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A NEW BRANCH OF THE LOGICAL ALGEBRA: UP-ALGEBRAS

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Abstract. In this paper, we introduce a new algebraic structure, called a UP-algebra (UP means the University of Phayao) and a concept of UP-ideals, UP-subalgebras, congruences and UP-homomorphisms in UP-algebras, and investigated some related properties of them. We also describe connections between UP-ideals, UP-subalgebras, congruences and UP-homomorphisms, and show that the notion of UP-algebras is a generalization of KU-algebras.

1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [5], BCI-algebras [6], BCH-algebras [4], KU-algebras [12], SU-algebras [7] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [6] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [5, 6] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

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In 2009, the notion of a KU-algebra was first introduced by Prabpayak and Leerawat [12] as follows:

Definition 1.1. [12] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *KU-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,
(KU-1): $(y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0$,
(KU-2): $0 \cdot x = x$,
(KU-3): $x \cdot 0 = 0$, and
(KU-4): $x \cdot y = y \cdot x = 0$ implies $x = y$.

They gave the concept of homomorphisms of KU-algebras and investigated some related properties.

Lemma 1.2. [11] *In a KU-algebra A, we have*

$$z \cdot (y \cdot x) = y \cdot (z \cdot x) \text{ for all } x, y, z \in A.$$

Several researches were conducted to investigate the characterizations of KU-algebras such as: In 2011, Mostafa, Abdel Naby and Elgendy [10] introduced the notion of intuitionistic fuzzy KU-ideals in KU-algebras and fuzzy intuitionistic image (preimage) of KU-ideals in KU-algebras. They also introduced the Cartesian product of two intuitionistic fuzzy KU-ideals in KU-algebras and investigated some results. In 2011, Mostafa, Abdel Naby and Elgendy [9] introduced the notion of interval-valued fuzzy KU-ideals in KU-algebras and studied some of their properties. In 2011, Mostafa, Abdel Naby and Yousef [11] introduced the notion of fuzzy KU-ideals in KU-algebras and their some properties are investigated. In 2012, Mostafa, Abdel Naby and Yousef [8] introduced the notion of anti-fuzzy KU-ideals in KU-algebras, several appropriate examples are provided and their some properties are investigated. In 2012, Sitharselvam, Priya and Ramachandran [14] introduced the concept of anti Q-fuzzy KU-ideals of KU-algebras, lower level cuts of a fuzzy set and proved that a Q-fuzzy set of a KU-algebra is a KU-ideal if and only if the complement of this Q-fuzzy set is an anti Q-fuzzy KU-ideal. In 2013, Yaqoob, Mostafa and Ansari [15] introduced the notion of cubic KU-ideals of KU-algebras and several results are presented in this regard. The image, preimage, and cartesian product of cubic KU-ideals of KU-algebras are defined. In 2013, Akram, Yaqoob and Gulistan [1] provided some new properties of cubic KU-subalgebras. In 2013, Sithar Selvam, Priya, Nagalakshmi and Ramachandran [13] introduced the concept of anti Q-fuzzy KU-subalgebras of KU-algebras. They discussed few results of KU-ideals of KU-algebras under homomorphisms and anti homomorphisms and some of its properties. In 2014, Gulistan, Shahzad and Ahmed [3] defined (α, β)-fuzzy KU-ideals of KU-algebras and then some useful characterizations have provided. Also, they introduced the concept of (α, β)-fuzzy KU-relations. In 2014, Akram, Yaqoob and Kavikumar [2] introduced the notion of interval-valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras and some related properties are investigated.

In this paper, we introduce a new algebraic structure, called a UP-algebra and a concept of UP-ideals, congruences and UP-homomorphisms in UP-algebras, and investigated some related properties of them. We also describe connections between UP-ideals, congruences and UP-homomorphisms, and present some connections between UP-algebras and KU-algebras.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.3. An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, and
(UP-4): $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.4. Let X be a universal set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A' = A' \cap B = B - A$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*. In fact, for any $A, B, C \in \mathcal{P}(X)$, we have

$$\begin{aligned}(A \cdot B) \cdot (A \cdot C) &= (B \cap A') \cdot (C \cap A') \\ &= (C \cap A') \cap (B \cap A')' \\ &= (C \cap A') \cap (B' \cup A) \\ &= ((C \cap A') \cap B') \cup ((C \cap A') \cap A) \\ &= ((C \cap A') \cap B') \cup \emptyset \\ &= (C \cap A') \cap B'.\end{aligned}$$

Thus

$$\begin{aligned}(B \cdot C) \cdot ((A \cdot B) \cdot (A \cdot C)) &= (B \cdot C) \cdot ((C \cap A') \cap B') \\ &= (C \cap B') \cdot ((C \cap A') \cap B') \\ &= ((C \cap A') \cap B') \cap (C \cap B')' \\ &= A' \cap (C \cap B') \cap (C \cap B')' \\ &= A' \cap \emptyset \\ &= \emptyset,\end{aligned}$$

(UP-1) holding. Also, $\emptyset \cdot A = A \cap \emptyset' = A \cap X = A$ and $A \cdot \emptyset = \emptyset \cap A' = \emptyset$, (UP-2) and (UP-3) are valid. Moreover, if $A \cdot B = B \cdot A = \emptyset$, then $B \cap A' = A \cap B' = \emptyset$. Thus $B \subseteq A$ and $A \subseteq B$ and so $A = B$, (UP-4) holding.

Example 1.5. Let X be a universal set. Define a binary operation $*$ on the power set of X by putting $A * B = B \cup A' = A' \cup B$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*. In fact, for any $A, B, C \in \mathcal{P}(X)$, we have

$$\begin{aligned}(A * B) * (A * C) &= (B \cup A') * (C \cup A') \\ &= (C \cup A') \cup (B \cup A')' \\ &= (C \cup A') \cup (B' \cap A) \\ &= ((C \cup A') \cup B') \cap ((C \cup A') \cup A) \\ &= ((C \cup A') \cup B') \cap X \\ &= (C \cup A') \cup B'.\end{aligned}$$

Thus

$$\begin{aligned}(B * C) * ((A * B) * (A * C)) &= (B * C) * ((C \cup A') \cup B') \\ &= (C \cup B') * ((C \cup A') \cup B') \\ &= ((C \cup A') \cup B') \cup (C \cup B')' \\ &= A' \cup (C \cup B') \cup (C \cup B')' \\ &= A' \cup X \\ &= X,\end{aligned}$$

(UP-1) holding. Also, $X * A = A \cup X' = A \cup \emptyset = A$ and $A * X = X \cup A' = X$, (UP-2) and (UP-3) are valid. Moreover, if $A * B = B * A = X$, then $B \cup A' = A \cup B' = X$. Thus $B \subseteq A \cup B'$ and $A \subseteq B \cup A'$ and so $B \subseteq A$ and $A \subseteq B$. Hence, $A = B$, (UP-4) holding.

We can easily show the following example.

Example 1.6. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$$\begin{array}{c|cccc} \cdot & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 3 \\ 3 & 0 & 1 & 2 & 0 \end{array} \tag{1.1}$$

Then $(A; \cdot, 0)$ is a UP-algebra.

The following proposition is very important for the study of UP-algebras.

Proposition 1.7. *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,
(3) $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and
(7) $x \cdot (y \cdot y) = 0$.

Proof. (1) By the definition of a UP-algebra, we have

$$\begin{aligned} 0 &= (0 \cdot x) \cdot ((0 \cdot 0) \cdot (0 \cdot x)) && \text{(By (UP-1))} \\ &= (0 \cdot x) \cdot (0 \cdot x) && \text{(By (UP-2))} \\ &= x \cdot x. && \text{(By (UP-2))} \end{aligned}$$

Hence, $x \cdot x = 0$.
(2) Assume that $x \cdot y = 0$ and $y \cdot z = 0$. Then

$$\begin{aligned} x \cdot z &= 0 \cdot (0 \cdot (x \cdot z)) && \text{(By (UP-2))} \\ &= (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) && \text{(By substituting)} \\ &= 0. && \text{(By (UP-1))} \end{aligned}$$

Hence, $x \cdot z = 0$.
(3) Assume that $x \cdot y = 0$. Then

$$\begin{aligned} (z \cdot x) \cdot (z \cdot y) &= 0 \cdot ((z \cdot x) \cdot (z \cdot y)) && \text{(By (UP-2))} \\ &= (x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) && \text{(By substituting)} \\ &= 0. && \text{(By (UP-1))} \end{aligned}$$

Hence, $(z \cdot x) \cdot (z \cdot y) = 0$.

(4) Assume that $x \cdot y = 0$. Then

$$\begin{aligned}(y \cdot z) \cdot (x \cdot z) &= (y \cdot z) \cdot (0 \cdot (x \cdot z)) && \text{(By (UP-2))}\\ &= (y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) && \text{(By substituting)}\\ &= 0. && \text{(By (UP-1))}\end{aligned}$$

Hence, $(y \cdot z) \cdot (x \cdot z) = 0$.
(5) By (UP-1), (UP-2) and (UP-3), we have $x \cdot (y \cdot x) = (0 \cdot x) \cdot ((y \cdot 0) \cdot (y \cdot x)) = 0$
(6) If $(y \cdot x) \cdot x = 0$, then by (5), $x \cdot (y \cdot x) = 0$. By (UP-4), $x = y \cdot x$. By (1), we have the converse.
(7) By (UP-3) and (1), we have $x \cdot (y \cdot y) = x \cdot 0 = 0$. □

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.2}$$

Proposition 1.8 obviously follows from Proposition 1.7.

Proposition 1.8. *In a UP-algebra A, the following properties hold: for any $x, y, z \in A$,*

(1) $x \leq x$,
(2) $x \leq y$ and $y \leq x$ imply $x = y$,
(3) $x \leq y$ and $y \leq z$ imply $x \leq z$,
(4) $x \leq y$ implies $z \cdot x \leq z \cdot y$,
(5) $x \leq y$ implies $y \cdot z \leq x \cdot z$,
(6) $x \leq y \cdot x$, and
(7) $x \leq y \cdot y$.

From Proposition 1.8 and (UP-3), we have Proposition 1.9.

Proposition 1.9. *Let A be a UP-algebra with a binary relation $\leq$ defined by (1.2). Then $(A, \leq)$ is a partially ordered set with 0 as the greatest element.*

We often call the partial ordering $\leq$ defined by (1.2) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

This means that a UP-algebra can be considered as a partially ordered set with some additional properties.

Proposition 1.10. *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ with a binary relation $\leq$ defined by (1.2) is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,*

(1) $(y \cdot z) \leq (x \cdot y) \cdot (x \cdot z)$,

(2) $0 \cdot x = x$,
(3) $x \leq 0$, *and*
(4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

The following theorem is an important result of KU-algebras for study in the connections between UP-algebras and KU-algebras.

Theorem 1.11. *Any KU-algebra is a UP-algebra.*

Proof. It only needs to show (UP-1). By Lemma 1.2, we have that any KU-algebra satisfies (UP-1). □

Example 1.12. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	0	0
2	0	2	0	0	0
3	0	2	2	0	0
4	0	2	2	4	0

(1.3)

By routine calculations it can be seen that $(A; \cdot, 0)$ is a UP-algebra. Since $(0 \cdot 3) \cdot ((3 \cdot 1) \cdot (0 \cdot 1)) = 3 \cdot (2 \cdot 1) = 3 \cdot 2 = 2$, we have that (KU-1) is not satisfied. Hence, $(A; \cdot, 0)$ is not a KU-algebra.

We give an example showing that the notion of UP-algebras is a generalization of KU-algebras.

Theorem 1.13. *An algebra* $A = (A; \cdot, 0)$ *of type* $(2, 0)$ *is a KU-algebra if and only if it satisfies the following conditions: for all* $x, y, z \in A$,

(1) (KU-1): $(y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0$,
(2) $y \cdot ((y \cdot x) \cdot x) = 0$,
(3) $x \cdot x = 0$,
(4) (KU-3): $x \cdot 0 = 0$, *and*
(5) (KU-4): $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

Proof. Necessity: It suffices to prove (2) and (3). By (KU-1) and (KU-2), we have

$$y \cdot ((y \cdot x) \cdot x) = (0 \cdot y) \cdot ((y \cdot x) \cdot (0 \cdot x) = 0$$

and

$$x \cdot x = 0 \cdot (x \cdot x) = (0 \cdot 0) \cdot ((0 \cdot x) \cdot (0 \cdot x) = 0,$$

(2) and (3) holding.

Sufficiency: It only needs to show (KU-2). Replacing y by 0 in (2), we get

$$0 \cdot ((0 \cdot x) \cdot x) = 0. \tag{1.4}$$

Substituting $0 \cdot x$ for y and x for z in (1), it follows

$$((0 \cdot x) \cdot x) \cdot ((x \cdot x) \cdot ((0 \cdot x) \cdot x)) = 0.$$

By (3), we have

$$((0 \cdot x) \cdot x) \cdot (0 \cdot ((0 \cdot x) \cdot x)) = 0. \tag{1.5}$$

An application of (1.4) to (1.5) gives

$$((0 \cdot x) \cdot x) \cdot 0 = 0. \tag{1.6}$$

Comparing (1.4) with (1.6) and using (5), we obtain

$$(0 \cdot x) \cdot x = 0. \tag{1.7}$$

Also, by (2) and (3), the following holds:

$$x \cdot (0 \cdot x) = x \cdot ((x \cdot x) \cdot x) = 0. \tag{1.8}$$

Now, combining (1.7) with (1.8) and using (5) once again, it yields $0 \cdot x = 0$, showing (KU-2). Hence, $A = (A; \cdot, 0)$ is a KU-algebra. □

Theorem 1.14. *In a UP-algebra A, the following statements are equivalent:*

(1) A is a KU-algebra,
(2) $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $x, y, z \in A$, and
(3) $x \cdot (y \cdot z) = 0$ implies $y \cdot (x \cdot z) = 0$ for all $x, y, z \in A$.

Proof. (1)⇒(2) By Theorem 1.13 (2), we get $x \leq (x \cdot z) \cdot z$, then by Proposition 1.8 (5) implies

$$((x \cdot z) \cdot z) \cdot (y \cdot z) \leq x \cdot (y \cdot z).$$

Substituting $x \cdot z$ for x in (KU-1), we have $(y \cdot (x \cdot z)) \cdot (((x \cdot z) \cdot z) \cdot (y \cdot z)) = 0$. Thus

$$y \cdot (x \cdot z) \leq ((x \cdot z) \cdot z) \cdot (y \cdot z).$$

The transitivity of $\leq$ gives

$$y \cdot (x \cdot z) \leq x \cdot (y \cdot z) \text{ for all } x, y, z \in A. \tag{1.9}$$

Replacing y by x and x by y in (1.9), we obtain

$$x \cdot (y \cdot z) \leq y \cdot (x \cdot z). \tag{1.10}$$

Hence, the anti-symmetry of $\leq$ implies that $x \cdot (y \cdot z) = y \cdot (x \cdot z)$.
(2)⇒(3) Assume that $x \cdot (y \cdot z) = 0$ where $x, y, z \in A$. By (2), we have $y \cdot (x \cdot z) = 0$.

(3)⇒(1) It only needs to show (KU-1). By (UP-1), we get $(y\cdot z)\cdot((x\cdot y)\cdot(x\cdot z)) = 0$ for all $x, y, z \in A$. By (3), we have $(x\cdot y)\cdot((y\cdot z)\cdot(x\cdot z)) = 0$, showing (KU-1). □

Theorem 1.15. *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,*

(1) (UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(2) $(y \cdot 0) \cdot x = x$, *and*
(3) (UP-4): $x \cdot y = y \cdot x = 0$ *implies* $x = y$.

Proof. Necessity: It suffices to prove (2). By (UP-2) and (UP-3), we have

$$(y \cdot 0) \cdot x = 0 \cdot x = x,$$

(2) holding.

Sufficiency: It suffices to show (UP-2) and (UP-3). Replacing y and z by 0 in (1) and using (2), we get

$$0 = (0 \cdot 0) \cdot ((x \cdot 0) \cdot (x \cdot 0)) = (0 \cdot 0) \cdot (x \cdot 0) = x \cdot 0, \tag{1.11}$$

(UP-3) holding. Combining (1.11) with (2), we obtain

$$0 \cdot x = (x \cdot 0) \cdot x = x,$$

showing (UP-2). Hence, $A = (A; \cdot, 0)$ is a UP-algebra. □

2. UP-Ideals and UP-Subalgebras

Definition 2.1. Let A be a UP-algebra. A subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

We can easily show the following example.

Example 2.2. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

·	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

(2.1)

Then $(A; \cdot, 0)$ is a UP-algebra and $\{0, 1, 2\}$ and $\{0, 1, 3\}$ are UP-ideals of A.

Theorem 2.3. *Let A be a UP-algebra and B a UP-ideal of A. Then the following statements hold: for any $x, a, b \in A$,*

(1) if $b \cdot x \in B$ and $b \in B$, then $x \in B$. Moreover, if $b \cdot X \subseteq B$ and $b \in B$, then $X \subseteq B$,
(2) if $b \in B$, then $x \cdot b \in B$. Moreover, if $b \in B$, then $X \cdot b \subseteq B$, and
(3) if $a, b \in B$, then $(b \cdot (a \cdot x)) \cdot x \in B$.

Proof. (1) Let $x, b \in A$ be such that $b \cdot x \in B$ and $b \in B$. By (UP-2), we get $0 \cdot (b \cdot x) = b \cdot x \in B$ and $b \in B$. Since B is a UP-ideal of A and (UP-2), we have $x = 0 \cdot x \in B$. If $b \cdot X \subseteq B$ and $b \in B$, then $b \cdot x \in B$ for all $x \in X$. From the previous result, $x \in B$ for all $x \in X$. Thus $X \subseteq B$.
(2) Let $x \in A$ and $b \in B$. By (UP-3) and using Proposition 1.7 (1), we have $x \cdot (b \cdot b) = x \cdot 0 = 0 \in B$. Since B is a UP-ideal of A and $b \in B$, we have $x \cdot b \in B$. If $b \in B$, then from the previous result, $x \cdot b \in B$ for all $x \in X$. Thus $X \cdot b \subseteq B$.
(3) Let $x \in A$ and $a, b \in B$. By Proposition 1.7 (1), we have $(a \cdot x) \cdot (a \cdot x) = 0 \in B$. Since B is a UP-ideal of A and $a \in B$, we have $(a \cdot x) \cdot x \in B$. By (UP-1), we have

$$((a \cdot x) \cdot x) \cdot ((b \cdot (a \cdot x)) \cdot (b \cdot x)) = 0 \in B.$$

It follows from (1) that $(b \cdot (a \cdot x)) \cdot (b \cdot x) \in B$. Since $b \in B$, it follows from the definition of a UP-ideal that $(b \cdot (a \cdot x)) \cdot x \in B$. □

Corollary 2.4. *Let A be a UP-algebra and B a UP-ideal of A. Then for any $x \in A$ and $b \in B$, $b \leq x$ implies $x \in B$.*

Proof. If $b \leq x$, then $b \cdot x = 0 \in B$. Since $b \in B$, it follows from Theorem 2.3 (1) that $x \in B$. □

Corollary 2.5. *Let A be a UP-algebra and B a UP-ideal of A. Then for any $x \in A$ and $a, b \in B$, $b \leq a \cdot x$ implies $x \in B$.*

Proof. If $b \leq a \cdot x$, then $b \cdot (a \cdot x) = 0 \in B$. Since $b \in B$, it follows from Theorem 2.3 (1) that $a \cdot x \in B$. Using Theorem 2.3 (1) again, $x \in B$. □

Theorem 2.6. *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Proof. Clearly, $0 \in B_i$ for all $i \in I$. Thus $0 \in \bigcap_{i \in I} B_i$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in \bigcap_{i \in I} B_i$ and $y \in \bigcap_{i \in I} B_i$. Then $x \cdot (y \cdot z) \in B_i$ and $y \in B_i$ for all $i \in I$. Since B_i is a UP-ideal of A, we have $x \cdot z \in B_i$ for all $i \in I$. Thus $x \cdot z \in \bigcap_{i \in I} B_i$. Hence, $\bigcap_{i \in I} B_i$ is a UP-ideal of A. □

From Theorem 2.6, the intersection of all UP-ideals of a UP-algebra A containing a subset X of A is the UP-ideal of A generated by X. For $X = \{a\}$, let $I(a)$ denote the UP-ideal of A generated by $\{a\}$. We see that the UP-ideal of A generated by $\emptyset$ and $\{0\}$ is $\{0\}$, and the UP-ideal of A generated by A is A.

Applying Theorem 2.3 and Proposition 1.8 (6), we can then easily prove the following Proposition.

Proposition 2.7. *Let A be a UP-algebra and B a UP-ideal of A. Then the following statements hold: for any $x, y \in A$,*

(1) if $x \in B$ and $x \leq y$, then $y \in B$,
(2) if $x \leq y$, then $I(y) \subseteq I(x)$,
(3) $I(y \cdot x) \subseteq I(x)$, and
(4) if $y \in I(y \cdot x)$, then $I(y \cdot x) = I(x)$.

Definition 2.8. Let $A = (A; \cdot, 0)$ be a UP-algebra. A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.7 (1), we can then easily prove the following Proposition.

Proposition 2.9. *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 2.10. *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A.*

Proof. Since $0 \in B_i$ for all $i \in I$, we have $0 \in \bigcap_{i \in I} B_i \neq \emptyset$. Let $x, y \in \bigcap_{i \in I} B_i$. Then $x, y \in B_i$ for all $i \in I$, it follows from Proposition 2.9 that $x \cdot y \in \bigcap_{i \in I} B_i$. Using Proposition 2.9 once again, $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A. □

Theorem 2.11. *Let A be a UP-algebra and B a UP-ideal of A. Then $A \cdot B \subseteq B$. In particular, B is a UP-subalgebra of A.*

Proof. Let $x \in A \cdot B$. Then $x = a \cdot b$ for some $a \in A$ and $b \in B$. By (UP-3) and Proposition 1.7 (1), we have $a \cdot (b \cdot b) = a \cdot 0 = 0 \in B$. Since B is a UP-ideal of A and $b \in B$, we have $x = a \cdot b \in B$. Hence, $A \cdot B \subseteq B$. Since $B \cdot B \subseteq A \cdot B \subseteq B$, we get B is a UP-subalgebra of A. □

Example 2.12. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$$\begin{array}{c|cccc} \cdot & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 2 & 2 \\ 2 & 0 & 1 & 0 & 2 \\ 3 & 0 & 1 & 0 & 0 \end{array} \tag{2.2}$$

Then $(A; \cdot, 0)$ is a UP-algebra. Let $S = \{0, 2\}$. Then S is a UP-subalgebra of A. Since $0 \cdot (2 \cdot 3) = 2 \in S$ and $2 \in S$, but $0 \cdot 3 = 3 \notin S$, we have S is not a UP-ideal of A.

By Theorem 2.11 and Example 2.12, we have that the notion of UP-subalgebras is a generalization of UP-ideals.

Theorem 2.13. *Let A be a UP-algebra and let B be a UP-subalgebra of A satisfying the property of the Theorem 1.14 (2), i.e., $x \cdot (y \cdot z) = y \cdot (x \cdot z)$ for all $x, y, z \in B$. If S is a subset of B that is satisfies the following properties:*

(1) the constant 0 of A is in S, and
(2) for any $x, b \in B$, if $b \cdot x \in S$ and $b \in S$, then $x \in S$.

Then S is a UP-ideal of B.

Proof. Let $x, y, z \in B$ be such that $x \cdot (y \cdot z) \in S$ and $y \in S$. Since $y \in S \subseteq B$ and B satisfies the property of the Theorem 1.14 (2), we get $y \cdot (x \cdot z) = x \cdot (y \cdot z) \in S$. Using (2), we obtain $x \cdot z \in S$. Hence, S is a UP-ideal of B. □

Theorem 2.14. *Let A be a UP-algebra and B a UP-subalgebra of A. If S is a subset of B that is satisfies the following properties:*

(1) the constant 0 of A is in S, and
(2) for any $x, a, b \in B$, if $a, b \in S$, then $(b \cdot (a \cdot x)) \cdot x \in S$.

Then S is a UP-ideal of B.

Proof. Let $x, y, z \in B$ be such that $x \cdot (y \cdot z) \in S$ and $y \in S$. Replacing b by 0, a by y and x by z in (2) and using (UP-2), we get $(y \cdot z) \cdot z = (0 \cdot (y \cdot z)) \cdot z \in S$. It follows from (UP-1), (UP-2), and (2) that

$$x \cdot z = 0 \cdot (x \cdot z) = (((y \cdot z) \cdot z) \cdot ((x \cdot (y \cdot z)) \cdot (x \cdot z)) \cdot (x \cdot z) \in S.$$

Hence, S is a UP-ideal of B. □

3. Congruences

Definition 3.1. Let A be a UP-algebra and B a UP-ideal of A. Define the binary relation $\sim_B$ on A as follows: for all $x, y \in A$,

$$x \sim_B y \text{ if and only if } x \cdot y \in B \text{ and } y \cdot x \in B. \tag{3.1}$$

We can easily show the following example.

Example 3.2. From Example 2.2, let $B = \{0, 1, 3\}$ be an UP-ideal of A. Then

$$\sim_B = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,1),(1,0),(0,3),(3,0),(1,3),(3,1)\}.$$

We can see that $\sim_B$ is an equivalence relation on A.

Definition 3.3. Let A be a UP-algebra. An equivalence relation ρ on A is called a *congruence* if for any $x, y, z \in A$,

$$x\rho y \text{ implies } x \cdot z\rho y \cdot z \text{ and } z \cdot x\rho z \cdot y.$$

Lemma 3.4. *Let A be a UP-algebra. An equivalence relation ρ on A is a congruence if and only if for any $x, y, u, v \in A$, $x\rho y$ and $u\rho v$ imply $x \cdot u\rho y \cdot v$.*

Proof. Assume that ρ is a congruence on A and let $x, y, u, v \in A$ be such that $x\rho y$ and $u\rho v$. Then $x \cdot u\rho y \cdot u$ and $y \cdot u\rho y \cdot v$. The transitivity of ρ gives $x \cdot u\rho y \cdot v$.

Conversely, let $x, y, z \in A$ be such that $x\rho y$. Since $z\rho z$, it follows from assumption that $x \cdot z\rho y \cdot z$ and $z \cdot x\rho z \cdot y$. Hence, ρ is a congruence on A. □

Proposition 3.5. *Let A be a UP-algebra and B a UP-ideal of A with a binary relation $\sim_B$ defined by (3.1). Then $\sim_B$ is a congruence on A.*

Proof. Reflexive: For all $x \in A$, it follows from Proposition 1.7 (1) that $x \cdot x = 0$. Since B is a UP-ideal of A, we have $x \cdot x = 0 \in B$. Thus $x \sim_B x$.
Symmetric: Let $x, y \in A$ be such that $x \sim_B y$. Then $x \cdot y \in B$ and $y \cdot x \in B$, so $y \cdot x \in B$ and $x \cdot y \in B$. Thus $y \sim_B x$.
Transitive: Let x, y, z be such that $x \sim_B y$ and $y \sim_B z$. Then $x \cdot y, y \cdot x, y \cdot z, z \cdot y \in B$. Since B is a UP-ideal of A and (UP-1), we get $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0 \in B$. Since $y \cdot z \in B$, it follows from Theorem 2.3 that $(x \cdot y) \cdot (x \cdot z) \in B$. Since $x \cdot y \in B$, it follows from Theorem 2.3 again that $x \cdot z \in B$. Similarly, since B is a UP-ideal of A and (UP-1), we get $(y \cdot x) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0 \in B$. Since $y \cdot x \in B$, it follows from Theorem 2.3 that $(z \cdot y) \cdot (z \cdot x) \in B$. Since $z \cdot y \in B$, it follows from Theorem 2.3 again that $z \cdot x \in B$. Thus $x \sim_B z$.

Therefore, $\sim_B$ is an equivalence relation on A. Finally, let $x, y, u, v \in A$ be such that $x \sim_B u$ and $y \sim_B v$. Then $x \cdot u, u \cdot x, y \cdot v, v \cdot y \in B$. Since B is a UP-ideal of A and (UP-1), we get $(v \cdot y) \cdot ((x \cdot v) \cdot (x \cdot y)) = 0 \in B$. Since $v \cdot y \in B$, it follows from Theorem 2.3 that $(x \cdot v) \cdot (x \cdot y) \in B$. Similarly, since B is a UP-ideal of A and (UP-1), we get $(y \cdot v) \cdot ((x \cdot y) \cdot (x \cdot v)) = 0 \in B$. Since $y \cdot v \in B$, it follows from Theorem 2.3 again that $(x \cdot y) \cdot (x \cdot v) \in B$. Thus $x \cdot y \sim_B x \cdot v$. On the other hand, since B is a UP-ideal of A and (UP-1), we get $(u \cdot v) \cdot ((x \cdot u) \cdot (x \cdot v)) = 0 \in B$. Since B is a UP-ideal of A and $x \cdot u \in B$, we have $(u \cdot v) \cdot (x \cdot v) \in B$. Similarly, since B is a UP-ideal of A and (UP-1), we get $(x \cdot v) \cdot ((u \cdot x) \cdot (u \cdot v)) = 0 \in B$. Since B is a UP-ideal of A and $u \cdot x \in B$, we have $(x \cdot v) \cdot (u \cdot v) \in B$. Thus $x \cdot v \sim_B u \cdot v$. The transitivity of $\sim_B$ gives $x \cdot y \sim_B u \cdot v$. Hence, $\sim_B$ is a congruence on A. □

Let A be a UP-algebra and ρ a congruence on A. If $x \in A$, then the ρ-class of x is the $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in A \mid y\rho x\}.$$

Then the set of all ρ-classes is called the *quotient set of A by ρ*, and is denoted by A/ρ. That is,

$$A/\rho = \{(x)_\rho \mid x \in A\}.$$

Theorem 3.6. *Let A be a UP-algebra and ρ a congruence on A. Then the following statements hold:*

(1) the ρ-class $(0)_\rho$ is a UP-ideal and a UP-subalgebra of A,
(2) a ρ-class $(x)_\rho$ is a UP-ideal of A if and only if $x\rho 0$, and
(3) a ρ-class $(x)_\rho$ is a UP-subalgebra of A if and only if $x\rho 0$.

Proof. (1) Since $0\rho 0$, $0 \in (0)_\rho$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in (0)_\rho$ and $y \in (0)_\rho$. Then $y\rho 0$ and

$$x \cdot (y \cdot z)\rho 0. \tag{3.2}$$

Since $x\rho x$ and $z\rho z$, it follows from Lemma 3.4 that $x \cdot (y \cdot z)\rho x \cdot (0 \cdot z)$. By (UP-2), we get $x \cdot (y \cdot z)\rho x \cdot z$ and so

$$x \cdot z\rho x \cdot (y \cdot z). \tag{3.3}$$

The transitivity of ρ gives $x \cdot z\rho 0$, so $x \cdot z \in (0)_\rho$. Hence, $(0)_\rho$ is a UP-ideal of A. Now, let $x, y \in (0)_\rho$. Then $x\rho 0$ and $y\rho 0$. By Lemma 3.4 and (UP-2), we have $x \cdot y\rho 0$. Thus $x \cdot y \in (0)_\rho$. Hence, $(0)_\rho$ is a UP-subalgebra of A.
(2) Assume that $(x)_\rho$ is a UP-ideal of A. Then $0 \in (x)_\rho$. Hence, the symmetry of ρ gives $x\rho 0$.

Converse, let $x\rho 0$. Then $(x)_\rho = (0)_\rho$. It follows from (1) that $(x)_\rho$ is a UP-ideal of A.

(3) Assume that $(x)_\rho$ is a UP-subalgebra of A. Since $x \in (x)_\rho$ and Proposition 1.7 (1), we have $0 = x \cdot x \in (x)_\rho$. Hence, the symmetry of ρ gives $x\rho 0$.

Converse, let $x\rho 0$. Then $(x)_\rho = (0)_\rho$. It follows from (1) that $(x)_\rho$ is a UP-subalgebra of A. □

Theorem 3.7. *Let A be a UP-algebra and B a UP-ideal of A. Then the following statements hold:*

(1) the $\sim_B$-class $(0)_{\sim_B}$ is a UP-ideal and a UP-subalgebra of A contained in B,
(2) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-ideal of A if and only if $x \in B$,
(3) a $\sim_B$-class $(x)_{\sim_B}$ is a UP-subalgebra of A if and only if $x \in B$, and
(4) $(A/\sim_B; *, (0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in A$, called the quotient UP-algebra of A induced by the congruence $\sim_B$.

Proof. (1) From Proposition 3.5 and Theorem 3.6 (1), we have $(0)_{\sim_B}$ is a UP-ideal and a UP-subalgebra of A. Now, let $x \in (0)_{\sim_B}$. Then $x \sim_B 0$, it follows from (UP-2) that $x = 0 \cdot x \in B$. Hence, $(0)_{\sim_B} \subseteq B$.
(2) It now follows directly from Proposition 3.5, Theorem 3.6 (2) and (UP-2).
(3) It now follows directly from Proposition 3.5, Theorem 3.6 (3) and (UP-2).
(4) Let $x, y, u, v \in A$ be such that $(x)_{\sim_B} = (y)_{\sim_B}$ and $(u)_{\sim_B} = (v)_{\sim_B}$. Since $\sim_B$ is an equivalence relation on A, we get $x \sim_B y$ and $u \sim_B v$. By Lemma 3.4, we have $x \cdot u \sim_B y \cdot v$. Hence, $(x)_{\sim_B} * (u)_{\sim_B} = (x \cdot u)_{\sim_B} = (y \cdot v)_{\sim_B} = (y)_{\sim_B} * (v)_{\sim_B}$, showing $*$ is well defined.
(UP-1): Let $x, y, z \in A$. By (UP-1), we have $((y)_{\sim_B} * (z)_{\sim_B}) * (((x)_{\sim_B} * (y)_{\sim_B}) * ((x)_{\sim_B} * (z)_{\sim_B})) = ((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)))_{\sim_B} = (0)_{\sim_B}$.
(UP-2): Let $x \in A$. By (UP-2), we have $(0)_{\sim_B} * (x)_{\sim_B} = (0 \cdot x)_{\sim_B} = (x)_{\sim_B}$.
(UP-3): Let $x \in A$. By (UP-3), we have $(x)_{\sim_B} * (0)_{\sim_B} = (x \cdot 0)_{\sim_B} = (0)_{\sim_B}$.
(UP-4): Let $x, y \in A$ be such that $(x)_{\sim_B} * (y)_{\sim_B} = (y)_{\sim_B} * (x)_{\sim_B} = (0)_{\sim_B}$. Then $(x \cdot y)_{\sim_B} = (y \cdot x)_{\sim_B} = (0)_{\sim_B}$, it follows from (1) that $x \cdot y, y \cdot x \in (0)_{\sim_B} \subseteq B$. Hence, $x \sim_B y$ and so $(x)_{\sim_B} = (y)_{\sim_B}$.
Hence, $(A/\sim_B; *, (0)_{\sim_B})$ is a UP-algebra. □

4. UP-Homomorphisms

Definition 4.1. Let $(A;\cdot,0)$ and $(A';\cdot',0')$ be UP-algebras. A mapping f from A to A' is called a *UP-homomorphism* if

$$f(x\cdot y)=f(x)\cdot' f(y) \text{ for all } x,y\in A.$$

A UP-homomorphism $f\colon A\to A'$ is called a

(1) *UP-epimorphism* if f is surjective,
(2) *UP-monomorphism* if f is injective,
(3) *UP-isomorphism* if f is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A\cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x)\mid x\in B\}$ is called the *image* of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the *image* of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x\in A\mid f(x)\in B'\}$ is said the *inverse image* of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the *kernel* of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f)=\{f(x)\in A'\mid x\in A\}$$

and

$$\mathrm{Ker}(f)=\{x\in A\mid f(x)=0'\}.$$

By using Microsoft Excel, we have the following example.

Example 4.2. Let $A=\{0,1,2,3,4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

(4.1)

and let $A'=\{0',a,b,c,d\}$ be a set with a binary operation $\cdot'$ defined by the following Cayley table:

$\cdot'$	$0'$	a	b	c	d
$0'$	$0'$	a	b	c	d
a	$0'$	$0'$	$0'$	$0'$	$0'$
b	$0'$	a	$0'$	c	$0'$
c	$0'$	a	$0'$	$0'$	$0'$
d	$0'$	a	b	c	$0'$

(4.2)

Then $(A;\cdot,0)$ and $(A';\cdot',0')$ are UP-algebras. We define a mapping $f\colon A\to A'$ as follows:

$$f(0)=0', f(1)=0', f(2)=0', f(3)=d, \text{ and } f(4)=a.$$

Then f is a UP-homomorphism with $\mathrm{Im}(f)=\{0',a,d\}$ and $\mathrm{Ker}(f)=\{0,1,2\}$.

In fact it is easy to show the following theorem.

Theorem 4.3. *Let A,B and C be UP-algebras. Then the following statements hold:*

(1) the identity mapping $I_A\colon A\to A$ is a UP-isomorphism,
(2) if $f\colon A\to B$ is a UP-isomorphism, then $f^{-1}\colon B\to A$ is a UP-isomorphism, and
(3) if $f\colon A\to B$ and $g\colon B\to C$ are UP-isomorphisms, then $g\circ f\colon A\to C$ is a UP-isomorphism.

Theorem 4.4. *Let A be a UP-algebra and B a UP-ideal of A. Then the mapping $\pi_B\colon A\to A/\sim_B$ defined by $\pi_B(x)=(x)_{\sim_B}$ for all $x\in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_B$.*

Proof. Let $x,y\in A$ be such that $x=y$. Then $(x)_{\sim_B}=(y)_{\sim_B}$, so $\pi_B(x)=\pi_B(y)$. Thus π_B is well defined. Note that by the definition of π_B, we have π_B is surjective. Let $x,y\in A$. Then

$$\pi_B(x\cdot y)=(x\cdot y)_{\sim_B}=(x)_{\sim_B}*(y)_{\sim_B}=\pi_B(x)*\pi_B(y).$$

Thus π_B is a UP-homomorphism. Hence, π_B is a UP-epimorphism. □

Theorem 4.5. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-homomorphism. Then the following statements hold:*

(1) $f(0_A)=0_B$,
(2) for any $x,y\in A$, if $x\leq y$, then $f(x)\leq f(y)$,
(3) if C is a UP-subalgebra of A, then the image $f(C)$ is a UP-subalgebra of B. In particular, $\mathrm{Im}(f)$ *is a UP-subalgebra of B,*
(4) if D is a UP-subalgebra of B, then the inverse image $f^{-1}(D)$ is a UP-subalgebra of A. In particular, $\mathrm{Ker}(f)$ *is a UP-subalgebra of A,*
(5) if C is a UP-ideal of A, then the image $f(C)$ is a UP-ideal of $f(A)$,
(6) if D is a UP-ideal of B, then the inverse image $f^{-1}(D)$ is a UP-ideal of A. In particular, $\mathrm{Ker}(f)$ *is a UP-ideal of A, and*
(7) $\mathrm{Ker}(f)=\{0_A\}$ *if and only if f is injective.*

Proof. (1) By Proposition 1.7 (1), we have

$$f(0_A) = f(0_A \cdot 0_A) = f(0_A) * f(0_A) = 0_B.$$

(2) If $x \leq y$, then $x \cdot y = 0_A$. By (1), we have $f(x) * f(y) = f(x \cdot y) = f(0_A) = 0_B$. Hence, $f(x) \leq f(y)$.
(3) Assume that C is a UP-subalgebra of A. Since $0_A \in C$, we have $f(0_A) \in f(C) \neq \emptyset$. Let $a, b \in f(C)$. Then $f(x) = a$ and $f(y) = b$ for some $x, y \in C$. Since C is closed under the $\cdot$ multiplication on A, we get $a * b = f(x) * f(y) = f(x \cdot y) \in f(C)$. By Proposition 2.9, we get $f(C)$ is a UP-subalgebra of B. In particular, since A is a UP-subalgebra of A, we obtain $\mathrm{Im}(f) = f(A)$ is a UP-subalgebra of B.
(4) Assume that D is a UP-subalgebra of B. Since $0_B \in D$, it follows from (1) that $0_A \in f^{-1}(D) \neq \emptyset$. Let $x, y \in f^{-1}(D)$. Then $f(x), f(y) \in D$. Since D is closed under the $*$ multiplication on B, we get $f(x \cdot y) = f(x) * f(y) \in D$. Thus $x \cdot y \in f^{-1}(D)$, it follows from Proposition 2.9 that $f^{-1}(D)$ is a UP-subalgebra of A. In particular, since $\{0_B\}$ is a UP-subalgebra of B, we obtain $\mathrm{Ker}(f) = f^{-1}(\{0_B\})$ is a UP-subalgebra of A.
(5) Assume that C is a UP-ideal of A. Since $0_A \in C$ and (1), we have $0_B = f(0_A) \in f(C)$. Let $a, b, c \in f(A)$ be such that $a * (b * c) \in f(C)$ and $b \in f(C)$. Then $f(u) = a * (b * c)$ and $f(y) = b$ for some $u, y \in C$, and $f(x) = a$ and $f(z) = c$ for some $x, z \in A$. By Proposition 1.7 (1), we have

$$0_B = (a*(b*c))*(a*(b*c)) = f(u)*(f(x)*(f(y)*f(z))) = f(u\cdot(x\cdot(y\cdot z))).$$

Put $v = (u \cdot (x \cdot (y \cdot z))) \cdot y$. Since $y \in C$, it follows from Theorem 2.3 (2) that $v \in C$. Thus $f(v) \in f(C)$. By (UP-2), we have

$$b = 0_B * b = f(u \cdot (x \cdot (y \cdot z))) * f(y) = f((u \cdot (x \cdot (y \cdot z))) \cdot y) = f(v).$$

Therefore, $b = f(v) \in f(C)$, proving $f(C)$ is a UP-ideal of $f(A)$.
(6) Assume that D is a UP-ideal of B. Since $0_B \in D$ and (1), we have $f(0_A) = 0_B \in D$. Thus $0_A \in f^{-1}(D)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in f^{-1}(D)$ and $y \in f^{-1}(D)$. Then $f(x \cdot (y \cdot z)) \in D$ and $f(y) \in D$. Since f is a UP-homomorphism, we have

$$f(x) * (f(y) * f(z)) = f(x \cdot (y \cdot z)) \in D.$$

Since D is a UP-ideal of B and $f(y) \in D$, we have $f(x \cdot z) = f(x) * f(z) \in D$. Thus $x \cdot z \in f^{-1}(D)$. Hence, $f^{-1}(D)$ is a UP-ideal of A. In particular, since $\{0_B\}$ is a UP-ideal of B, we obtain $\mathrm{Ker}(f) = f^{-1}(\{0_B\})$ is a UP-ideal of A.

(7) Assume that $\mathrm{Ker}(f) = \{0_A\}$. Let $x, y \in A$ be such that $f(x) = f(y)$. By Proposition 1.7 (1), we have

$$f(x \cdot y) = f(x) * f(y) = f(y) * f(y) = 0_B$$

and

$$f(y \cdot x) = f(y) * f(x) = f(y) * f(y) = 0_B.$$

Thus $x \cdot y, y \cdot x \in \mathrm{Ker}(f) = \{0_A\}$, so $x \cdot y = y \cdot x = 0_A$. By (UP-4), we have $x = y$. Hence, f is injective.

Conversely, assume that f is injective. By (1), we obtain $\{0_A\} \subseteq \mathrm{Ker}(f)$. Let $x \in \mathrm{Ker}(f)$. Then $f(x) = 0_B = f(0_A)$, so $x = 0_A$ because f is injective. Hence, $\mathrm{Ker}(f) = \{0_A\}$. □

5. Conclusions

In the present paper, we have introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals, UP-subalgebras, congruences and UP-homomorphisms in UP-algebras and investigated some of its essential properties. We present some connections between UP-algebras and KU-algebras and show that the notion of UP-algebras is a generalization of KU-algebras. We think this work would enhance the scope for further study in a new concept of UP-algebras and related algebraic systems. It is our hope that this work would serve as a foundation for the further study in a new concept of UP-algebras.

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