Now, there are few other questions:

- For which positive integer n, do there exist only reduced rings with n non-units?
- Given a positive integer n, do there exist non-reduced rings with n non-units?
- If we determine a non-reduced ring R with n non-units, then what is the value of $|J(R)|$, where $J(R)$ denotes the Jacobson radical of R. Whether it depends on prime factorization of n or not?

In Section 2, we find some conditions on $|J(R)|$ such that for a given positive integer n, there does not exist a non-reduced ring with n non-units. In Section 3, we present tables listing all non-reduced decomposable rings $R = \prod_{i=1}^{k} R_i$ (up to cardinalities of constituent local rings R_i's) with n non-unit elements, where $2 \leq n \leq 100$. In Section 4, we discuss that for which positive integer n, $1 \leq n \leq 7500$, $|Center(\Gamma(R))|$ attains the bounds in the inequality $1 \leq |Center(\Gamma(R))| \leq n$ and for which n, $2 \leq n \leq 100$, $|Center(\Gamma(R))|$ attains the value between the bounds. Throughout the paper, ring shall mean a finite commutative ring with nonzero identity.

2. Non-reduced Rings

We begin the section with some results which are established for zero-divisors. In view of the fact that every non-unit is a zero-divisor in a finite ring R, we are restating them for non-units.

- [5, Theorem 2] Let R be a commutative ring of cardinality α having n non-units, where $1 < n \leq \alpha$. Then $\alpha < n^2$.
- [5, Theorem 3] Suppose that p is prime and s and t are integers such that $0 < s < t$. Then there exists a local ring of order p^t having maximal ideal of cardinality p^s if and only if $t - s$ divides s.
- [7, Proposition 2.1] Let R be a finite commutative reduced ring.
  (1) If k is the smallest positive integer such that $|R| < 2^k$, then R is a product of $k-1$ or fewer fields.
  (2) Suppose R has n non-units. Let k be the smallest positive integer such that $n < 2^k - 1$. Then R is a product of $k-1$ or fewer fields.

If R is a finite ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k$, then $R \cong \prod_{i=1}^{k} R_i$, where R_i is a finite local ring with maximal ideal, say $\mathfrak{n}_i$ for all i. Also, $|R_i| = p_i^{m_i\alpha_i}$ for some prime p_i, where m_i is the length of R_i and $|R_i/\mathfrak{n}_i| = p_i^{\alpha_i}$ for all i. If $\mathfrak{m}_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{n}_i \times R_{i+1} \times \cdots \times R_k$, then

$$|\mathfrak{m}_i| = p_i^{(m_i-1)\alpha_i} \prod_{\substack{j=1 \\ j\neq i}}^{k} p_j^{m_j\alpha_j} = p_i^{-\alpha_i}|R|$$

for all i, and

$$|J(R)| = | \cap_{i=1}^{k} \mathfrak{m}_i| = \prod_{i=1}^{k} p_i^{(m_i-1)\alpha_i}.$$

Also

$$| \cup_{i=1}^{k} \mathfrak{m}_i| = |J(R)| \left\{ \prod_{i=1}^{k} p_i^{\alpha_i} - \prod_{i=1}^{k} (p_i^{\alpha_i} - 1) \right\} \qquad (2.1)$$

In the next two propositions we show that under certain conditions there does not exist any finite, non-reduced ring R with n non-units.

Proposition 2.1. *Let p and q be distinct primes, $p^l < q$ and $n = p^l q$ for some $l \in \mathbb{N}$. Then there does not exist any finite, non-reduced ring R with n non-units and $|J(R)| = q$.*

Proof. Suppose that R is a finite ring with $p^l q$ non-units. Let $|J(R)| = q$ and $p^l < q$. Since R is a finite ring, it will have finitely many maximal ideals, say k. Then, in the decomposition of R as a finite direct product of finite local rings, that is, $R \cong \prod_{i=1}^{k} R_i$, all R_i's are field except one, say R_k, which is a local ring with maximal ideal of cardinality q, and hence by [5, Theorem 3], $|R_k| = q^2$.

Thus, equation (2.1) becomes

$$p^l = q \prod_{i=1}^{k-1} p_i^{\alpha_i} - (q-1) \prod_{i=1}^{k-1} (p_i^{\alpha_i} - 1) \tag{2.2}$$

which is not possible as $p^l < q$. Thus there does not exist a non-reduced ring with $p^l q$ non-units and $|J(R)| = q$. □

Proposition 2.2. *Let p, q, and r be distinct primes, $p < q < r$ and $n = pqr$. Then there does not exist any finite, non-reduced ring R with n non-units satisfying the following:*

(i) $|J(R)| = r$ *if* $pq < r$;
(ii) $|J(R)| = qr$;
(iii) $|J(R)| = pr$.

Proof. Suppose that R is a finite ring with pqr non-units. Since R is a finite ring, it will have finitely many maximal ideals, say k.

Let us assume that $|J(R)| = r$. Then, in the decomposition of R as a finite direct product of finite local rings, that is, $R \cong \prod_{i=1}^{k} R_i$, all R_i's are field except one, say R_k, which is a local ring with maximal ideal of cardinality r, and hence by [5, Theorem 3], $|R_k| = r^2$.

Thus, equation (2.1) becomes

$$pq = r \prod_{i=1}^{k-1} p_i^{\alpha_i} - (r-1) \prod_{i=1}^{k-1} (p_i^{\alpha_i} - 1) \tag{2.3}$$

which is not possible as $pq < r$.

Next assume that $|J(R)| = qr$. Then, in the decomposition of R as a finite direct product of finite local rings, that is, $R \cong \prod_{i=1}^{k} R_i$, all R_i's are field except two, say R_{k-1}, R_k, which are local rings with maximal ideals of cardinality q and r, respectively and hence by [5, Theorem 3], $|R_{k-1}| = q^2, |R_k| = r^2$.

Thus, equation (2.1) becomes

$$p = qr \prod_{i=1}^{k-2} p_i^{\alpha_i} - (q-1)(r-1) \prod_{i=1}^{k-2} (p_i^{\alpha_i} - 1) \tag{2.4}$$

which is not possible as $p < q, p < r$. Thus there does not exist a non-reduced ring with pqr non-units and $|J(R)| = qr$. Similarly for $|J(R)| = pr$, there does not exist a non-reduced ring. □

Remark 2.3. Thus equation (2.1) gives a useful criteria to determine the non-existence of a non-reduced ring with $n = p_1p_2\cdots p_m$ non-units and appropriate $|J(R)|$.

3. The List

In this section, we present tables listing all non-reduced decomposable rings $R = \prod_{i=1}^{k} R_i$ (up to cardinalities of constituent local rings R_i's) having n non-units, where $2 \leq n \leq 100$. For $n = 1$, we have a field, which is a reduced ring. Next, if $n = p^s$, where p is prime and s is a positive integer, then by [2, Theorem 2], we have either local rings of order p^t, $0 < s < t$ or reduced ring for $1 \leq s < 3$. For $s \geq 3$, we have non-reduced decomposable rings listed in the table:

Table 1. $n = p^s$

Non-units	R	Non-units	R	Non-units	R
$2^3 = 8$	$\mathbb{F}_3 \times \mathbb{Z}_4$	$2^4 = 16$	$\mathbb{F}_3 \times \mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{F}_7$	$2^5 = 32$	$\mathbb{F}_3 \times \mathbb{Z}_{16}$ $\mathbb{F}_5 \times \mathbb{F}_4[x]/(x^2)$ $\mathbb{F}_7 \times \mathbb{Z}_8$ $\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_5$
$2^6 = 64$	$\mathbb{F}_3 \times \mathbb{Z}_{32}$ $\mathbb{Z}_4 \times \mathbb{F}_{31}$ $\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_{13}$ $\mathbb{F}_7 \times \mathbb{Z}_{16}$ $\mathbb{F}_2 \times \mathbb{F}_5 \times \mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{F}_5$ $\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{Z}_4$	$3^3 = 27$	$\mathbb{F}_7 \times \mathbb{Z}_9$	$3^4 = 81$	$\mathbb{F}_7 \times \mathbb{Z}_{27}$ $\mathbb{Z}_9 \times \mathbb{F}_{25}$

Now, suppose n is not a prime power. First we consider simple prime factorization $n = pq$. We may also assume that $p < q$. Then by Proposition 2.1, there does not exist a non-reduced ring with $|J(R)| = q$. Let us find a non-reduced ring with $|J(R)| = p$. Now, if $|J(R)| = p$, then the equation (2.1) becomes

$$q = p\prod_{i=1}^{k-1} p_i^{\alpha_i} - (p-1)\prod_{i=1}^{k-1}(p_i^{\alpha_i} - 1) \tag{3.1}$$

Thus, for the existence of a non-reduced ring with pq non-units and $|J(R)| = p < q$, p and q should satisfy the equation (3.1). To elaborate this, consider the following example:

Suppose $n = 2 \cdot 3 = 6$. Since $n \leq 2^3 - 1$, by [7, Proposition 2.1], we have $k \leq 2$. Thus, the equation (3.1) becomes

$$3 = p_1^{\alpha_1} + 1.$$

This implies that $p_1 = 2$ and $\alpha_1 = 1$. Thus $\mathbb{F}_2 \times \mathbb{Z}_4$ is non-reduced ring with $n = 6$ and $|J(R)| = 2$.

By applying the same argument to $n \in \{22, 38, 51, 69, 74, 78, 82, 94, 95\}$, we conclude that there does not exist a non-reduced ring with n non-units.

We now give a list of non-reduced decomposable rings with n $(2 \leq n \leq 100)$ non-units, where $n \notin \{22, 38, 51, 69, 74, 78, 82, 94, 95\}$ and is not a prime power.

Table 2. $n = pq$

Non-units	R	Non-units	R	Non-units	R
$2 \cdot 3 = 6$	$\mathbb{F}_2 \times \mathbb{Z}_4$	$2 \cdot 5 = 10$	$\mathbb{Z}_4 \times \mathbb{F}_4$	$2 \cdot 7 = 14$	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4$
$2 \cdot 13 = 26$	$\mathbb{F}_2 \times \mathbb{F}_4 \times \mathbb{Z}_4$	$2 \cdot 17 = 34$	$\mathbb{Z}_4 \times \mathbb{F}_{16}$	$2 \cdot 23 = 46$	$\mathbb{Z}_4 \times \mathbb{F}_4 \times \mathbb{F}_4$
$2 \cdot 29 = 58$	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_4$	$2 \cdot 31 = 62$	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4$	$2 \cdot 43 = 86$	$\mathbb{F}_4 \times \mathbb{Z}_4 \times \mathbb{F}_8$
$3 \cdot 5 = 15$	$\mathbb{F}_3 \times \mathbb{Z}_9$	$3 \cdot 7 = 21$	$\mathbb{F}_5 \times \mathbb{Z}_9$	$3 \cdot 11 = 33$	$\mathbb{F}_9 \times \mathbb{Z}_9$
$3 \cdot 13 = 39$	$\mathbb{Z}_9 \times \mathbb{F}_{11}$	$3 \cdot 19 = 57$	$\mathbb{Z}_9 \times \mathbb{F}_{17}$ $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{Z}_9$	$3 \cdot 29 = 87$	$\mathbb{Z}_9 \times \mathbb{F}_{27}$ $\mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{Z}_9$
$3 \cdot 31 = 93$	$\mathbb{Z}_9 \times \mathbb{F}_{29}$	$5 \cdot 7 = 35$	$\mathbb{F}_3 \times \mathbb{Z}_{25}$	$5 \cdot 11 = 55$	$\mathbb{F}_7 \times \mathbb{Z}_{25}$
$5 \cdot 13 = 65$	$\mathbb{F}_9 \times \mathbb{Z}_{25}$	$5 \cdot 17 = 85$	$\mathbb{F}_{13} \times \mathbb{Z}_{25}$	$7 \cdot 11 = 77$	$\mathbb{F}_5 \times \mathbb{Z}_{49}$
$7 \cdot 13 = 91$	$\mathbb{F}_7 \times \mathbb{Z}_{49}$				

Table 3. $n = p^2q$

Non-units	R	Non-units	R	Non-units	R
$2^2 \cdot 3 = 12$	$\mathbb{F}_2 \times \mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{Z}_4$ $\mathbb{Z}_4 \times \mathbb{F}_5$ $\mathbb{F}_2 \times \mathbb{Z}_9$	$2^2 \cdot 5 = 20$	$\mathbb{F}_4 \times \mathbb{Z}_8$ $\mathbb{F}_2 \times \mathbb{F}_4[x]/(x^2)$ $\mathbb{Z}_4 \times \mathbb{F}_9$ $\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_4$	$2^2 \cdot 7 = 28$	$\mathbb{Z}_4 \times \mathbb{F}_{13}$ $\mathbb{F}_4 \times \mathbb{F}_4[x]/(x^2)$ $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_8$ $\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{Z}_4$
$2^2 \cdot 11 = 44$	$\mathbb{F}_8 \times \mathbb{F}_4[x]/(x^2)$ $\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_7$ $\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{F}_5$ $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_4$	$2^2 \cdot 13 = 52$	$\mathbb{Z}_4 \times \mathbb{F}_{25}$ $\mathbb{F}_2 \times \mathbb{F}_4 \times \mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{F}_4$ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4[x]/(x^2)$	$2^2 \cdot 17 = 68$	$\mathbb{Z}_8 \times \mathbb{F}_{16}$ $\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_{11}$ $\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{F}_8$ $\mathbb{Z}_4 \times \mathbb{F}_5 \times \mathbb{F}_5$
$2^2 \cdot 19 = 76$	$\mathbb{Z}_4 \times \mathbb{F}_{37}$ $\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_{16}$ $\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{F}_9$ $\mathbb{Z}_4 \times \mathbb{F}_4 \times \mathbb{F}_7$	$2^2 \cdot 23 = 92$	$\mathbb{F}_2 \times \mathbb{F}_4 \times \mathbb{F}_4[x]/(x^2)$ $\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_8$ $\mathbb{Z}_4 \times \mathbb{F}_5 \times \mathbb{F}_7$ $\mathbb{Z}_4 \times \mathbb{F}_3 \times \mathbb{F}_{11}$ $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{Z}_4$ $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_4$	$3^2 \cdot 2 = 18$	$\mathbb{Z}_4 \times \mathbb{F}_8$ $\mathbb{F}_4 \times \mathbb{Z}_9$
$3^2 \cdot 5 = 45$	$\mathbb{F}_3 \times \mathbb{Z}_{27}$ $\mathbb{F}_5 \times \mathbb{Z}_{25}$ $\mathbb{Z}_9 \times \mathbb{F}_{13}$ $\mathbb{Z}_9 \times \mathbb{Z}_9$	$3^2 \cdot 7 = 63$	$\mathbb{F}_3 \times \mathbb{Z}_{49}$ $\mathbb{F}_5 \times \mathbb{Z}_{27}$ $\mathbb{Z}_9 \times \mathbb{F}_{19}$	$3^2 \cdot 11 = 99$	$\mathbb{F}_3 \times \mathbb{F}_9[x]/(x^2)$ $\mathbb{F}_9 \times \mathbb{Z}_{27}$ $\mathbb{Z}_9 \times \mathbb{F}_{31}$
$5^2 \cdot 2 = 50$	$\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_8$	$5^2 \cdot 3 = 75$	$\mathbb{Z}_9 \times \mathbb{F}_{23}$ $\mathbb{F}_{11} \times \mathbb{Z}_{25}$	$7^2 \cdot 2 = 98$	$\mathbb{F}_8 \times \mathbb{Z}_{49}$ $\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_{16}$

TABLE 4. $n = p^3q$

Non-units	R	Non-units	R	Non-units	R
$2^3 \cdot 3 = 24$	$\mathbb{F}_2 \times \mathbb{Z}_{16}$	$2^3 \cdot 5 = 40$	$\mathbb{F}_4 \times \mathbb{Z}_{25}$	$2^3 \cdot 7 = 56$	$\mathbb{F}_2 \times \mathbb{Z}_{49}$
	$\mathbb{Z}_4 \times \mathbb{Z}_9$		$\mathbb{F}_4 \times \mathbb{Z}_{16}$		$\mathbb{Z}_4 \times \mathbb{F}_{27}$
	$\mathbb{Z}_4 \times \mathbb{F}_{11}$		$\mathbb{Z}_4 \times \mathbb{F}_{19}$		$\mathbb{Z}_8 \times \mathbb{F}_{13}$
	$\mathbb{F}_5 \times \mathbb{Z}_8$		$\mathbb{Z}_8 \times \mathbb{F}_9$		$\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_{11}$
	$\mathbb{F}_3 \times \mathbb{F}_4[x]/(x^2)$		$\mathbb{Z}_4 \times \mathbb{F}_4[x]/(x^2)$		$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_{16}$
	$\mathbb{Z}_4 \times \mathbb{Z}_8$		$\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_7$		$\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
			$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_8$		$\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_9$
			$\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$		$\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{Z}_8$
					$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$
					$\mathbb{F}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
$2^3 \cdot 11 = 88$	$\mathbb{Z}_4 \times \mathbb{F}_{43}$	$3^3 \cdot 2 = 54$	$\mathbb{F}_4 \times \mathbb{Z}_{27}$		
	$\mathbb{F}_4 \times \mathbb{F}_8[x]/(x^2)$		$\mathbb{Z}_9 \times \mathbb{F}_{16}$		
	$\mathbb{F}_{19} \times \mathbb{F}_4[x]/(x^2)$		$\mathbb{F}_2 \times \mathbb{F}_4 \times \mathbb{Z}_9$		
	$\mathbb{F}_2 \times \mathbb{F}_7 \times \mathbb{Z}_8$				
	$\mathbb{F}_3 \times \mathbb{F}_5 \times \mathbb{Z}_8$				
	$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_7$				
	$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$				
	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_8$				

TABLE 5. $n = p^4q,\ p^5q$

Non-units	R	Non-units	R	Non-units	R
$2^4.3 = 48$	$\mathbb{F}_2 \times \mathbb{Z}_{32}$	$2^4.5 = 80$	$\mathbb{F}_9 \times \mathbb{Z}_{16}$	$2^5 \cdot 3 = 96$	$\mathbb{F}_2 \times \mathbb{Z}_{64}$
	$\mathbb{Z}_4 \times \mathbb{F}_{23}$		$\mathbb{F}_4 \times \mathbb{Z}_{32}$		$\mathbb{Z}_4 \times \mathbb{F}_{47}$
	$\mathbb{F}_5 \times \mathbb{Z}_{16}$		$\mathbb{Z}_8 \times \mathbb{F}_{19}$		$\mathbb{F}_5 \times \mathbb{Z}_{32}$
	$\mathbb{Z}_8 \times \mathbb{Z}_9$		$\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_{17}$		$\mathbb{Z}_8 \times \mathbb{F}_{23}$
	$\mathbb{Z}_8 \times \mathbb{F}_{11}$		$\mathbb{Z}_8 \times \mathbb{F}_4[x]/(x^2)$		$\mathbb{Z}_9 \times \mathbb{Z}_{16}$
	$\mathbb{Z}_8 \times \mathbb{Z}_8$		$\mathbb{F}_3 \times \mathbb{F}_8[x]/(x^2)$		$\mathbb{F}_{11} \times \mathbb{Z}_{16}$
	$\mathbb{F}_4[x]/(x^2) \times \mathbb{F}_9$		$\mathbb{F}_2 \times \mathbb{F}_4(+)\mathbb{F}_4[x]/(x^2)$		$\mathbb{Z}_4 \times \mathbb{Z}_{32}$
	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$		$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_{25}$		$\mathbb{Z}_8 \times \mathbb{Z}_{16}$
			$\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_{13}$		$\mathbb{F}_3 \times \mathbb{F}_4(+)\mathbb{F}_4[x]/(x^2)$
			$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_{16}$		$\mathbb{F}_5 \times \mathbb{F}_8[x]/(x^2)$
			$\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_8$		$\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_4[x]/(x^2)$
					$\mathbb{F}_4 \times \mathbb{Z}_4 \times \mathbb{F}_9$
					$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_9$

TABLE 6. $n = p^2q^2,\ p^3q^2$

Non-units	R	Non-units	R	Non-units	R
$2^2 \cdot 3^2 = 36$	$\mathbb{F}_2 \times \mathbb{Z}_{27}$	$2^2 \cdot 5^2 = 100$	$\mathbb{Z}_4 \times \mathbb{F}_{49}$	$2^3 \cdot 3^2 = 72$	$\mathbb{Z}_4 \times \mathbb{Z}_{27}$
	$\mathbb{Z}_4 \times \mathbb{F}_{17}$		$\mathbb{F}_{16} \times \mathbb{Z}_{25}$		$\mathbb{F}_8 \times \mathbb{Z}_{16}$
	$\mathbb{F}_8 \times \mathbb{Z}_8$		$\mathbb{F}_2 \times \mathbb{F}_8 \times \mathbb{Z}_8$		$\mathbb{F}_4[x]/(x^2) \times \mathbb{Z}_9$
	$\mathbb{Z}_4 \times \mathbb{F}_3 \times \mathbb{F}_4$		$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{F}_8$		$\mathbb{Z}_8 \times \mathbb{F}_{17}$
			$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_7$		$\mathbb{F}_2 \times \mathbb{F}_8[x]/(x^2)$
					$\mathbb{F}_3 \times \mathbb{F}_4 \times \mathbb{Z}_9$
					$\mathbb{F}_3 \times \mathbb{F}_4 \times \mathbb{Z}_8$
					$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4[x]/(x^2)$
					$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{F}_5$

Table 7. $n = pqr,\ p^2qr$

Non-units	R	Non-units	R
$2 \cdot 3 \cdot 5 = 30$	$\mathbb{F}_2 \times \mathbb{Z}_{25}$	$2^2 \cdot 3 \cdot 5 = 60$	$\mathbb{Z}_4 \times \mathbb{Z}_{25}$
	$\mathbb{F}_8 \times \mathbb{Z}_9$		$\mathbb{Z}_4 \times \mathbb{F}_{29}$
	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_9$		$\mathbb{F}_8 \times \mathbb{Z}_{25}$
	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4$		$\mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9$
			$\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{F}_7$
			$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$
			$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_8$
$2 \cdot 3 \cdot 7 = 42$	$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{Z}_9$	$2^2 \cdot 3 \cdot 7 = 84$	$\mathbb{Z}_4 \times \mathbb{F}_{41}$
			$\mathbb{F}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_9$
			$\mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4 \times \mathbb{Z}_4$
$2 \cdot 3 \cdot 11 = 66$	$\mathbb{Z}_4 \times \mathbb{F}_{32}$	$3^2 \cdot 2 \cdot 5 = 90$	$\mathbb{F}_2 \times \mathbb{F}_9[x]/(x^2)$
	$\mathbb{F}_2 \times \mathbb{F}_5 \times \mathbb{Z}_9$		$\mathbb{F}_8 \times \mathbb{Z}_{27}$
	$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_9$		$\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{Z}_{27}$
			$\mathbb{F}_2 \times \mathbb{F}_7 \times \mathbb{Z}_9$
			$\mathbb{F}_4 \times \mathbb{F}_4 \times \mathbb{Z}_9$
$2 \cdot 5 \cdot 7 = 70$	$\mathbb{F}_4 \times \mathbb{Z}_{49}$		

4. Center and Median

4.1. **Center.** We begin this section with the following definition from [1].

Definition 4.1. Center : The set of vertices with minimum eccentricity of a graph G is called the center of G. It is denoted by $Center(G)$.

Note that if R is a commutative ring with nonzero identity having n non-units, then maximal graph $\Gamma(R)$ has n vertices. As $\Gamma(R)$ may be a complete graph for some n, we have the following inequality:

$$1 \leq |Center(\Gamma(R))| \leq n \tag{4.1}$$

In the view of (6), the following question may arises:

Question 4.2. Given a positive integer n do there exist maximal graphs $\Gamma(R)$ of order n such that

(1) $|Center(\Gamma(R))|$ attains the bounds in the Inequality (6)?
(2) $1 < |Center(\Gamma(R))| < n$?

Note that for any maximal graph $\Gamma(R)$ of order n, the following are equivalent:

(i) $|Center(\Gamma(R))| = n$.
(ii) $\Gamma(R)$ is a complete graph.

(iii) R is a local ring.

Similarly, the following are equivalent:

(i) $|Center(\Gamma(R))| = 1$.

(ii) There exists exactly one vertex $v \in V((\Gamma(R)))$ such that $deg(v) = n - 1$.

(iii) R is a reduced ring.

Therefore, for $1 < |Center(\Gamma(R))| < n$, R must be a non-reduced and non-local ring.

If $n = p^s$, where p is prime and s is a positive integer, then by [5, Theorem 3], there exists a local ring R with maximal ideal of cardinality p^s and hence $|Center(\Gamma(R))| = p^s$. If n is not a prime power, then there is no ring R with n non-units and $|Center(\Gamma(R))| = n$.

In [6], it was shown that for $1 \leq n \leq 7500$, there always exist a reduced ring except $n \in \{2, 1206, 1210, 1806, 3342, 5466, 6462, 6534, 6546, 7430\}$. Thus for $1 \leq n \leq 7500$, $n \notin \{2, 1206, 1210, 1806, 3342, 5466, 6462, 6534, 6546, 7430\}$ there always exist ring R such that $\Gamma(R)$ is of order n and $|Center(\Gamma(R))| = 1$.

In general, we cannot say that there always exist a maximal graph whose center attains the value between the bounds, that is, there exists a non-reduced ring having n non-units. However, from the list given in Section 3, we conclude that there does not exist a ring R for which $\Gamma(R)$ is of order n and $1 < |Center(\Gamma(R))| < n$ for $n \in \{22,\ 38,\ 51,\ 69,\ 74,\ 78,\ 82,\ 94,\ 95\}$. Clearly, for all the rings R listed in Section 3, we have $1 < |Center(\Gamma(R))| < n$.

4.2. **Median.** Let G be a connected graph. For any vertex x of G, the status of x, is the sum of the distances from x to all the other vertices of G, and is denoted by $s(x)$, that is, $s(x) = \sum\{d(x, y) : y \in V(G)\}$. The set of vertices with minimal status is called the median of the graph. If G has no edges, then we shall say the median of G is $V(G)$.

Although both the center and the median relate to the topic of centrality in a graph, they need not coincide. One can easily construct examples where the center is a proper subset of the median, or the median is a proper subset of the center. In general, finding the median of a graph is more involved than finding the center. However, the following theorem gives a relationship between the center and median, in the case of maximal graphs of finite commutative rings with identity.

Theorem 4.3. *Let R be a finite commutative ring with nonzero identity. Then the median and center of $\Gamma(R)$ are equal.*

Proof. Let $|V(\Gamma(R))| = n$. Then for any $x \in V(\Gamma(R))$, $s(x) \geq n - 1$ as $\Gamma(R)$ is a connected graph. Also, for all $x \in J(R)$, $s(x) = n - 1$, and for all $x \in V(\Gamma(R)) \setminus J(R)$, $s(x) \geq n$. Since $Center(\Gamma(R)) = J(R)$, by [4, Proposition 2.8], the result follows. □

Remark 4.4. Note that $Center(\Gamma(R)) = J(R) = Median(\Gamma(R))$, by Theorem 4.3 and [4, Proposition 2.8].

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