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$\mathcal{N}$-FUZZY UP-ALGEBRAS AND ITS LEVEL SUBSETS

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ABSTRACT. In this paper, $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras are introduced and considered their generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). Further, we discuss the relations between $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) and its level subsets.

1. Introduction

A fuzzy subset f of a set S is a function from S to a closed interval $[0,1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [19] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [19], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2001, Lele et al. [9] studied fuzzy ideals and weak ideals in BCK-algebras. Jun [5] studied Q-fuzzy subalgebras of BCK/BCI-algebras. In 2002, Jun et al. [7] studied fuzzy B-algebras in B-algebras. Yonglin and Xiaohong [18] studied fuzzy a-ideals in BCI-algebras. In 2005, Akram

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and Dar [1] studied T-fuzzy subalgebras and T-fuzzy H-ideals in BCI-algebras. In 2007, Jun [6] studied fuzzy subalgebras with thresholds of BCK/BCI-algebras. Akram and Dar [2] studied fuzzy ideals in K-algebras. In 2010, Song et al. [16] studied fuzzy ideals in BE-algebras. In 2011, Mostafa et al. [10] studied fuzzy KU-ideals in KU-algebras. In 2012, Mostafa et al. [11] studied fuzzy Q-ideals in Q-algebras. In 2014, Yamini and Kailasavalli [17] studied fuzzy B-ideals in B-algebras. In 2015, Senapati [12] introduced the notion of T-fuzzy subalgebras of KU-algebras. In 2016, Somjanta et al. [15] studied fuzzy UP-subalgebras, fuzzy UP-ideals and fuzzy UP-filters of UP-algebras.

Iampan [4] introduced a new branch of the logical algebra, called a UP-algebra. Later Guntasow et al. [3] studied fuzzy translations of a fuzzy set in UP-algebras. Senapati et al. [13, 14] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras. In this paper, we introduce the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). Further, we discuss the relations between $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) and its level subsets.

2. Basic Results on UP-Algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 2.1. [4] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, and
(UP-4): $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

Example 2.2. [4] Let X be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of X by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, \emptyset)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the *power UP-algebra of type 1* and the *power UP-algebra of type 2*, respectively.

Example 2.3. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Then $(A, \cdot, 0)$ is a UP-algebra.

In what follows, let A be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.4. [4] *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

Definition 2.5. [4] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Iampan [4] proved the useful criteria that a nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.

Definition 2.6. [4] A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.7. [15] A subset F of A is called a *UP-filter* of A if it satisfies the following properties:

(1) the constant 0 of A is in F, and
(2) for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.8. [3] A subset C of A is called a *strongly UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in C, and
(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Guntasow et al. [3] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. $\mathcal{N}$-Fuzzy Sets

In this section, we introduce the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras, provide the necessary examples and prove its generalizations.

Definition 3.1. [8] A *negative fuzzy set* (briefly *$\mathcal{N}$-fuzzy set*) in a nonempty set X (or a negative fuzzy subset (briefly *$\mathcal{N}$-fuzzy subset*) of X) is an arbitrary function from the set X into $[-1, 0]$ where $[-1, 0]$ is the unit segment of the real line. If $A \subseteq X$, the *characteristic $\mathcal{N}$-fuzzy set* χ_A of X is a function of X into $\{-1, 0\}$ defined as follows:

$$\chi_A(x) = \begin{cases} -1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Hence, χ_A is an $\mathcal{N}$-fuzzy set in X.

Lemma 3.2. *Let A be a subset of a nonempty set X. Then χ_A is constant if and only if $A = X$ or $A = \emptyset$.*

Proof. Assume that χ_A is constant and $A \neq \emptyset$. Then there exists $a \in A$, that is, $\chi_A(a) = -1$. Thus $\chi_A(x) = -1$ for all $x \in X$, so $x \in A$. Hence, $X = A$.

Conversely, assume that $A = X$ or $A = \emptyset$. Then $\chi_A(x) = \chi_X(x) = -1$ for all $x \in X$ or $\chi_A(x) = \chi_\emptyset(x) = 0$ for all $x \in X$. Hence, χ_A is constant. $\square$

Definition 3.3. Let f be an $\mathcal{N}$-fuzzy set in a nonempty set X. The $\mathcal{N}$-fuzzy set $\overline{f}$ defined by $\overline{f}(x) = -1 - f(x)$ for all $x \in X$ is said to be the *complement* of f in X.

Definition 3.4. An $\mathcal{N}$-fuzzy set f in A is called an *$\mathcal{N}$-fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$

By Proposition 2.4 (1), we have $f(0) = f(x \cdot x) \leq \max\{f(x), f(x)\} = f(x)$ for all $x \in A$.

Example 3.5. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	2	4
2	0	1	0	1	4
3	0	0	0	0	4
4	0	1	2	3	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.7, f(2) = -0.5, f(3) = -0.3, \text{ and } f(4) = -0.1.$$

Then f is an $\mathcal{N}$-fuzzy UP-subalgebra of A.

Definition 3.6. An $\mathcal{N}$-fuzzy set f in A is called an *$\mathcal{N}$-fuzzy UP-filter* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $f(0) \leq f(x)$, and
(2) $f(y) \leq \max\{f(x \cdot y), f(x)\}$.

Example 3.7. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	1	2	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.8, f(2) = -0.7, f(3) = -0.9, \text{ and } f(4) = -0.1.$$

Then f is an $\mathcal{N}$-fuzzy UP-filter of A.

Definition 3.8. An $\mathcal{N}$-fuzzy set f in A is called an *$\mathcal{N}$-fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \leq f(x)$, and
(2) $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Example 3.9. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.7, f(2) = -0.5, \text{ and } f(3) = -0.3.$$

Then f is an $\mathcal{N}$-fuzzy UP-ideal of A.

Definition 3.10. An $\mathcal{N}$-fuzzy set f in A is called an *$\mathcal{N}$-fuzzy strongly UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \leq f(x)$, and
(2) $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

Example 3.11. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(x) = -0.7 \text{ for all } x \in A.$$

Then f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A.

Theorem 3.12. *An $\mathcal{N}$-fuzzy set in A is constant if and only if it is an $\mathcal{N}$-fuzzy strongly UP-ideal of A.*

Proof. Assume that f is a constant $\mathcal{N}$-fuzzy set in A. Then for all $x \in A$, $f(0) = f(x)$ and so $f(0) \leq f(x)$. For all $x, y, z \in A$, $f(x) = f((z \cdot y) \cdot (z \cdot x)) = f(y)$, so $f(x) = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Hence, f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A.

Conversely, assume that f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. Then $f(0) \leq f(x)$ and $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$ for all

$x, y, z \in A$. For any $x \in A$, we choose $z = x$ and $y = 0$. Then

$$\begin{aligned} f(x) &\leq \max\{f((x \cdot 0) \cdot (x \cdot x)), f(0)\} \\ &= \max\{f(0 \cdot (x \cdot x)), f(0)\} && \text{((UP-3))} \\ &= \max\{f(x \cdot x), f(0)\} && \text{((UP-2))} \\ &= \max\{f(0), f(0)\} && \text{(Proposition 2.4 (1))} \\ &= f(0). \end{aligned}$$

Thus $f(0) = f(x)$ for all $x \in A$. Hence, f is a constant $\mathcal{N}$-fuzzy set in A. ∎

Theorem 3.13. *Every $\mathcal{N}$-fuzzy strongly UP-ideal of A is an $\mathcal{N}$-fuzzy UP-ideal.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. By Theorem 3.12, we have $f(x) = f(0)$ for all $x \in A$. For any $x, y, z \in A$, we have $f(0) \leq f(x)$ and

$$f(x \cdot z) \leq f(0) = \max\{f(0), f(0)\} = \max\{f(x \cdot (y \cdot z)), f(y)\}.$$

Hence, f is an $\mathcal{N}$-fuzzy UP-ideal of A. ∎

Example 3.14. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	3	4
3	0	0	2	0	4
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.8, f(2) = -0.5, f(3) = -0.2, \text{ and } f(4) = -0.1.$$

Then f is an $\mathcal{N}$-fuzzy UP-ideal of A. Since $f(2) = -0.5 > -0.8 = \max\{f((2 \cdot 1) \cdot (2 \cdot 2)), f(1)\}$, we have f is not an $\mathcal{N}$-fuzzy strongly UP-ideal of A.

Theorem 3.15. *Every $\mathcal{N}$-fuzzy UP-ideal of A is an $\mathcal{N}$-fuzzy UP-filter.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-ideal of A. Then for all $x, y \in A$, $f(0) \leq f(x)$ and

$$\begin{aligned} f(y) &= f(0 \cdot y) && \text{((UP-2))} \\ &\leq \max\{f(0 \cdot (x \cdot y)), f(x)\} \\ &= \max\{f(x \cdot y), f(x)\}. && \text{((UP-2))} \end{aligned}$$

Hence, f is an $\mathcal{N}$-fuzzy UP-filter of A. $\square$

Example 3.16. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.9, f(2) = -0.7, f(3) = -0.5, \text{ and } f(4) = -0.5.$$

Then f is an $\mathcal{N}$-fuzzy UP-filter of A. Since $f(3 \cdot 4) = -0.5 > -0.7 = \max\{f(3 \cdot (2 \cdot 4)), f(2)\}$, we have f is not an $\mathcal{N}$-fuzzy UP-ideal of A.

Theorem 3.17. *Every $\mathcal{N}$-fuzzy UP-filter of A is an $\mathcal{N}$-fuzzy UP-subalgebra.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-filter of A. Then for all $x, y \in A$,

$$\begin{aligned} f(x \cdot y) &\leq \max\{f(y \cdot (x \cdot y)), f(y)\} \\ &= \max\{f(0), f(y)\} && \text{(Proposition 2.4 (5))} \\ &= f(y) \\ &\leq \max\{f(x), f(y)\}. \end{aligned}$$

Hence, f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. $\square$

Example 3.18. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	1	2	4
2	0	0	0	2	4
3	0	0	0	0	4
4	0	0	0	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.7, f(2) = -0.4, f(3) = -0.2, \text{ and } f(4) = -0.1.$$

Then f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Since $f(3) = -0.2 > -0.4 = \max\{f(2 \cdot 3), f(2)\}$, we have f is not an $\mathcal{N}$-fuzzy UP-filter of A.

By Theorem 3.13, 3.15, and 3.17 and Example 3.14, 3.16, and 3.18, we have that the notion of $\mathcal{N}$-fuzzy UP-subalgebras is a generalization of $\mathcal{N}$-fuzzy UP-filters, the notion of $\mathcal{N}$-fuzzy UP-filters is a generalization of $\mathcal{N}$-fuzzy UP-ideals, and the notion of $\mathcal{N}$-fuzzy UP-ideals is a generalization of $\mathcal{N}$-fuzzy strongly UP-ideals.

4. Characteristic $\mathcal{N}$-Fuzzy Sets

In this section, we prove several theorems for characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals).

Theorem 4.1. *A nonempty subset S of A is a UP-subalgebra of A if and only if the characteristic $\mathcal{N}$-fuzzy set χ_S is an $\mathcal{N}$-fuzzy UP-subalgebra of A.*

Proof. Assume that S is a UP-subalgebra of A. Let $x, y \in A$.

Case 1: $x, y \in S$. Then $\chi_S(x) = -1$ and $\chi_S(y) = -1$. Thus $\max\{\chi_S(x), \chi_S(y)\} = \max\{-1, -1\} = -1$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so $\chi_S(x \cdot y) = -1$. Therefore, $\chi_S(x \cdot y) = -1 \leq -1 = \max\{\chi_S(x), \chi_S(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S(x) = 0$ or $\chi_S(y) = 0$. Thus $\max\{\chi_S(x), \chi_S(y)\} = 0$. Therefore, $\chi_S(x \cdot y) \leq 0 = \max\{\chi_S(x), \chi_S(y)\}$.

Hence, χ_S is an $\mathcal{N}$-fuzzy UP-subalgebra of A.

Conversely, assume that χ_S is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $x, y \in S$. Then $\chi_S(x) = -1$ and $\chi_S(y) = -1$. Thus $\chi_S(x \cdot y) \leq \max\{\chi_S(x), \chi_S(y)\} = -1$ and so $\chi_S(x \cdot y) = -1$. Hence, $x \cdot y \in S$, that is, S is a UP-subalgebra of A. □

Lemma 4.2. *The constant 0 of A is in a nonempty subset B of A if and only if $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$.*

Proof. If $0 \in B$, then $\chi_B(0) = -1$. Thus $\chi_B(0) = -1 \leq \chi_B(x)$ for all $x \in A$.

Conversely, assume that $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$. Since B is a nonempty subset of A, we have $a \in B$ for some $a \in A$. Then $\chi_B(0) \leq \chi_B(a) = -1$, so $\chi_B(0) = -1$. Hence, $0 \in B$. □

Theorem 4.3. *A nonempty subset F of A is a UP-filter of A if and only if the characteristic $\mathcal{N}$-fuzzy set χ_F is an $\mathcal{N}$-fuzzy UP-filter of A.*

Proof. Assume that F is a UP-filter of A. Since $0 \in F$, it follows from Lemma 4.2 that $\chi_F(0) \leq \chi_F(x)$ for all $x \in A$. Next, let $x, y \in A$.

Case 1: $x, y \in F$. Then $\chi_F(x) = -1$ and $\chi_F(y) = -1$. Thus $\chi_F(y) = -1 \leq \chi_F(x \cdot y) = \max\{\chi_F(x \cdot y), -1\} = \max\{\chi_F(x \cdot y), \chi_F(x)\}$.

Case 2: $x \notin F$ or $y \notin F$. Then $\chi_F(x) = 0$ or $\chi_F(y) = 0$.

Case 2.1: If $x \notin F$, then $\chi_F(x) = 0$. Thus $\chi_F(y) \leq 0 = \max\{\chi_F(x \cdot y), 0\} = \max\{\chi_F(x \cdot y), \chi_F(x)\}$.

Case 2.2: If $y \notin F$, then $\chi_F(y) = 0$. Since F is a UP-filter of A, we have $x \cdot y \notin F$ or $x \notin F$. Then $\chi_F(x \cdot y) = 0$ or $\chi_F(x) = 0$. Thus $\chi_F(y) \leq 0 = \max\{\chi_F(x \cdot y), \chi_F(x)\}$.

Hence, χ_F is an $\mathcal{N}$-fuzzy UP-filter of A.

Conversely, assume that χ_F is an $\mathcal{N}$-fuzzy UP-filter of A. Since $\chi_F(0) \leq \chi_F(x)$ for all $x \in A$, it follows from Lemma 4.2 that $0 \in F$. Next, let $x, y \in A$ be such that $x \cdot y \in F$ and $x \in F$. Then $\chi_F(x \cdot y) = -1$ and $\chi_F(x) = -1$. Thus $\chi_F(y) \leq \max\{\chi_F(x \cdot y), \chi_F(x)\} = -1$, so $\chi_F(y) = -1$. Therefore, $y \in F$ and so F is a UP-filter of A. □

Theorem 4.4. *A nonempty subset B of A is a UP-ideal of A if and only if the characteristic $\mathcal{N}$-fuzzy set χ_B is an $\mathcal{N}$-fuzzy UP-ideal of A.*

Proof. Assume that B is a UP-ideal of A. Since $0 \in F$, it follows from Lemma 4.2 that $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$. Next, let $x, y, z \in A$.

Case 1: $x \cdot (y \cdot z) \in B$ and $y \in B$. Then $\chi_B(x \cdot (y \cdot z)) = -1$ and $\chi_B(y) = -1$. Thus $\max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = -1$. Since $x \cdot (y \cdot z) \in B$ and $y \in B$, we have $x \cdot z \in B$ and so $\chi_B(x \cdot z) = -1$. Therefore, $\chi_B(x \cdot z) = -1 \leq -1 = \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\}$.

Case 2: $x \cdot (y \cdot z) \notin B$ or $y \notin B$. Then $\chi_B(x \cdot (y \cdot z)) = 0$ or $\chi_B(y) = 0$. Thus $\max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = 0$. Therefore, $\chi_B(x \cdot z) \leq 0 = \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\}$.

Hence, χ_B is an $\mathcal{N}$-fuzzy UP-ideal of A.

Conversely, assume that χ_B is an $\mathcal{N}$-fuzzy UP-ideal of A. Since $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$, it follows from Lemma 4.2 that $0 \in B$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in B$ and $y \in B$. Then $\chi_B(x \cdot (y \cdot z)) = -1$ and $\chi_B(y) = -1$. Thus $\chi_B(x \cdot z) \leq \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = -1$, so $\chi_B(x \cdot z) = -1$. Therefore, $x \cdot z \in B$ and so B is a UP-ideal of A. □

Theorem 4.5. *A nonempty subset C of A is a strongly UP-ideal of A if and only if the characteristic $\mathcal{N}$-fuzzy set χ_C is an $\mathcal{N}$-fuzzy strongly UP-ideal of A.*

Proof. Assume that C is a strongly UP-ideal of A. Then $C = A$. Thus $\chi_C = \chi_A$, so χ_C is constant. It follows from Theorem 3.12 that χ_C is an $\mathcal{N}$-fuzzy strongly UP-ideal of A.

Conversely, assume that χ_C is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. By Theorem 3.12, we have χ_C is constant. By Lemma 3.2, we have $C = A$. Hence, C is a strongly UP-ideal of A. $\square$

5. Level Subsets of an $\mathcal{N}$-Fuzzy Set

In this section, we discuss the relationships among $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) and and its level subsets.

Definition 5.1. Let f be an $\mathcal{N}$-fuzzy set in A. For any $t \in [-1, 0]$, the sets

$$U(f;t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f;t) = \{x \in A \mid f(x) > t\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The sets

$$L(f;t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f;t) = \{x \in A \mid f(x) < t\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively. The set

$$E(f;t) = \{x \in A \mid f(x) = t\}$$

is called an *equal t-level subset* of f. Then

$$U(f;t) = U^+(f;t) \cup E(f;t) \text{ and } L(f;t) = L^-(f;t) \cup E(f;t).$$

The following lemma is easily verified, therefore the proof is omitted.

Lemma 5.2. *Let f be an $\mathcal{N}$-fuzzy set in A and $t \in [-1, 0]$. Then the following statements hold:*

(1) $U(f;t) = L(\overline{f}; -1-t)$,
(2) $L(f;t) = U(\overline{f}; -1-t)$,
(3) $U^+(f;t) = L^-(\overline{f}; -1-t)$, *and*
(4) $L^-(f;t) = U^+(\overline{f}; -1-t)$.

5.1. **Lower t-Level Subsets of an $\mathcal{N}$-Fuzzy Set.**

Theorem 5.3. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-subalgebra of A if and only if for all $t \in [-1, 0]$, $L(f;t)$ is a UP-subalgebra of A if $L(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $t \in [-1, 0]$ be such that $L(f;t) \neq \emptyset$ and let $x, y \in L(f;t)$. Then $f(x) \leq t$ and $f(y) \leq t$, so t is an upper bound of $\{f(x), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t$ and thus $x \cdot y \in L(f;t)$. Hence, $L(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [-1, 0]$, $L(f;t)$ is a UP-subalgebra of A if $L(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [-1, 0]$.

Choose $t = \max\{f(x), f(y)\}$. Then $f(x) \leq t$ and $f(y) \leq t$. Thus $x, y \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in L(f;t)$. So $f(x \cdot y) \leq t = \max\{f(x), f(y)\}$. Hence, f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. □

Theorem 5.4. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-filter of A if and only if for all $t \in [-1, 0]$, $L(f;t)$ is a UP-filter of A if $L(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-filter of A. Let $t \in [-1, 0]$ be such that $L(f;t) \neq \emptyset$ and let $a \in L(f;t)$. Then $f(a) \leq t$. Since f is an $\mathcal{N}$-fuzzy UP-filter of A, we have $f(0) \leq f(a) \leq t$. Thus $0 \in L(f;t)$. Next, let $x, y \in A$ be such that $x \cdot y \in L(f;t)$ and $x \in L(f;t)$. Then $f(x \cdot y) \leq t$ and $f(x) \leq t$, so t is an upper bound of $\{f(x \cdot y), f(x)\}$. Since f is an $\mathcal{N}$-fuzzy UP-filter of A, we have $f(y) \leq \max\{f(x \cdot y), f(x)\} \leq t$. Thus $y \in L(f;t)$. Hence, $L(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [-1, 0]$, $L(f;t)$ is a UP-filter of A if $L(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \leq t$. Thus $x \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a UP-filter of A. Then $0 \in L(f;t)$. Thus $f(0) \leq t = f(x)$. Next, let $x, y \in A$. Then $f(x \cdot y), f(x) \in [-1, 0]$. Choose $t = \max\{f(x \cdot y), f(x)\}$. Then $f(x \cdot y) \leq t$ and $f(x) \leq t$. Thus $x \cdot y, x \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a UP-filter of A. So $y \in L(f;t)$. Hence, $f(y) \leq t = \max\{f(x \cdot y), f(x)\}$. Therefore, f is an $\mathcal{N}$-fuzzy UP-filter of A. □

Theorem 5.5. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-ideal of A if and only if for all $t \in [-1, 0]$, $L(f;t)$ is a UP-ideal of A if $L(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-ideal of A. Let $t \in [-1, 0]$ be such that $L(f;t) \neq \emptyset$ and let $a \in L(f;t)$. Then $f(a) \leq t$. Since f is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $f(0) \leq f(a) \leq t$. Thus $0 \in L(f;t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(f;t)$ and $y \in L(f;t)$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$, so t is an upper bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t$. Thus $x \cdot z \in L(f;t)$. Hence $L(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $L(f;t)$ is a UP-ideal of A if $L(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \leq t$. Thus $x \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a UP-ideal of A. Then $0 \in L(f;t)$. Thus $f(0) \leq t = f(x)$. Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [-1, 0]$. Choose $t = \max\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. Thus

$x \cdot (y \cdot z), y \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a UP-ideal of A. So $x \cdot z \in L(f;t)$. Hence, $f(x \cdot z) \leq t = \max\{f(x \cdot (y \cdot z)), f(y)\}$. Therefore, f is an $\mathcal{N}$-fuzzy UP-ideal of A. □

Theorem 5.6. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy strongly UP-ideal of A if and only if for all $t \in [-1, 0]$, $L(f;t)$ is a strongly UP-ideal of A if $L(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. Let $t \in [-1, 0]$ be such that $L(f;t) \neq \emptyset$ and let $a \in L(f;t)$. Then $f(a) \leq t$. Since f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $f(0) \leq f(a) \leq t$. Thus $0 \in L(f;t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in L(f;t)$ and $y \in L(f;t)$. Then $f((z \cdot y) \cdot (z \cdot x)) \leq t$ and $f(y) \leq t$, so t is an upper bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \leq t$. Thus $x \in L(f;t)$. Hence, $L(f;t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $L(f;t)$ is a strongly UP-ideal of A if $L(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \leq t$. Thus $x \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a strongly UP-ideal of A. Then $0 \in L(f;t)$. Thus $f(0) \leq t = f(x)$. Next, let $x, y, z \in A$. Then $f((z \cdot y) \cdot (z \cdot x)), f(y) \in [-1, 0]$. Choose $t = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Then $f((z \cdot y) \cdot (z \cdot x)) \leq t$ and $f(y) \leq t$. Thus $(z \cdot y) \cdot (z \cdot x), y \in L(f;t) \neq \emptyset$. By assumption, we have $L(f;t)$ is a strongly UP-ideal of A. So $x \in L(f;t)$. Hence, $f(x) \leq t = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Therefore, f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. □

5.2. **Lower t-Strong Level Subsets of an $\mathcal{N}$-Fuzzy Set.**

Theorem 5.7. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-subalgebra of A if and only if for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-subalgebra of A if $L^-(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $t \in [-1, 0]$ be such that $L^-(f;t) \neq \emptyset$ and let $x, y \in L^-(f;t)$. Then $f(x) < t$ and $f(y) < t$, so t is an upper bound of $\{f(x), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} < t$, so $x \cdot y \in L^-(f;t)$. Hence, $L^-(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-subalgebra of A if $L^-(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $f(x \cdot y) > \max\{f(x), f(y)\}$. Then $f(x \cdot y) \in [-1, 0]$. Choose $t = f(x \cdot y)$. Then $f(x) < t$ and $f(y) < t$. Thus $x, y \in L^-(f;t) \neq \emptyset$. By assumption, we have $L^-(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in L^-(f;t)$. So $f(x \cdot y) < t = f(x \cdot y)$, a contradiction. Hence,

$f(x \cdot y) \leq \max\{f(x), f(y)\}$, for all $x, y \in A$. Therefore, f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. □

Theorem 5.8. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-filter of A if and only if for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-filter of A if $L^-(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-filter of A. Let $t \in [-1, 0]$ be such that $L^-(f;t) \neq \emptyset$ and let $a \in L^-(f;t)$. Then $f(a) < t$. Since f is an $\mathcal{N}$-fuzzy UP-filter of A, we have $f(0) \leq f(a) < t$. Thus $0 \in L^-(f;t)$. Next, let $x, y \in A$ be such that $x \in L^-(f;t)$ and $x \cdot y \in L^-(f;t)$ Then $f(x) < t$ and $f(x \cdot y) < t$, so t is an upper bound of $\{f(x), f(x \cdot y)\}$. Since f is an $\mathcal{N}$-fuzzy UP-filter of A, we have $f(y) \leq \max\{f(x), f(x \cdot y)\} < t$. Thus $y \in L^-(f;t)$. Hence, $L^-(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-filter of A if $L^-(f;t)$ is nonempty. Assume that there exist $x \in A$ such that $f(0) > f(x)$. Then $f(0) \in [-1, 0]$. Choose $t = f(0)$. Then $f(x) < t$. Thus $x \in L^-(f;t) \neq \emptyset$. By assumption, we have $L^-(f;t)$ is a UP-filter of A and thus $0 \in L^-(f;t)$. So $f(0) < t = f(0)$, a contradiction. Hence, $f(0) \leq f(x)$, for all $x \in A$. Assume that there exist $x, y \in A$ such that $f(y) > \max\{f(x), f(x \cdot y)\}$. Then $f(y) \in [-1, 0]$. Choose $t = f(y)$. Then $f(x) < t$ and $f(x \cdot y) < t$. Thus $x, x \cdot y \in L^-(f;t) \neq \emptyset$. By assumption, we have $L^-(f;t)$ is a UP-filter of A and thus $y \in L^-(f;t)$. So $f(y) < t = f(y)$, a contradiction. Hence, $f(y) \leq \max\{f(x), f(x \cdot y)\}$, for all $x, y \in A$. Therefore, f is an $\mathcal{N}$-fuzzy UP-filter of A. □

Theorem 5.9. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy UP-ideal of A if and only if for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-ideal of A if $L^-(f;t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-ideal of A. Let $t \in [-1, 0]$ be such that $L^-(f;t) \neq \emptyset$ and let $a \in L^-(f;t)$. Then $f(a) < t$. Since f is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $f(0) \leq f(a) < t$. Thus $0 \in L^-(f;t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(f;t)$ and $y \in L^-(f;t)$ Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$, so t is an upper bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$. Thus $x \cdot z \in L^-(f;t)$. Hence, $L^-(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $L^-(f;t)$ is a UP-ideal of A if $L^-(f;t)$ is nonempty. Assume that there exist $x \in A$ such that $f(0) > f(x)$. Then $f(0) \in [-1, 0]$. Choose $t = f(0)$. Then $f(x) < t$. Thus $x \in L^-(f;t) \neq \emptyset$. By assumption, we have $L^-(f;t)$ is a UP-ideal of A and thus $0 \in L^-(f;t)$. So $f(0) < t = f(0)$, a contradiction.

Hence, $f(0) \leq f(x)$, for all $x \in A$. Assume that there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot z) \in [-1, 0]$. Choose $t = f(x \cdot z)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. Thus $x \cdot (y \cdot z), y \in L^-(f; t) \neq \emptyset$. By assumption, we have $L^-(f; t)$ is a UP-ideal of A and thus $x \cdot z \in L^-(f; t)$. So $f(x \cdot z) < t = f(x \cdot z)$, a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$, for all $x, y \in A$. Therefore, f is an $\mathcal{N}$-fuzzy UP-ideal of A. □

Theorem 5.10. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy strongly UP-ideal of A if and only if for all $t \in [-1, 0]$, $L^-(f; t)$ is a strongly UP-ideal of A if $L^-(f; t)$ is nonempty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. Let $t \in [-1, 0]$ be such that $L^-(f; t) \neq \emptyset$ and let $a \in L^-(f; t)$. Then $f(a) < t$. Since f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $f(0) \leq f(a) < t$. Thus $0 \in L^-(f; t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in L^-(f; t)$ and $y \in L^-(f; t)$ Then $f((z \cdot y) \cdot (z \cdot x)) < t$ and $f(y) < t$, so t is an upper bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Since f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} < t$. Thus $x \in L^-(f; t)$. Hence, $L^-(f; t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $L^-(f; t)$ is a strongly UP-ideal of A, if $L^-(f; t)$ is nonempty. Assume that there exist $x \in A$ such that $f(0) > f(x)$. Then $f(0) \in [-1, 0]$. Choose $t = f(0)$. Then $f(x) < t$. Thus $x \in L^-(f; t) \neq \emptyset$. By assumption, we have $L^-(f; t)$ is a strongly UP-ideal of A and thus $0 \in L^-(f; t)$. So $f(0) < t = f(0)$, a contradiction. Hence, $f(0) \leq f(x)$, for all $x \in A$. Assume that there exist $x, y, z \in A$ such that $f(x) > \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f((z \cdot y) \cdot (z \cdot x)) < t$ and $f(y) < t$. Thus $(z \cdot y) \cdot (z \cdot x), y \in L^-(f; t) \neq \emptyset$. By assumption, we have $L^-(f; t)$ is a strongly UP-ideal of A and thus $x \in L^-(f; t)$. So $f(x) < t = f(x)$, a contradiction. Hence, $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$, for all $x, y \in A$. Therefore, f is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. □

5.3. **Upper t-Level Subsets of an $\mathcal{N}$-Fuzzy Set.**

The following lemma is easily proved.

Lemma 5.11. *Let f be an $\mathcal{N}$-fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $-1 - \max\{f(x), f(y)\} = \min\{-1 - f(x), -1 - f(y)\}$, *and*
(2) $-1 - \min\{f(x), f(y)\} = \max\{-1 - f(x), -1 - f(y)\}$.

Theorem 5.12. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-subalgebra of A if and only if for all $t \in [-1, 0]$, $U(f; t)$ is a UP-subalgebra of A if $U(f; t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $t \in [-1,0]$ be such that $U(f;t) \neq \emptyset$ and let $x, y \in U(f;t)$. Then $f(x) \geq t$ and $f(y) \geq t$, so t is a lower bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x \cdot y) \leq \max\{-1 - f(x), -1 - f(y)\} = -1 - \min\{f(x), f(y)\}$. Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$. So $x \cdot y \in U(f;t)$. Hence, $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [-1,0]$, $U(f;t)$ is a UP-subalgebra of A if $U(f;t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [-1,0]$. Choose $t = \min\{f(x), f(y)\}$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus $x, y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A and thus $x \cdot y \in U(f;t)$. So $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. By Lemma 5.11 (2), we have

$$\begin{aligned}
\overline{f}(x \cdot y) &= -1 - f(x \cdot y) \\
&\leq -1 - \min\{f(x), f(y)\} \\
&= \max\{-1 - f(x), -1 - f(y)\} \\
&= \max\{\overline{f}(x), \overline{f}(y)\}.
\end{aligned}$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A. □

Theorem 5.13. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-filter of A if and only if for all $t \in [-1,0]$, $U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A. Let $t \in [-1,0]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$.Then $f(a) \geq t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f;t)$. Next, let $x, y \in A$ be such that $x \in U(f;t)$ and $x \cdot y \in U(f;t)$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$, so t is a lower bound of $\{f(x), f(x \cdot y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A, we have $\overline{f}(y) \leq \max\{\overline{f}(x), \overline{f}(x \cdot y)\}$. By Lemma 5.11 (2), we have $-1 - f(y) \leq \max\{-1 - f(x), -1 - f(x \cdot y)\} = -1 - \min\{f(x), f(x \cdot y)\}$. Thus $f(y) \geq \min\{f(x), f(x \cdot y)\} \geq t$. So $y \in U(f;t)$. Hence, $U(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [-1,0]$, $U(f;t)$ is a UP-filter of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1,0]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A and thus $0 \in U(f;t)$. So $f(0) \geq f(x)$. Hence, $\overline{f}(0) = -1 - f(0) \leq -1 - f(x) = \overline{f}(a)$. Next, Let $x, y \in A$. Then $f(x), f(x \cdot y) \in [-1,0]$. Choose $t = \min\{f(x), f(x \cdot y)\}$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$. Thus $x, x \cdot y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-filter of A and thus $y \in U(f;t)$. Thus $f(y) \geq t = \min\{f(x), f(x \cdot y)\}$. By Lemma 5.11 (2), we have

$$\begin{aligned}\overline{f}(y) &= -1 - f(y) \\ &\leq -1 - \min\{f(x), f(x \cdot y)\} \\ &= \max\{-1 - f(x), -1 - f(x \cdot y)\} \\ &= \max\{\overline{f}(x), \overline{f}(x \cdot y)\}.\end{aligned}$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A. □

Theorem 5.14. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-ideal of A if and only if for all $t \in [-1, 0]$, $U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A. Let $t \in [-1, 0]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$.Then $f(a) \geq t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$, so t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x \cdot z) \leq \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$. So $x \cdot z \in U(f;t)$. Hence, $U(f;t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $U(f;t)$ is a UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A and thus $0 \in U(f;t)$. So $f(0) \geq f(x)$. Hence, $\overline{f}(0) = -1 - f(0) \leq -1 - f(x) = \overline{f}(a)$. Next, Let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [-1, 0]$. Choose $t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus $x \cdot (y \cdot z), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-ideal of A and thus $x \cdot z \in U(f;t)$. Thus $f(x \cdot z) \geq t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 5.11 (2), we have

$$\begin{aligned}\overline{f}(x \cdot z) &= -1 - f(x \cdot z) \\ &\leq -1 - \min\{f(x \cdot (y \cdot z)), f(y)\} \\ &= \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} \\ &= \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.\end{aligned}$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A. □

Theorem 5.15. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy strongly UP-ideal of A if and only if for all $t \in [-1, 0]$, $U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. Let $t \in [-1, 0]$ be such that $U(f;t) \neq \emptyset$ and let $a \in U(f;t)$.Then $f(a) \geq t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f;t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U(f;t)$ and $y \in U(f;t)$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t$ and $f(y) \geq t$, so t is a lower bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $\overline{f}(x) \leq \max\{\overline{f}((z \cdot y) \cdot (z \cdot x)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x) \leq \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\} = -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \geq t$. So $x \in U(f;t)$. Hence, $U(f;t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $U(f;t)$ is a strongly UP-ideal of A if $U(f;t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A and thus $0 \in U(f;t)$. So $f(0) \geq f(x)$. Hence, $\overline{f}(0) = -1 - f(0) \leq -1 - f(x) = \overline{f}(a)$. Next, Let $x, y, z \in A$. Then $f((z \cdot y) \cdot (z \cdot x)), f(y) \in [-1, 0]$. Choose $t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t$ and $f(y) \geq t$. Thus $(z \cdot y) \cdot (z \cdot x), y \in U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a strongly UP-ideal of A and thus $y \in U(f;t)$. Thus $f(y) \geq t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. By Lemma 5.11 (2), we have

$$\begin{aligned}
\overline{f}(x) &= -1 - f(x) \\
&\leq -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \\
&= \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\} \\
&= \max\{\overline{f}((z \cdot y) \cdot (z \cdot x)), \overline{f}(y)\}.
\end{aligned}$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. □

5.4. **Upper t-Strong Level Subsets of an $\mathcal{N}$-Fuzzy Set.**

Theorem 5.16. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-subalgebra of A if and only if for all $t \in [-1, 0]$, $U^+(f;t)$ is a UP-subalgebra of A if $U^+(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $t \in [-1, 0]$ be such that $U^+(f;t) \neq \emptyset$ and let $x, y \in U^+(f;t)$. Then $f(x) > t$ and $f(y) > t$, so t is a lower bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. By

Lemma 5.11 (2), we have $-1-f(x\cdot y) \leq \max\{-1-f(x), -1-f(y)\} = -1-\min\{f(x), f(y)\}$. Thus $f(x\cdot y) \geq \min\{f(x), f(y)\} > t$. So $x\cdot y \in U^+(f;t)$. Hence, $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f;t)$ is a UP-subalgebra of A if $U^+(f;t)$ is nonempty. Assume that there exist $x, y \in A$ such that $\overline{f}(x\cdot y) > \max\{\overline{f}(x), \overline{f}(y)\}$. By Lemma 5.11 (2), we have

$$-1-f(x\cdot y) > \max\{-1-f(x), -1-f(y)\} = -1-\min\{f(x), f(y)\}.$$

Thus $f(x\cdot y) < \min\{f(x), f(y)\}$. Now $f(x\cdot y) \in [-1, 0]$, we choose $t = f(x\cdot y)$. Then $f(x) > t$ and $f(y) > t$. Thus $x, y \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a UP-subalgebra of A and thus $x\cdot y \in U^+(f;t)$. So $f(x\cdot y) > t = f(x\cdot y)$, a contradiction. Hence, $\overline{f}(x\cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$, for all $x, y \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of A. $\square$

Theorem 5.17. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-filter of A if and only if for all $t \in [-1, 0]$, $U^+(f;t)$ is a UP-filter of A if $U^+(f;t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A. Let $t \in [-1, 0]$ be such that $U^+(f;t) \neq \emptyset$ and let $a \in U^+(f;t)$. Then $f(a) > t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1-f(0) \leq -1-f(a)$. So $f(0) \geq f(a) > t$. Hence, $0 \in U^+(f;t)$. Next, let $x, y \in A$ be such that $x \in U^+(f;t)$ and $x\cdot y \in U^+(f;t)$. Then $f(x) > t$ and $f(x\cdot y) > t$, so t is a lower bound of $\{f(x), f(x\cdot y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A, we have $\overline{f}(y) \leq \max\{\overline{f}(x), \overline{f}(x\cdot y)\}$. By Lemma 5.11 (2), we have $-1-f(y) \leq \max\{-1-f(x), -1-f(x\cdot y)\} = -1-\min\{f(x), f(x\cdot y)\}$. So $f(y) \geq \min\{f(x), f(x\cdot y)\} > t$ and thus $y \in U^+(f;t)$. Hence, $U(f;t)$ is a UP-filter of A.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f;t)$ is a UP-filter of A if $U^+(f;t)$ is nonempty. Assume that there exist $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Then $-1-f(0) > -1-f(x)$. Thus $f(0) < f(x)$. Now $f(0) \in [-1, 0]$, we choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a UP-filter of A and thus $0 \in U^+(f;t)$. So $f(0) > t = f(0)$, a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$, for all $x \in A$. Assume that there exist $x, y \in A$ such that $\overline{f}(y) > \max\{\overline{f}(x), \overline{f}(x\cdot y)\}$. By Lemma 5.11 (2), we have

$$-1-f(y) > \max\{-1-f(x), -1-f(x\cdot y)\} = -1-\min\{f(x), f(x\cdot y)\}.$$

Thus $f(y) < \min\{f(x), f(x\cdot y)\}$. Now $f(y) \in [-1, 0]$, we choose $t = f(y)$. Then $f(x) > t$ and $f(x\cdot y) > t$. Then $x, x\cdot y \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a UP-filter of A and thus $y \in U^+(f;t)$.

So $f(y) > t = f(y)$, a contradiction. Hence, $\overline{f}(y) \leq \max\{\overline{f}(x), \overline{f}(x \cdot y)\}$, for all $x, y \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of A. $\square$

Theorem 5.18. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy UP-ideal of A if and only if for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-ideal of A if $U^+(f; t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A. Let $t \in [-1, 0]$ be such that $U^+(f; t) \neq \emptyset$ and let $a \in U^+(f; t)$. Then $f(a) > t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$. So $f(0) \geq f(a) > t$. Hence, $0 \in U^+(f; t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(f; t)$ and $y \in U^+(f; t)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$, so t is a lower bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A, we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x \cdot z) \leq \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t$ and thus $x \cdot z \in U^+(f; t)$. Hence, $U(f; t)$ is a UP-ideal of A.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-filter of A if $U^+(f; t)$ is nonempty. Assume that there exist $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Then $-1 - f(0) > -1 - f(x)$. Thus $f(0) < f(x)$. Now $f(0) \in [-1, 0]$, we choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-filter of A and thus $0 \in U^+(f; t)$. So $f(0) > t = f(0)$, a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$, for all $x \in A$. Assume that there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x \cdot z) > \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Now $f(x \cdot z) \in [-1, 0]$, we choose $t = f(x \cdot z)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$. Thus $x \cdot (y \cdot z), y \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-ideal of A and thus $x \cdot z \in U^+(f; t)$. So $f(x \cdot z) > t = f(x \cdot z)$, a contradiction. Hence, $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$, for all $x, y, z \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of A. $\square$

Theorem 5.19. *The complement $\overline{f}$ in A is an $\mathcal{N}$-fuzzy strongly UP-ideal of A if and only if for all $t \in [-1, 0]$, $U^+(f; t)$ is a strongly UP-ideal of A if $U^+(f; t)$ is nonempty.*

Proof. Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. Let $t \in [-1, 0]$ be such that $U^+(f; t) \neq \emptyset$ and let $a \in U^+(f; t)$. Then $f(a) > t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$. So $f(0) \geq f(a) > t$. Hence, $0 \in U^+(f; t)$. Next, let $x, y, z \in A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U^+(f; t)$ and $y \in$

$U^+(f;t)$. Then $f((z\cdot y)\cdot(z\cdot x)) > t$ and $f(y) > t$, so t is a lower bound of $\{f((z\cdot y)\cdot(z\cdot x)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A, we have $\overline{f}(x) \leq \max\{\overline{f}((z\cdot y)\cdot(z\cdot x)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1-f(x) \leq \max\{-1-f((z\cdot y)\cdot(z\cdot x)), -1-f(y)\} = -1-\min\{f((z\cdot y)\cdot(z\cdot x)), f(y)\}$. So $f(x) \geq \min\{f((z\cdot y)\cdot(z\cdot x)), f(y)\} > t$ and thus $x \in U^+(f;t)$. Hence, $U(f;t)$ is a strongly UP-ideal of A.

Conversely, assume that for all $t \in [-1,0]$, $U^+(f;t)$ is a strongly UP-ideal of A if $U^+(f;t)$ is nonempty. Assume that there exist $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Then $-1-f(0) > -1-f(x)$. Thus $f(0) < f(x)$. Now $f(0) \in [-1,0]$, we choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a strongly UP-ideal of A and thus $0 \in U^+(f;t)$. So $f(0) > t = f(0)$, a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$, for all $x \in A$. Assume that there exist $x, y, z \in A$ such that $\overline{f}(x) > \max\{\overline{f}((z\cdot y)\cdot(z\cdot x)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1-f(x) > \max\{-1-f((z\cdot y)\cdot(z\cdot x)), -1-f(y)\} = -1-\min\{f((z\cdot y)\cdot(z\cdot x)), f(y)\}$. Thus $f(x) < \min\{f((z\cdot y)\cdot(z\cdot x)), f(y)\}$. Now $f(x) \in [-1,0]$, we choose $t = f(x)$. Then $f((z\cdot y)\cdot(z\cdot x)) > t$ and $f(y) > t$. Then $(z\cdot y)\cdot(z\cdot x), y \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a strongly UP-ideal of A and thus $x \in U^+(f;t)$. So $f(x) > t = f(x)$, a contradiction. Hence, $\overline{f}(x) \leq \max\{\overline{f}((z\cdot y)\cdot(z\cdot x)), \overline{f}(y)\}$, for all $x, y, z \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of A. □

5.5. **Equal t-Level Subsets of an $\mathcal{N}$-Fuzzy Set.**

Corollary 5.20. *If f is an $\mathcal{N}$-fuzzy UP-subalgebra of A, then for all $t \in [-1,0]$, $E(f;t)$ is a UP-subalgebra of A where $E(f;t)$ is nonempty and $L^-(f;t)$ is empty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-subalgebra of A. Let $t \in [-1,0]$ be such that $E(f;t) \neq \emptyset$ and $L^-(f;t) = \emptyset$. Since $E(f;t) \subseteq L(f;t)$, we have $L(f;t) \neq \emptyset$. By Theorem 5.3, we have $E(f;t) = \emptyset \cup E(f;t) = L^-(f;t) \cup E(f;t) = L(f;t)$ is a UP-subalgebra of A. □

Example 5.21. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	1	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1,0]$ as follows:

$$f(0) = -1, f(1) = -0.1, f(2) = -0.2, \text{ and } f(3) = -0.2.$$

If $t \neq -1$, then $L^-(f;t) \neq \emptyset$. If $t = -1$, then $L^-(f;t) = \emptyset$ and $E(f;t) = \{0\}$. Clearly, $E(f;t)$ is a UP-subalgebra of A. Since $f(3 \cdot 2) = -0.1 > -0.2 = \max\{f(3), f(2)\}$, we have f is not an $\mathcal{N}$-fuzzy UP-subalgebra of A.

Corollary 5.22. *If f is an $\mathcal{N}$-fuzzy UP-filter of A, then for all $t \in [-1, 0]$, $E(f;t)$ is a UP-filter of A where $E(f;t)$ is nonempty and $L^-(f;t)$ is empty.*

Proof. Assume that f is an $\mathcal{N}$-fuzzy UP-filter of A. Let $t \in [-1, 0]$ be such that $E(f;t) \neq \emptyset$ and $L^-(f;t) = \emptyset$. Since $E(f;t) \subseteq L(f;t)$, we have $L(f;t) \neq \emptyset$. By Theorem 5.4, we have $E(f;t) = \emptyset \cup E(f;t) = L^-(f;t) \cup E(f;t) = L(f;t)$ is a UP-filter of A. □

Example 5.23. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	1	3
2	0	0	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.9, f(2) = -0.7, \text{ and } f(3) = -0.5.$$

If $t \neq -1$, then $L^-(f;t) \neq \emptyset$. If $t = -1$, then $L^-(f;t) = \emptyset$ and $E(f;t) = \{0\}$. Clearly, $E(f;t)$ is a UP-filter of A. Since $f(2) = -0.7 > -0.9 = \max\{f(1 \cdot 2), f(1)\}$, we have f is not an $\mathcal{N}$-fuzzy UP-filter of A.

Corollary 5.24. *If f is an $\mathcal{N}$-fuzzy UP-ideal of A, then for all $t \in [-1, 0]$, $E(f;t)$ is a UP-ideal of A where $E(f;t)$ is nonempty and $L^-(f;t)$ is empty.*

Proof. Let f is an $\mathcal{N}$-fuzzy UP-ideal of A. Let $t \in [-1, 0]$ be such that $E(f;t) \neq \emptyset$ and $L^-(f;t) = \emptyset$. Since $E(f;t) \subseteq L(f;t)$, we have $L(f;t) \neq \emptyset$. By Theorem 5.5, we have $E(f;t) = \emptyset \cup E(f;t) = L^-(f;t) \cup E(f;t) = L(f;t)$ is a UP-ideal of A. □

Example 5.25. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	2
3	0	1	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$$f(0) = -1, f(1) = -0.8, f(2) = -0.5, \text{ and } f(3) = -0.2.$$

If $t \neq -1$, then $L^{-}(f;t) \neq \emptyset$. If $t = -1$, then $L^{-}(f;t) = \emptyset$ and $E(f;t) = \{0\}$. Clearly, $E(f;t)$ is a UP-ideal of A. Since $f(1 \cdot 3) = -0.2 > -0.5 = \max\{f(1 \cdot (2 \cdot 3)), f(2)\}$, we have f is not an $\mathcal{N}$-fuzzy UP-ideal of A.

Corollary 5.26. *An $\mathcal{N}$-fuzzy set f in A is an $\mathcal{N}$-fuzzy strongly UP-ideal of A if and only if for all $t \in [-1, 0]$, $E(f;t)$ is a strongly UP-ideal of A where $E(f;t)$ is nonempty.*

Proof. It is straightforward by Theorem 3.12 and 5.6, and A is the only one strongly UP-ideal of itself. □

6. Conclusions

In the present paper, we have introduced the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras and proved its generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). We think this work would enhance the scope for further study in UP-algebras and related algebraic systems. It is our hope that this work would serve as a foundation for the further study in a new concept of UP-algebras.

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