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On the mild solution for nonlocal impulsive fractional semilinear differential inclusion in Banach spaces

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Abstract. This paper gives existence results for impulsive fractional semilinear differential inclusions involving Caputo derivative in Banach spaces. We are concerned with the case when the linear part generates a semigroup not necessarily compact, and the multivalued function is upper semicontinuous and compact. The methods used throughout the paper range over applications of Hausdorff measure of noncompactness, and multivalued fixed point theorems. Finally, we provide an example to clarify our results.

Keywords: Impulsive fractional differential inclusions, nonlocal conditions, fixed point theorems, mild solutions.
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1 Introduction

Fractional differential equations and fractional differential inclusions have gained considerable attention since two decades due to thier wide use as mathematical modeling in various areas such as physics, biology, mechanics and engineering, medical field, industry and technology. Moreover, fractional differential equations and inclusions serve as an effective tool for the description of hereditary properties of various materials and processes. For more details, we refer to [15, 18, 22, 25] and the references therein.

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In this paper, we shall be concerned with the following impulsive differential inclusion with nonlocal condition:

$$(Q)\begin{cases} {}^cD^\alpha x(t) \in Ax(t) + F(t, x(t)),\ t \in J = [0, b],\quad t \neq t_i, i = 1, \dots, m,\\ x(t_i^+) = x(t_i) + I_i(x(t_i)),\quad i = 1, \dots, m,\\ x(0) = g(x),\end{cases}$$

where ${}^cD^\alpha$ is the Caputo derivative of order α, $A : D(A) \subseteq E \to E$ is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on a real separable Banach space E, $F : J \times E \to 2^E$ is an upper-Caratheodory multifunction, 2^E is the power set of E, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, for every $i = 1, 2, \dots, m$, $I_i : E \to E$ impulsive functions which characterize the jump of the solutions at impulse points, $g : PC(J, E) \to E$, is a function related to the nonlocal condition at the origin and $x(t_i^+), x(t_i^-)$ are the right and left limits of x at the point t_i respectively and $PC(J, E)$ will be specified later.

Impulsive differential equations and impulsive differential inclusions have played a significant role in development of modeling impulsive problems in various areas; physics, technology, optimal control, and so forth. The reason for this applicability arises from the fact that impulsive differential problems are an appropriate modle for describing process which at certain moments change their state rapidly and which cannot described using the classical differential problems. For some applications, one can see [2,6,27]. Along with the applied development, the basic of general theory of impulsive differential equations and inclusions has been discussed in details, see the book of Benchohra et al. [8], the papers [10,13] and the references therein.

On the other hand, impulsive semilinear differential problems, with nonlocal conditions are often motivated by physical problems, for instance see [5,9,14]. The work of abstract nonlocal differential problems was firstly investigated by Byszewski [9]. Nonlocal problems have received much more attention after it was demonstrated that nonlocal conditions can be more descriptive with better effects than the classical ones in applications, see for example [12]. However, dealing with the compactness of the solution operator at zero is the main difficulty of problems involving nonlocal conditions. In this direction, various methods and techniques have been adopted by some authors. We refer readers to [3, 10, 11, 13, 17, 19, 20, 23, 26, 29–31]. Among them, Wang et al. [30] introduced a new concept of PC-mild solutions for (Q). They obtained existence results when F is a Lipschitz single-valued function or continuous and maps bounded sets into bounded sets and $\{T(t)\}_{t>0}$ is compact.

Using the Housdorff measure of noncompactness, Li [23] gave existence results concerning nonlocal fractional differential equations, where the semigroup is equicontinuous and the nonlocal term is compact. Moreover, Ibrahim and Alsarori [19] established sufficient conditions which guarantee the existence of mild solutions for the problem (Q) when the semigroup is compact. Recently, Lian et al. [24] discussed the existence results of mild solutions for (Q) without impulsive when the operator semigroup is not necessarily compact.

Motivated by the importance of impulsive problems with nonlocal conditions as a siginficant modeling tool in many fields as we mentioned before as well as most of privous works contained the assumption of compactness of the operator semigroup. We study such kind of problems in order to prove the exitence of mild solution. In our study, we extend the results shown in [24] to nonlocal differential inclusions undergoing impulse effects scenario. Also we generalize the condition assumed by Ibrahim and Alsarori [19] on the semigroup $\{T(t)\}_{t>0}$, that is, the compactness condition is not necessary in our results.

After presenting some definitions and facts related to fractional calculus, the Hausdorff measure of noncompactness, and the set-valued analysis in Section 2. Section 3 proceeds to prove the existence results of PC-mild solutions for (Q), where PC-mild solutions as introduced in [30]. The results are derived by techniques and methods of noncompactness Hausdorff measure, and multivalued fixed point theorems. In Section 4, An example is given to demonstrate the applicability of our results.

2 Preliminaries and notations

During this section, we state some previous known results so that we can use them later throughout this paper. Let $C(J,E)$ be the space of $E-$valued continuous function on J with the uniform norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$, $L^1(J,E)$ the space of $E-$valued Bochner integrable functions on J with the norm $\|x\|_{L^1(J,E)} = \int_0^b \|f(t)\|dt$. We denote

$$\begin{aligned} P_b(E) &= \{B \subseteq E : B \text{ is nonemty and bounded}\}, \\ P_{cl}(E) &= \{B \subseteq E : B \text{ is nonemty and closed}\}, \\ P_k(E) &= \{B \subseteq E : B \text{ is nonemty and compact}\}, \\ P_{cl,cv}(E) &= \{B \subseteq E : B \text{ is nonemty, closed and convex}\}, \\ P_{ck}(E) &= \{B \subseteq E : B \text{ is nonemty, convex and compact}\}, \end{aligned}$$

and $\overline{conv}(B)$ be the convex closed hull in E of subset B.

Definition 1. *([21]).* The Hausdorff measure of noncompactness on E, $\chi : P_b(E) \to [0, +\infty)$ is defined as
$\chi(B) = \inf\{\varepsilon > 0 : B$ can be covered by finitely many balls of radius $\leq \varepsilon\}$.

Lemma 1. *([21]). The Hausdorff measure of noncompactness on E satisfies the following properties:*

1. *monotone if $B_0, B_1 \in P_b(E), B_0 \subset B_1$ implies $\chi(B_0) \leq \chi(B_1)$;*
2. *nonsingular if $\chi(\{a\} \cup B) = \chi(B)$, for every $a \in E, B \in P_b(E)$;*
3. *invariant with respect to union with compact sets if for any compact subset $K \subset E$ and any $B \in P_b(E), \chi(B \cup K) = \chi(B)$;*
4. *algebraic semiadditive if $\chi(B_1 + B_2) \leq \chi(B_1) + \chi(B_2)$, for every $B_1, B_2 \in P_b(E)$; where $B_1 + B_2 = \{a + b : a \in B_1, b \in B_2\}$;*
5. *semiadditive if $\chi(B_1 \cup B_2) = max\{\chi(B_1), \chi(B_2)\}$, for every $B_1, B_2 \in P_b(E)$;*
6. *B is relatively compact if and only if $\chi(B) = 0$, for every $B \in P_b(E)$;*
7. *the Lipschitz property: $\mid \chi(B_1) - \chi(B_2) \mid \leq h(B_1, B_2)$, for every $B_1, B_2 \in P_b(E)$; where h is the Hausdorff distance;*
8. *$\chi(tB) =\mid t \mid \chi(B)$, $t \in \mathbb{R}, B \in P_b(E)$;*
9. *let $L : E \to E$ be a bounded linear operator. Then, $\chi(L(B)) \leq \|L\|\chi(B)$, for every $B \in P_b(E)$.*

Considering a partition on $[0, b]$, i.e., a finite set $\{t_0, t_1, \ldots, t_{m+1}\} \subset [0, b]$ such that $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$. Let $J_0 = [0, t_1], J_i =]t_i, t_{i+1}]$ and $x(t_i^+) = \lim_{s \to t_i^+} x(s)$, $i = 1, \ldots, m$. For $0 \leq i \leq m$, we define

$$PC(J, E) = \{x : J \to E \text{ and } x_{|_{J_i}} \in C(J_i, E),\ x(t_i^+) \text{ and } x(t_i^-) \text{ exist}\}.$$

Note that $PC(J, E)$ with $\|x\|_{PC(J,E)} = \sup\{\|x(t)\| : t \in J\}$ is a Banach space. Also, let us consider the map $\chi_{PC} : P_b(PC(J, E)) \to [0, \infty[$, defined by

$$\chi_{PC}(B) = \max_{i=0,1,\ldots,m} \chi_i(B_{|_{\overline{J_i}}}),\ B \in P_b(PC(J, E),$$

where χ_i is the Hausdorff measure of noncompactness on the Banach space $C(\overline{J_i}, E)$ and

$$B_{|_{\overline{J_i}}} = \{x^* : \overline{J_i} \longrightarrow E : x^*(t) = x(t), t \in J_i, x^*(t_i) = x(t_i^+), x \in B\}.$$

Note that $B_{|_{\overline{J_0}}} = \{x_{|_{\overline{J_0}}} : x \in B\}$. It is easy to see that χ_{PC} is the Hausdorff measure of noncompactness on $PC(J, E)$. For more information about measure of noncompactness, we refer to [16, 21].

Definition 2. According to the Riemann-Liouville approach, the fractional integral of order $\alpha \in (0, 1)$ of a function $f \in L^1(J, E)$ is defined by

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds, \quad t > 0,$$

provided the right side is defined on J, where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Definition 3. The Caputo derivative of order $\alpha \in (0, 1)$ of continuously differentiable function $f : J \to E$ is defined by

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s)ds = I^{(1-\alpha)} f^{(1)}.$$

Note that the integral appeared in the two previous definitions are taken in Bochner sense and ${}^cD^\alpha I^\alpha f(t) = f(t)$ for all $t \in J$. For more details about the fractional calculus, see [22, 25].

Definition 4. A function $x \in PC(J, E)$ is an impulsive mild solution for (Q) if

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \displaystyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \displaystyle\sum_{i=1}^{i=m} \mathcal{T}_\alpha(t-t_i)y_i + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, \ t \in J_i, \end{cases}$$

where $y_i = I_i(x(t_i^-)), i = 1, 2, \ldots, m$, f is an integrable selection for $F(\cdot, x(\cdot))$,

$$\begin{aligned} \mathcal{T}_\alpha(t) &= \int_0^\infty \xi_\alpha(\theta)T(t^\alpha\theta)d\theta, S_\alpha(t) = \alpha \int_0^\infty \theta\xi_\alpha(\theta)T(t^\alpha\theta)d\theta, \\ \xi_\alpha(\theta) &= \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{\frac{-1}{\alpha}}) \geq 0, \\ \varpi_\alpha(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-\alpha n-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \end{aligned}$$

$\theta \in (0, \infty)$ and ξ is a probability density function defined on $(0, \infty)$, that is $\int_0^\infty \xi_\alpha(\theta)d\theta = 1$.

Remark 1. *Since $\mathcal{T}_\alpha(\cdot)$ and $S_\alpha(\cdot)$ are associated with the number α, there are no analogue of the semigroup property, i.e. $\mathcal{T}_\alpha(t+s) \neq \mathcal{T}_\alpha(t)\mathcal{T}_\alpha(s)$ and $S_\alpha(t+s) \neq S_\alpha(t)S_\alpha(s)$.*

In the following we recall the properties of $\mathcal{T}_\alpha(\cdot)$ and $S_\alpha(\cdot)$.

Lemma 2. *([31]).*

(i) For any fixed $t \geq 0, \mathcal{T}_\alpha(t), S_\alpha(t)$ are linear bounded operators.

(ii) For $\gamma \in [0,1], \int_0^\infty \theta^\gamma \xi_\alpha(\theta)d\theta = \dfrac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)}$.

(iii) If $\|T(t)\| \leq M, t \geq 0$, then for any $x \in E, \|\mathcal{T}_\alpha(t)x\| \leq M\|x\|$ and $\|S_\alpha(t)x\| \leq \dfrac{M}{\Gamma(\alpha)}\|x\|$.

(iv) For any fixed $t \geq 0, \mathcal{T}_\alpha(t), S_\alpha(t)$ are strongly continuous.

(v) If $T(t), t > 0$ is compact, then $\mathcal{T}_\alpha(t), S_\alpha(t)$ are compact.

(vi) Both $\mathcal{T}_\alpha(t)$ and $S_\alpha(t)$ are equicontinuous for $t \in J$ if $\{T(t)\}_{t\geq 0}$ is equicontinuous.

Lemma 3. *([10]) (Generalized Cantor's intersection). If $(W_n)_{n\geq 1}$ is a decreasing sequence of bounded and closed nonempty subsets of E and $lim_{n\to\infty}\chi(W_n) = 0$, then $\cap_{n=1}^\infty W_n$ is nonempty and compact in E.*

Lemma 4. *([7]). If $W \subseteq C(J,E)$ is bounded and equicontinuous, then $\chi(W(t))$ is continuous on J and $\chi(W) = \sup_{t\in J}\chi(W(t))$.*

Lemma 5. *([16]). If $\{u_n\}_{n=1}^\infty \subset L^1(J,E)$ is uniformly integrable, then $\chi(\{u_n(t)\}_{n=1}^\infty)$ is measurable, and*

$$\chi(\{\int_0^t u_n(s)ds\}_{n=1}^\infty) \leq 2\int_0^t \chi(\{u_n(s)\}_{n=1}^\infty)ds.$$

Lemma 6. *([24]). If $B \subseteq E$ is bounded, then for each $\epsilon > 0$, there is a sequence $\{u_n\}_{n=1}^\infty$ in B such that $\chi(B) \leq 2\chi(\{u_n\}_{n=1}^\infty) + \epsilon$.*

Lemma 7. *([4]). Let $(W_n)_{n\geq 1} \subset W \subset E$ be a sequence of subsets where W is a compact in the separable Banach space E. Then,*

$$\overline{conv}(\limsup_{n\to\infty} W_n) = \cap_{N>0}\overline{conv}(\cup_{n\geq N}W_n).$$

Definition 5. *([16, 21]).* Let X and Y be two topological spaces. A multifunction $F : X \to P(Y)$ is said to be:

1. Upper semicontinuous ($u.s.c$)if $F^{-1}(V) = \{x \in X : F(x) \subseteq V\}$ is an open subset of X for every open $V \subseteq Y$.

2. Closed if its graph $\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is closed subset of the topological space $X \times Y$, that is, $x_n \to x, y_n \to y$ and $y_n \in F(x_n)$ imply $y \in F(x)$.

3. Completely continuous if $F(B)$ is relatively compact for every bounded subset B of X.

4. If the multifunction F is completely continuous with non empty compact values, then F is $u.s.c.$ if and only if F is closed.

5. F is said to have a fixed point if there is $x \in X$ such that $x \in F(x)$.

Remark 2. *If $U \subset X$ is closed, $F(x)$ is closed for all $x \in U$, and $\overline{F(U)}$ is compact, then F is u.s.c. if and only if F is closed.*

Lemma 8. *(Theorem 1.3.5, [21]). Let X, Y be (not necessarily separable) Banach space, and let $F : J \times X \to P_k(Y)$ be such that*

(i) for every $x \in X$ the multifunction $F(\cdot, x)$ has a strongly measurable selection;

(ii) for a.e. $t \in J$ the multifunction $F(t, \cdot)$ is upper semicontinuous.

Then, for every strongly measurable function $z : J \to X$ there exists a strongly measurable function $f : J \to Y$ such that $f(t) \in F(t, z(t))$ a.e..

Remark 3. *(Theorem 1.3.1, [21]). For single-valued or compact valued multifunction acting on a separable Banach space the notions measurability and strongly measurable coincide. So, if X, Y be separable Banach spaces we can replace strongly measurable with measurable in the previous lemma.*

Definition 6. A sequence $\{f_n : n \in \mathbb{N}\} \subset L^1(J, E)$ is said to be semi-compact if:

1. It is integrably bounded i.e. there is $q \in L^1(J, \mathbb{R}^+)$ such that $\|f_n(t)\| \leq q(t)$ *a.e.* $t \in J$.

2. The set $\{f_n : n \in \mathbb{N}\}$ is relatively compact in E *a.e.* $t \in J$.

Lemma 9. *([21]). Every semi-compact sequence in $L^1(J, E)$ is weakly compact in $L^1(J, E)$.*

Lemma 10. *([30], Lemma 2.10). For $\delta \in (0, 1]$ and $0 < e \leq c$, we have $|e^\delta - c^\delta| \leq (c - e)^\delta$.*

Theorem 1. *([1]). If W is a bounded, closed, convex and compact nonempty subset of E and the map $G : W \to 2^W$ is upper semicontinuous with $G(x)$ is a closed and convex nonempty subset of W for each $x \in W$, then G has at least one fixed point in W.*

3 Main results

By using the Hausdorff measure of noncompactness and multivalued fixed point theorem we will prove the existence of mild solutions for the problem (Q).

Theorem 2. *Assume the following conditions:*
(HA) *The C_0-semigroup $\{T(t) : t \geq 0\}$ generated by A is equicontinuous and there exists a constant $M > 0$ such that $\sup_{t\in J} \|T(t)\| \leq M$.*
(HF) *Let $F : J \times E \to P_{ck}(E)$ be a multifunction satisfies the following hypotheses:*

1. *F is measurable to t for every $x \in E$ and u.s.c. to x for a.e. $t \in J$.*
2. *There exists a function $\varsigma \in L^{\frac{1}{q}}(J, \mathbb{R}^+)$ with $q \in (0, \alpha)$ such that for any $x \in E$, $\|F(t, x)\| \leq \varsigma(t)$ for a.e. $t \in J$.*
3. *There exists a constant $L > 0$ with $\dfrac{4MLb^\alpha}{\Gamma(1+\alpha)} < 1$ such that for any bounded subset D of E, we have $\chi(F(t, D)) \leq L\chi(D)$ for a.e. $t \in J$.*

(Hg) *$g : PC(J, E) \to E$ is continuous, compact and there exists a constant $N > 0$ such that $\|g(x)\| \leq N$ for all $x \in PC(J, E)$.*
(HI) *$I_i : E \to E$ for every $i = 1, 2 \cdots, m$ is continuous, compact and there exists a nondecreasing function $h_i : [0, +\infty) \to [0, +\infty)$ such that $\|I_i(x)\| \leq h_i\|x\|$, $x \in E$.*
(Hζ) *There exists a function $\zeta \in C(J, \mathbb{R}^+)$ such that for each constant $\varpi \in (-1, 0)$, $t \in J$ we have*

$$MN + M\sum_{i=1}^{m} h_i(\zeta(t_i)) + \frac{\alpha M t^{(1+\varpi)(1-q)}}{\Gamma(1+\alpha)(1+\varpi)^{(1-q)}} \|\varsigma\|_{L^{\frac{1}{q}}_{([0,t],\mathbb{R}^+)}} \leq \zeta(t).$$

Then there exists at least one impulsive mild solution for the problem (Q).

Proof. From (HF)(1), Lemma 8 and Remark 3 the set

$$S^1_{F(\cdot,x(\cdot))} = \{f \in L^1(J, E) : f(t) \in F(t, x(t)) \text{ a.e.}\},$$

is nonempty, for any $x \in PC(J,E)$. Therefore, we can define a multifunction $G : PC(J,E) \to 2^{PC(J,E)}$, as follows: $y \in G(x)$ if and only if

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) ds, \ t \in J_i, 1 \le i \le m, \end{cases} \tag{1}$$

where $f \in S^1_{F(\cdot,x(\cdot))}$. Clearly, any fixed point for G is a mild solution for the problem (Q). So, we will prove that G satisfies all the conditions of Theorem 1. We will give the proof in six steps.

Step 1. We will prove that the values of G are convex subsets in $PC(J,E)$. Let $x \in PC(J,E)$, $y_1, y_2 \in G(x)$ and $\lambda \in (0,1)$. Let $t \in J_0$, from the definition of G we have

$$\lambda y_1(t) + (1-\lambda) y_2(t) = \mathcal{T}_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[\lambda f_1(s) + (1-\lambda) f_2(s)] ds,$$

where $f_1, f_2 \in S^1_{F(.,x(.))}$. Easily, one can see that $S^1_{F(.,x(.))}$ is convex because F has convex values. Then, $[\lambda f_1 + (1-\lambda) f_2] \in S^1_{F(.,x(.))}$. Thus, $\lambda y_1(t) + (1-\lambda) y_2(t) \in G(x), t \in J_0$. Similarly, we can prove that $\lambda y_1(t) + (1-\lambda) y_2(t) \in G(x)$ for $t \in J_i, i = 1, 2, \ldots, m$. Which means that $G(x)$ is convex for each $x \in PC(J,E)$.

Step 2. We will show that $G(x)$ is closed for every $x \in PC(J,E)$.
Let $x \in PC(J,E)$ and $\{z_n\}_{n=1}^\infty$ be a sequence in $G(x)$ such that $z_n \to z$ as $n \to \infty$. We need to prove that $z \in G(x)$. From the definition of G, there exists a sequence $\{f_n\}_{n=1}^\infty \subset S^1_{F(.,x(.))}$ such that

$$z_n(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) ds, \ t \in J_i, 1 \le i \le m. \end{cases} \tag{2}$$

By (HF)(2) we have for every $n \ge 1$ and *a.e.* $t \in J$, $\|f_n(t)\| \le \varsigma(t)$. So, $\{f_n : n \ge 1\}$ is integrable bounded. Also, $\{f_n : n \ge 1\}$ is relatively compact in E for *a.e.* $t \in J$ since $\{f_n(t) : n \ge 1\} \subset F(t, x(t))$. Then, the set $\{f_n : n \ge 1\}$ semicompact. By Lemma 9, it is weakly compact in $L^1(J,E)$. We can suppose that the sequence $(f_n)_{n\ge1}$ converges weakly to a function

$f \in L^1(J,E)$. By Mazur's Lemma there is a sequence $\{v_n\}_{n=1}^{\infty} \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that v_n converges strongly to f. Because the values of F are convex and compact, the set $S^1_{F(\cdot,x(\cdot))}$ is convex and compact. Therefore, $\{v_n\}_{n=1}^{\infty} \subseteq S^1_{F(\cdot,x(\cdot))}$ and $f \in S^1_{F(\cdot,x(\cdot))}$. Also, by using Holder inequality it can be shown that for all $t \in J, s \in (0,t]$ and every $n \geq 1$,

$$\|(t-s)^{\alpha-1}S_\alpha(t-s)f_n(s)\| \leq |t-s|^{\alpha-1}\frac{M\alpha}{\Gamma(\alpha+1)}\varsigma(s) \in L^1(J,\mathbb{R}^+).$$

Therefore, by the Lebesgue dominated convergence theorem, taking $n \to \infty$ on both sides of (2), we get

$$z(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \displaystyle\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \displaystyle\sum_{k=1}^{k=i}\mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ \qquad + \displaystyle\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds, \ t \in J_i, 1 \leq i \leq m, \end{cases} \tag{3}$$

which means that $z(t) \in G(x)$.

Step 3. Let us set $B_0 = \{x \in PC(J,E) : \|x(t)\| \leq \zeta(t), t \in J\}$, since $\zeta \in C(J,\mathbb{R}^+)$, then B_0 is a bounded subset of $PC(J,E)$. Moreover, B_0 is closed and convex. We need to prove that $G(B_0) \subset B_0$. In fact, for fixed $y \in G(B_0)$, let $x \in B_0$ such that $y \in G(x)$. Then, by using Lemma 2, (HF)(2), (Hg), $(H\zeta)$ and Holder's inequality for $t \in J_0$, we have

$$\begin{aligned} \|y(t)\| &\leq \|\mathcal{T}_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds\| \leq \|\mathcal{T}_\alpha(t)g(x)\| \\ &\quad + \|\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds\| \\ &\leq MN + \frac{\alpha M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\varsigma(s)ds \\ &\leq MN + \frac{\alpha M}{\Gamma(1+\alpha)}\frac{t^{(1+\omega)(1-q)}}{(1+\omega)^{(1-q)}}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \leq \zeta(t), \end{aligned}$$

where $\omega = \dfrac{\alpha-1}{1-q} \in (-1,0)$. Similarly, by using (HI) in addition and for $t \in J_i$, $i = 1,\ldots,m$ we get

$$\|y(t)\| \leq MN + M\sum_{k=1}^{k=i}h_k(\zeta(t_k)) + \frac{\alpha M}{\Gamma(1+\alpha)}\frac{t^{(1+\omega)(1-q)}}{(1+\omega)^{(1-q)}}\|\varsigma\|_{L^{\frac{1}{q}}(J,\mathbb{R}^+)} \leq \zeta(t).$$

Which follows that $y \in B_0$. Then, $G(B_0) \subset B_0$.
Step 4. We want to show that $G(B_0)_{|_{\overline{J_i}}}$ is equicontinuous for every $i = 0, 1, \ldots, m$, where

$$\begin{aligned} G(B_0)_{|_{\overline{J_i}}} &= \{y^* \in C(\overline{J_i}, E) : y^*(t) = y(t), t \in J_i = (t_i, t_{i+1}], \\ &\qquad y^*(t_i) = y(t_i^+), y \in G(B_0)\}. \end{aligned}$$

Let $y \in G(B_0)$. Then there exists $x \in B_0$ with $y \in G(x)$. From (1), there is $f \in S^1_{F(.,x(.))}$ such that

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ \qquad + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, t \in J_i, \ 1 \le i \le m. \end{cases}$$

We consider the following cases:
Case 1. When $i = 0$, let $t, t+\tau \in J_0 = [0, t_1]$. Then

$$\begin{aligned} \|y^*(t+\tau) - y^*(t)\| &= \|y(t+\tau) - y(t)\| \\ &\le \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(t)\| \\ &\quad + \|\int_0^{t+\tau} (t+\tau-s)^{\alpha-1} S_\alpha(t+\tau-s) f(s)ds \\ &\quad - \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds\| \\ &\le G_1 + G_2 + G_3 + G_4, \end{aligned}$$

where

$$\begin{aligned} G_1 &= \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(t)\|, \\ G_2 &= \|\int_0^t [(t+\tau-s)^{\alpha-1} - (t-s)^{\alpha-1}] S_\alpha(t+\tau-s) f(s)ds\|, \\ G_3 &= \|\int_0^t (t-s)^{\alpha-1} [S_\alpha(t+\tau-s) - S_\alpha(t-s)] f(s)ds\|, \\ G_4 &= \|\int_t^{t+\tau} (t+\tau-s)^{\alpha-1} S_\alpha(t+\tau-s) f(s)ds\|. \end{aligned}$$

We will show that $G_i \to 0$ as $\tau \to o$ for $i = 1, 2, 3, 4$. By (HA) and Lemma 2 we have

$$\lim_{\tau\to 0} G_1 = \lim_{\tau\to 0} \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(t)\| \le N \lim_{\tau\to 0} \|\mathcal{T}_\alpha(t+\tau) - \mathcal{T}_\alpha(t)\| = 0,$$

not dependent on x. For G_2 and G_4, one can see the proof in details in Theorem 4 of [19]. For G_3, from the equicontinuity of $\{S_\alpha(t) : t \in J\}$, we can get

$$G_3 \leq \int_0^t \|(t-s)^{\alpha-1}[S_\alpha(t+\tau-s) - S_\alpha(t-s)]f(s)\|ds \to 0, \text{ as } \tau \to 0.$$

Therefore,

$$\lim_{\tau\to 0} \|y^*(t+\tau) - y^*(t)\| = 0. \tag{4}$$

Case 2. When $t \in J_i, i \in \{1, 2, \cdots, m\}$. Let $t, t+\tau$ two points in J_i, according the definition of G we have

$$\begin{aligned}\|y^*(t+\tau) - y^*(t)\| &= \|y(t+\tau) - y(t)\| \leq \|\mathcal{T}_\alpha(t+\tau)g(x) - \mathcal{T}_\alpha(t)g(x)\| \\ &+ \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t+\tau-t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))\| \\ &+ \|\int_0^{t+\tau}(t+\tau-s)^{\alpha-1}S_\alpha(t+\tau-s)f(s)ds \\ &- \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)ds\|.\end{aligned}$$

Arguing as in the Case 1, we obtain

$$\lim_{\tau\to 0} \|y^*(t+\tau) - y^*(t)\| = 0. \tag{5}$$

Case 3. When $t = t_i,\ i = 1, 2, \cdots, m$. Let $\tau > 0$ and $\delta > 0$ such that $t_i + \tau \in J_i$ and $t_i < \delta < t_i + \tau \leq t_{i+1}$, then we have

$$\|y^*(t_i+\tau) - y^*(t_i)\| = \lim_{\delta\to t_i^+} \|y(t_i+\tau) - y(\delta)\|.$$

From the definition of G, we obtain

$$\begin{aligned}\|y(t_i+\tau) - y(\delta)\| &\leq \|\mathcal{T}_\alpha(t_i+\tau)g(x) - \mathcal{T}_\alpha(\delta)g(x)\| \\ &+ \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t_i+\tau-t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(\delta-t_k)I_k(x(t_k^-))\| \\ &+ \|\int_0^{t_i+\tau}(t_i+\tau-s)^{\alpha-1}S_\alpha(t_i+\tau-s)f(s)ds \\ &- \int_0^\delta (\delta-s)^{\alpha-1}S_\alpha(\delta-s)f(s)ds\|.\end{aligned}$$

With similar argument as in Case 1, we have

$$\lim_{\substack{\tau\to 0\\ \delta\to t_i^+}} \|y(t_i+\tau)-y(\delta)\| = 0. \tag{6}$$

From (4), (5) and (6) we conclude that $G(B_0)_{|\overline{J_i}}$ is equicontinuous for every $i = 0,1,\cdots,m$, and thus $G(B_0)$ is equicontinuous on J.

Now, we define a sequence $B_n = \overline{conv}G(B_{n-1})$, $n \geq 1$. From Step 1 and Step 2, we know that B_n is nonempty, closed and convex in $PC(J,E)$. Moreover, $B_1 = \overline{conv}G(B_0) \subset B_0$. By induction, $(B_n)_{n=1}^{\infty}$ is decreasing sequence of closed, bounded, convex and equicontinuous subsets of $PC(J,E)$. Set $B = \cap_{n=1}^{\infty} B_n$. So, B is a closed, bounded, convex and equicontinuous subset of $PC(J,E)$ and $G(B) \subset B$. We want to prove that B is nonempty and compact in $PC(J,E)$. By light of Lemma 3, it is enough to show that $\lim_{n\to\infty} \chi_{PC}(B_n) = 0$, where χ_{PC} is the Housdorff measure of noncompactness on $PC(J,E)$ as defined in Section 2. By Lemma 6, for arbitrary $\varepsilon > 0$ there exist sequence $\{y_k\}_{k=1}^{\infty}$ in $G(B_{n-1})$ such that

$$\chi_{PC}(B_n) = \chi_{PC}G(B_{n-1}) \leq 2\chi_{PC}\{y_k : k \geq 1\} + \varepsilon.$$

From the definition of χ_{PC},

$$\chi_{PC}(B_n) \leq 2 \max_{0\leq i\leq m} \chi_i(v_{|\overline{J_i}}) + \varepsilon,$$

where $v = \{y_k : k \geq 1\}$ and χ_i is the noncompactness on $C(\overline{J_i}, E)$. By using the equicontinuity $B_{n|\overline{J_i}}$, $i = 0,1,\cdots,m$, we can apply Lemma 4 and we get

$$\chi_i(v_{|\overline{J_i}}) = \sup_{t\in\overline{J_i}} \chi(v(t)),$$

where χ is the Hausdorff measure of noncompactenss on E. Therefore, by using the nonsinglarity of χ we get

$$\chi_{PC}(B_n) \leq 2 \max_{i=0,1,\cdots,m} [\sup_{t\in\overline{J_i}} \chi(v(t))] + \varepsilon = 2 \sup_{t\in J} \chi(v(t)) + \varepsilon.$$

Then,

$$\chi_{PC}(B_n) \leq 2 \sup_{t\in J} \chi\{y_k : k \geq 1\} + \varepsilon. \tag{7}$$

Since $y_k \in G(B_{n-1}), k \geq 1$ there is $x_k \in B_{n-1}$ such that $y_k \in G(x_k), k \geq 1$. From the definition of G, there exist $f_k \in S^1_{F(\cdot,x_k(\cdot))}$. So, (7) can be written as

$$\chi_{PC}(B_n) \leq 2\sup_{t\in J}\chi\{y_k : k\geq 1\}$$
$$\leq \begin{cases} \chi(\mathcal{T}_\alpha(t)g(x_k)) + \chi(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f_k(s)ds), \ t\in J_0, \\ \chi(\mathcal{T}_\alpha(t)g(x_k)) + \sum_{r=1}^{r=i}\chi(\mathcal{T}_\alpha(t-t_r)I_r(x_k(t_r^-))) \\ \qquad +\chi(\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f_k(s)ds), t\in J_i, \ 1\leq i\leq m. \end{cases}$$

Since, g and I_i for every $i=1,2,\cdots,m$ are compact, by Lemma 1 we have

$$\chi\{\mathcal{T}_\alpha(t)g(x_k) : k\geq 1\} = 0, \tag{8}$$
$$\chi\{\mathcal{T}_\alpha(t-t_r)I_r(x_k(t_r^-)) : k\geq 1\} = 0. \tag{9}$$

Hence, by (8) and (9) for every $t\in J$ we have

$$\chi_{PC}(B_n) \leq \varepsilon + 2\sup_{t\in J}\chi\{\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f_k(s)ds : k\geq 1\}.$$

By Lemma 1, Lemma 5 and (HF)(3),

$$\begin{aligned}\chi_{PC}(B_n) &\leq 4\int_0^t (t-s)^{\alpha-1}\chi\{S_\alpha(t-s)f_k(s) : k\geq 1\}ds + \varepsilon \\ &\leq \frac{4\alpha M}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\chi(F(s,B_{n-1}(s)))ds + \varepsilon \\ &\leq \frac{4\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\chi(B_{n-1}(s))ds + \varepsilon \\ &\leq \frac{4\alpha ML}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\chi_{PC}(B_{n-1})ds + \varepsilon \\ &\leq \frac{4\alpha ML}{\Gamma(1+\alpha)}\chi_{PC}(B_{n-1})\int_0^t (t-s)^{\alpha-1}ds + \varepsilon \\ &\leq \frac{4MLb^\alpha}{\Gamma(1+\alpha)}\chi_{PC}(B_{n-1}) + \varepsilon.\end{aligned}$$

Since ε is arbitrary, we find

$$\chi_{PC}(B_n) \leq r\chi_{PC}(B_{n-1}),$$

where $r = \dfrac{4ML}{\Gamma(1+\alpha)} < 1$. Clearly, by means of finite number of steps we can write

$$0 \leq \chi_{PC}(B_n) \leq r^{n-1}\chi_{PC}(B_1). \tag{10}$$

Now, if we take the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \chi_{PC}(B_n) = 0.$$

Thus, it follows from Lemma 3 that $B = \cap_{n=1}^{\infty} B_n$ is nonempty and compact.
Step 5. We will prove that the graph of $G_{|B} : B \to 2^B$ is closed. Let $\{x_n\}_{n=1}^{\infty}$ in B with $x_n \to x$ as $n \to \infty$, $y_n \in G(x_n)$ and $y_n \to y$ as $n \to \infty$. We need to show that $y \in G(x)$. Because $y_n \in G(x_n)$, for any $n \geq$ there exists $f_n \in S^1_{F(\cdot,x_n(\cdot))}$ such that

$$y_n(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x_n) + \displaystyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s)ds, t \in J_0, \\ \mathcal{T}_\alpha(t)g(x_n) + \displaystyle\sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x_n(t_k^-)) \\ \qquad + \displaystyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s)ds, t \in J_i. \end{cases} \tag{11}$$

We know that for every $n \geq 1, \|f_n(t)\| \leq \varsigma(t)$ for a.e. $t \in J$. This show that the set $\{f_n : n \geq 1\}$ is integrably bounded. Moreover, (HF)(3) and convergence of $\{x_n\}_{n=1}^{\infty}$ implies that

$$\chi\{f_n : n \geq 1\} \leq \chi(F(t, \{x_n(t) : n \geq 1\}) \leq L\chi\{x_n(t) : n \geq 1\} = 0.$$

This means that the sequence $\{f_n : n \geq 1\}$ is relatively compact in E for a.e. $t \in J$. Therefore, the sequence $\{f_n : n \geq 1\}$ is semicompact and by Lemma 9 it is weakly compact in $L^1(J, E)$. We can assume that f_n converges weakly to a function $f \in L^1(J, E)$. Then by Mazur's Lemma, there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that u_n converges strongly to f. Since F is $u.s.c.$ with convex and compact values, so by Lemma 7 we get

$$\begin{aligned} f(t) \in \cap_{k\geq 1}\overline{\{u_n(t) : n \leq k\}} &\subseteq \cap_{k\geq 1}\overline{conv}\{f_n : n \geq k\} \\ &\subseteq \cap_{k\geq 1}\overline{conv}\{\cup_{n\geq k} F(t, x_n(t))\} \\ &= \overline{conv} \lim_{n\to\infty} \sup F(t, x_n(t)) \subseteq F(t, x(t)). \end{aligned}$$

Then, by continuity of $g, \mathcal{T}_\alpha, S_\alpha, I_i$ $(i = 1, \ldots, m)$ and by the same arguments used in Step 2, we conclude that

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)g(x) + \displaystyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, \ t \in J_0, \\ \mathcal{T}_\alpha(t)g(x) + \displaystyle\sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k) I_k(x(t_k^-)) \\ \qquad + \displaystyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)ds, t \in J_i, \ 1 \leq i \leq m. \end{cases}$$

Hence, $y \in G(x)$. This means that the graph of $G_{|B}$ is closed.
Step 6. We will show that G is $u.s.c.$ on B.
From Steps 1 - 5, we have that B is closed and $G(x)$ is closed for every $x \in B$. Moreover, the set $\overline{G(B)} \subseteq B$ is compact and G is closed. Therefore, by Remark 2, we conclude that G is $u.s.c.$.

At the end, by Theorem 1, G has at least one point x such that $x \in G(x)$ and x is mild solution for the problem (Q). □

4 Example

We study the following impulsive partial differential system with nonlocal conditions:

$$\begin{cases} \partial_t^\alpha y(t,z) \in \partial_z^2 y(t,z) + R(t, y(t,z)), \quad t \in [0,1],\, t \neq t_i, i = 1, \dots, m, z \in [0,1], \\ y(t,0) = y(t,1) = 0, \\ y((\frac{i}{m+1})^+, z) = y(\frac{i}{m+1}, z) + \frac{1}{2^i}, \quad i = 1, \dots, m, z \in [0,1], \\ y(0,z) = \sum_{j=0}^{j=q} \int_0^1 k_j(z,v) \tan^{-1}(y(s_j, v)) dv, \quad z \in [0,1], \end{cases} \tag{12}$$

where q is a positive integer, $0 < s_0 < s_1 < \cdots < s_q < 1$, $k_j \in C([0,1] \times [0,1], \mathbb{R})$, $j = 0, 1, \dots, q, \partial_t^\alpha$ is the Caputo fractional partial derivative of order α, where $0 < \alpha < 1$ and $R : [0,1] \times E \to P(E)$.

In order to rewrite (12) in the abstract form, we put $E = L^2([0,1], \mathbb{R})$, and A is the Laplace operator, i.e., $A = \frac{\partial^2}{\partial z^2}$ on the domain $D(A) = \{x \in E : x, x'$ are absolutely continuous, and $x'' \in E, x(0) = x(1) = 0\}$. From [28], A is the infinitesimal generator of an analytic and compact semigroup $\{T(t)\}_{t \geq 0}$ in E. This implies that A satisfies the assumption (HA).

For every $i = 1, \dots, m$ define $I_i : E \to E$ by

$$I_i(x)(z) = \frac{1}{2^i}, z \in [0,1].$$

Note that the assumption (HI) is satisfied.

For every $j = 0, 1, \dots, q$, define $H_j : E \to E$ as

$$(H_j(x))(z) = \int_0^1 k_j(z,v) \tan^{-1}(x(v)) dv, \quad z \in [0,1].$$

Now take $g : PC([0,1], E) \to E$ as

$$g(x) = \sum_{j=0}^{j=q} H_j(x(s_j)).$$

Finally, let $F(t,x)(z) = R(t,x(z))$ and $x(t)(z) = x(t,z)$, where $z \in [0,1]$. Then, the system (12) can be rewritten as

$$\begin{cases} {}^cD^\alpha x(t) \in Ax(t) + F(t,x(t)), \;\; t \in J = [0,1], t \neq t_i, i = 1,\ldots,m, \\ x(t_i^+) = x(t_i) + I_i(x(t_i^-)), \;\; i = 1,\ldots,m, \\ x(0) = g(x), \end{cases}$$

If we put some conditions on F as in Theorem 2, then (12) has at least one mild solution on $[0,1]$.

5 Conclusion

The present article discussed the existence of PC-mild solutions of nonlocal impulsive differential inclusions in Banach space when the operator semigroup is not necessarily compact. We used methods and results of Hausdorff measure of noncompactness, and multivalued fixed point theorems in order to establish sufficient conditions that guarantee the existence of PC-mild solutions for (Q). The results presented in this paper developed and extended some previous results. An example was presented to support our main results.

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