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Numerical method for singularly perturbed fourth order ordinary differential equations of convection-diffusion type

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Abstract. In this paper, we have proposed a numerical method for singularly perturbed fourth order ordinary differential equations of convection-diffusion type. The numerical method combines boundary value technique, asymptotic expansion approximation, shooting method and finite difference method. In order to get a numerical solution for the derivative of the solution, the given interval is divided into two subintervals called inner region (boundary layer region) and outer region. The shooting method is applied to inner region whereas for the outer region, standard finite difference method is applied. Necessary error estimates are derived. Computational efficiency and accuracy are verified through numerical examples.

Keywords: singularly perturbed problems, fourth order ordinary differential equations, boundary value technique, asymptotic expansion approximation, shooting method, finite difference scheme, parallel computation.
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1 Introduction

Singular Perturbation Problems (SPPs) appear in many branches of applied mathematics, and for more than three decades quite a good number

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of research works on the qualitative and quantitative analysis of these problems for both ODEs and Partial Differential Equations (PDEs) have been reported in the literature. Such problems have been investigated by many researchers. In recent years, a variety of numerical methods are available in the literature to solve Singularly Perturbed Boundary Value Problems (SPBVPs) for second order ODEs, but for higher order equations only few results are reported in the literature.

Analytical treatment of SPBVPs for higher order non-linear ODEs which have important applications in fluid dynamics is available in [3], [7], [11], [13], [15]- [17], [29]. Niederdrenk and Yserentant [11] have considered convection-diffusion type problems and derived conditions for the uniform stability of the discrete and continuous problems. Gartland [3] has shown that the uniform stability of the discrete BVP follows from the uniform stability of the associated discrete IVP and the uniform consistency of the scheme. In [15], an iterative method is described.

In [15, 22], a FEM for convection and reaction type problems is described. Feckan [7] has considered higher order problems and his works are based on the non-linear analysis involving the fixed point theory, Leray-Schauder theory, etc. Howes [4] has considered the higher order problems and discussed the existence, uniqueness and asymptotic estimates of the solution. Weili [29], has considered a more general class of third order non-linear SPBVPs and discussed the existence, uniqueness of the solution and obtained asymptotic estimates using the theory of the differential inequalities. In fact Weili [30] has derived results on third order non-linear SPPs using differential inequality theorems. Robert's [14] and Valarmathi [24–27] have suggested methods of finding approximate solutions for third order SPBVPs.

As far as author's knowledge goes, only few results are reported in the literature in the case of fourth order differential equations. Sember [17], Roos [16] and O'Malley [13] have considered fourth order equations and applied a standard FEM. In [19] authors reported a numerical method known as Boundary Value Technique (BVT) and in [18, 20, 21] authors described an asymptotic numerical method for solving fourth order Singularly Perturbed Ordinary Differential Equations (SPODEs) of reaction diffusion and convection diffusion types.

Following the Boundary Value Technique (BVT) of Roberts [14], Vigo-Aguiar [28], Valarmathi [24], Santhi [19] and using the basic idea underlying the method suggested in Jayakumar [5] and Natesan [9] we in the present paper, suggest a new computational method which makes use of the zero order asymptotic expansion approximation, BVT and shooting method to obtain a numerical solution for the derivative of SPBVPs for fourth order ODEs of convection-diffusion type of the form:

$$\varepsilon y^{iv}(x) + a(x)y'''(x) - b(x)y''(x) + c(x)y(x) = f(x), \quad x \in \Omega, \tag{1}$$

$$y(0) = p, \quad y'''(0) = q, \quad y(1) = r, \quad -y''(1) = s \tag{2}$$

where $0 < \varepsilon \ll 1,\ a(x),\ b(x),\ c(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha,\ \alpha > 0, \tag{3}$$

$$b(x) \geq 0, \tag{4}$$

$$0 \geq c(x) \geq -\gamma,\ \gamma > 0, \tag{5}$$

$$\alpha > 3\gamma, \tag{6}$$

with $\Omega = (0,1)$, $\bar{\Omega} = [0,1]$ and $y \in C^{(4)}(\Omega) \cap C^{(3)}(\bar{\Omega})$. The boundary conditions (2) of the problem (1) are one of the types of boundary conditions discussed in [7].

In order to get a numerical solution for the derivative of the solution of SPBVP (1)-(2), we divide the interval $[0,1]$ into two subintervals $[0,\tau]$ and $[\tau,1]$. An inner region problem defined in the interval $[0,\tau]$ is solved by shooting method and BVP corresponding to the outer region is solved based on the standard finite difference scheme. It is quite natural to take τ as the width of the boundary layer which can be obtained or estimated [6]. The problems defined in the intervals $[0,\tau]$ and $[\tau,1]$ are independent of each other. Therefore, these problems can be solved simultaneously, that is, more suitable for parallel architectures.

This method is easy to implement, and further, we could give a full-fledged theory (consistency, stability, convergence and error estimates) for the same. In Section 2 some analytical results for the SPBVPs(1)-(2) are presented. Section 3 deals with derivative estimates of the solution. In Section 4 some analytical and numerical results are derived for auxiliary second order SPBVPs of convection diffusion type and description of the numerical method is also given. The error estimates for the method are discussed in detail in Section 5. Section 6 deals with non-linear problems. Numerical examples are presented in Section 7. Conclusions are drawn in the last section.

Through out this paper, we use C, with or without subscript to denote a generic positive constant, which is independent of N and ε. We use h_1 for mesh size for the inner region problem and h_2 for mesh size for the outer region problem. Error estimates are derived. Numerical examples are presented to illustrate the method. We define $||.||$ of $\bar{w} = (w_1, w_2)^T \in \mathbb{R}^2$ as $||\bar{w}|| = \max\{|w_1|, |w_2|\}$.

2 Preliminaries

The SPBVPs (1)-(2) can be transformed into an equivalent weakly coupled system of the form:

$$\begin{cases} P_1\bar{y}(x) \equiv -y_1''(x) - y_2(x) = 0, \;\; x \in \Omega, \\ P_2\bar{y}(x) \equiv -\varepsilon y_2''(x) - a(x)y_2'(x) + b(x)y_2(x) \\ \qquad\qquad +c(x)y_1(x) = f(x), \;\; x \in \Omega, \end{cases} \tag{7}$$

$$y_1(0) = p, \quad -y_2'(0) = q, \quad y_1(1) = r, \quad y_2(1) = s, \tag{8}$$

where $\bar{y} = (y_1, y_2)^T$, the functions $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the conditions (3)-(6).

This transformation makes it possible to establish the maximum principle theorems and stability results for the continuous problem.

Remark 1. The solution of the problem (7)-(8) exhibits a boundary layer at $x = 0$ which is less severe because the boundary conditions are prescribed [15] for the derivative of the solution. The condition (3) says that the problem (7)-(8) is a non turning point problem. The condition (5) is known as the quasi monotonicity condition [15]. The maximum principle for the above system (7)-(8) and for the corresponding discrete problem are established using the conditions (3)-(6).

2.1 Analytical results

This section presents the maximum principle for the problem (7)-(8). Using this principle, a stability result is derived. Further, an asymptotic expansion approximation is constructed for the solution and a theorem is presented to establish its accuracy.

Theorem 1. (Maximum Principle). *Consider the BVPs (7)-(8). Let* $y_1(0) \geq 0, \;\; y_2'(0) \leq 0, \;\; y_1(1) \geq 0$ *and* $y_2(1) \geq 0, \; P_1\bar{y}(x) \geq 0, \; P_2\bar{y}(x) \geq 0, \; \forall x \in \Omega$. *Then,* $\bar{y}(x) \geq 0, \; \forall x \in \bar{\Omega}$.

Proof. Define the test functions $\bar{s}(x) = (s_1(x), s_2(x))^T$ as

$$s_1(x) = 2(1+\eta)(1 - x^2/2), \;\; s_2(x) = 2 - x, \quad 0 < \eta \ll 1/2, \quad x \in \bar{\Omega}.$$

Clearly, $s_1(0) = 2(1+\eta) > 0, \;\; s_2'(0) = -1 < 0, \;\; s_1(1) = 1+\eta > 0, \; s_2(1) = 1 > 0$. We can easily prove that $P_1\bar{s} > 0$ and $P_2\bar{s} > 0$, for $x \in \Omega$.

Assume that the theorem is not true. We define

$$\xi = \max\left\{ \max_{x\in\bar{\Omega}} \left(-\frac{y_1}{s_1} \right)(x), \max_{x\in\bar{\Omega}} \left(-\frac{y_2}{s_2} \right)(x) \right\}.$$

Then $\xi > 0$. Also $(y_1+\xi s_1)(x) \geq 0$ and $(y_2+\xi s_2)(x) \geq 0$ for $x \in \bar{\Omega}$. Furthermore, there exists a point $x_0 \in \bar{\Omega}$ such that

$$(y_1 + \xi s_1)(x_0) = 0 \quad \text{for} \quad x_0 \in \Omega \quad \text{or} \quad (y_2 + \xi s_2)(x_0) = 0 \quad \text{for} \quad x_0 \in \Omega.$$

Case 1: $(y_1 + \xi s_1)(x_0) = 0$ for $x_0 \in \Omega$.
This implies that $y_1 + \xi s_1$ attains its minimum at $x = x_0$. Therefore,

$$0 < P_1(\bar{y} + \xi \bar{s})(x_0) = -(y_1 + \xi s_1)''(x_0) - (y_2 + \xi s_2)(x_0) \leq 0,$$

which is a contradiction.
Case 2: $(y_2 + \xi s_2)(x_0) = 0$ for $x_0 \in \Omega$.
This implies that $y_2 + \xi s_2$ attains its minimum at $x = x_0$. Therefore,

$$\begin{aligned} 0 < P_2(\bar{y} + \xi \bar{s})(x_0) &= -\varepsilon(y_2 + \xi s_2)''(x_0) - a(x)(y_2 + \xi s_2)'(x_0) \\ &\quad + b(x)(y_2 + \xi s_2)(x_0) + c(x)(y_1 + \xi s_1)(x_0) \leq 0, \end{aligned}$$

which is a contradiction. Hence it can be concluded that $\bar{y}(x) \geq 0, \forall x \in \bar{\Omega}$. □

Lemma 1. (Stability Result). *If $\bar{y}(x)$ is the solution of the BVPs (7)-(8) then*

$$||\bar{y}(x)|| \leq C \max\{|y_1(0)|, |y_2'(0)|, |y_1(1)|, |y_2(1)|, \max_{x\in\Omega} |P_1\bar{y}(x)|, \max_{x\in\Omega} |P_2\bar{y}(x)|\}, \quad \forall x \in \bar{\Omega}$$

Proof. Set

$$M = C \max\{|y_1(0)|, |y_2'(0)|, |y_1(1)|, |y_2(1)|, \max_{x\in\Omega} |P_1\bar{y}(x)|, \max_{x\in\Omega} |P_2\bar{y}(x)|\}.$$

Define two barrier functions $\bar{w}^{\pm}(x) = (w_1^{\pm}(x), w_2^{\pm}(x))^T$ as

$$w_1^{\pm}(x) = M\{2(1+\eta)(1 - \frac{x^2}{2})\} \pm y_1(x) \text{ and } w_2^{\pm}(x) = M(2-x) \pm y_2(x),$$

we have

$$P_1\bar{w}^{\pm}(x) = -w_1^{\pm\prime\prime}(x) - w_2^{\pm}(x) > M\eta \pm P_1\bar{y}(x) \geq 0,$$

and

$$\begin{aligned} P_2\bar{w}^{\pm}(x) &= -\varepsilon w_2^{\pm\prime\prime}(x) - a(x)w_2^{\pm\prime}(x) + b(x)w_2^{\pm}(x) + c(x)w_1^{\pm}(x) \\ &> M(\alpha - 3\gamma) \pm P_2\bar{y}(x) \geq 0, \end{aligned}$$

by a proper choice of C. Furthermore, we have

$$w_1^{\pm}(0) = 2M(1+\eta) \pm y_1(0) \geq 0, \quad w_2^{\pm\prime}(0) = -M \pm y_2'(0) \leq 0,$$
$$w_1^{\pm}(1) = M(1+\eta) \pm y_1(1) \geq 0 \quad \text{and} \quad w_2^{\pm}(1) = M \pm y_2(1) \geq 0,$$

by a proper choice of C. Applying Theorem 1 to the barrier functions $\bar{w}^{\pm}(x)$, we get the desired bound for $\bar{y}(x)$. □

2.2 Asymptotic expansion approximation

We look for an asymptotic expansion solution of the BVPs (7)-(8) in the form $\bar{y}_{as}(x,\varepsilon) = \bar{u}_0(x) + \bar{v}_0(x) + O(\varepsilon)$. By the method of stretching variable [8], one can obtain the zero order asymptotic approximation as $\bar{y}_{as} = \bar{u}_0(x)+\bar{v}_0(x)$, where $\bar{u}_0(x) = (u_{0_1}(x), u_{0_2}(x))^T$ is the solution of the reduced problem of the BVPs (7)-(8) given by

$$\begin{cases} -u''_{0_1}(x) - u_{0_2}(x) = 0, \\ -a(x)u'_{0_2}(x) + b(x)u_{0_2}(x) + c(x)u_{0_1}(x) = f(x), \\ u_{0_1}(0) = p, \quad u_{0_1}(1) = r, \quad u_{0_2}(1) = s \end{cases} \tag{9}$$

and $\bar{v}_0(x) = (v_{0_1}(x), v_{0_2}(x))^T$ is a layer correction term with

$$\begin{cases} v_{0_1}(x) = (\varepsilon/a(0))^3(q + u'_{0_2}(0)) \exp[-(a(0)/\varepsilon)(x)], \\ v_{0_2}(x) = (\varepsilon/a(0))(q + u'_{0_2}(0)) \exp[-(a(0)/\varepsilon)(x)], \end{cases} \tag{10}$$

where $\bar{v}_0(x)$ satisfies

$$\begin{cases} -v''_{0_1}(x) - v_{0_2}(x) = 0, \\ -\varepsilon v''_{0_2}(x) - a(0)v'_{0_2}(x) = 0, \\ v_{0_1}(0) = -(\varepsilon/a(0))^3 v'_{0_2}(0), \quad v_{0_1}(1) = (\varepsilon/a(0))^2 v_{0_2}(1), \\ B_0 v_{0_2}(0) \equiv -v'_{0_2}(0) = (q + u'_{0_2}(0)), \\ v_{0_2}(1) = -(\varepsilon/a(0))v'_{0_2}(0) \exp[-(a(0)/\varepsilon)]. \end{cases} \tag{11}$$

The following theorem gives the error bound for the difference between the solution of the BVPs (7)-(8) and its zero order asymptotic expansion approximation.

Theorem 2. *The zero order asymptotic expansion approximation $\bar{y}_{as} = \bar{u}_0(x) + \bar{v}_0(x)$ of the solution $\bar{y}(x)$ of the BVPs (7)-(8) defined by (9)-(11) satisfies the inequality*

$$||\bar{y}(x) - \bar{y}_{as}(x)|| \le C\varepsilon, \quad \forall x \in \bar{\Omega}.$$

Proof. It is easy to prove that

$$\begin{aligned} &|(y_1 - y_{1as})(0)| = |v_{0_1}(0)| \le C\varepsilon^3, \quad |(y_2 - y_{2as})'(0)| = 0, \\ &|(y_1 - y_{1as})(1)| = |v_{0_1}(1)| \le C\varepsilon^3, \quad |y_2(1) - y_{2as}(1)| = |v_{0_2}(1)| \le C\varepsilon e^{-\alpha/\varepsilon}. \end{aligned}$$

Applying the differential operator on $\bar{y}(x) - \bar{y}_{as}(x)$ and using the fact that $te^{-t} \le e^{-t/2}$, $\forall\, t \ge 0$, we have,

$$|P_1(\bar{y} - \bar{y}_{as})(x)| = 0 \quad \text{and} \quad |P_2(\bar{y} - \bar{y}_{as})(x)| \le C\varepsilon + Ce^{-\alpha x/2\varepsilon}.$$

Define the barrier functions $\bar{\phi}^{\pm}(x) = (\phi_1^{\pm}(x), \phi_2^{\pm}(x))^T$ for $x \in \bar{\Omega}$ by

$$\phi_1^{\pm}(x) = C_1[2(1-\frac{x^2}{2})]\varepsilon + C_2\varepsilon^2[1-(1/2)e^{-\alpha x/2\varepsilon}] \pm (y_1 - y_{1as})(x),\ 0 < \eta \ll 1/2$$

and

$$\phi_2^{\pm}(x) = C_1(2-x)\varepsilon + C_2\varepsilon^2 e^{-\alpha x/2\varepsilon} \pm (y_2 - y_{2as})(x),$$

where C_1 and C_2 are positive constants to be chosen suitably, so that the following expressions are satisfied:

$$\phi_1^{\pm}(0) \geq 0, \quad {\phi_2}^{\pm\prime}(0) < 0, \quad \phi_1^{\pm}(1) \geq 0, \quad \phi_2^{\pm}(1) \geq 0,$$

$$P_1\bar{\phi}^{\pm}(x) = -\phi_1^{\pm\prime\prime}(x) - \phi_2^{\pm}(x) > 0$$

and

$$P_2\bar{\phi}^{\pm}(x) = -\varepsilon\phi_2^{\pm\prime\prime}(x) - a(x)\phi_2^{\pm\prime}(x) + b(x)\phi_2^{\pm}(x) + c(x)\phi_1^{\pm}(x) \geq 0 \text{ for } x \in \Omega.$$

Then, applying Theorem 1 to the functions $\bar{\phi}^{\pm}(x)$, it follows that $\bar{\phi}^{\pm}(x) \geq 0$, $\forall x \in \Omega$, and consequently $||\bar{y}(x) - \underline{a}ry_{as}(x)|| \leq C\varepsilon$, $\forall x \in \bar{\Omega}$. □

Corollary 1. *If $y_1(x)$ is the solution of the BVP (7)-(8) and $u_{0_1}(x)$ is the solution of the problem (9) then $|y_1(x) - u_{0_1}(x)| \leq C\varepsilon$, $\forall x \in \bar{\Omega}$.*

Proof. From the above theorem, $|y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| \leq C_1\varepsilon$. Therefore,

$$\begin{aligned} |y_1(x) - u_{0_1}(x)| &= |y_1(x) - u_{0_1}(x) + v_{0_1}(x) - v_{0_1}(x)| \\ &\leq |y_1(x) - (u_{0_1}(x) + v_{0_1}(x))| + |v_{0_1}(x)| \\ &\leq C_1\varepsilon + C_2\varepsilon^2 \leq C\varepsilon, \end{aligned}$$

which completes the proof. □

3 Estimates for the derivatives

Theorem 3. *Let $\bar{y}(x)$ be the solution of the BVP (7)-(8). Then $y_2(x)$ satisfies*

$$|y_2^{(k)}(x)| \leq C(1 + \varepsilon^{-(k-1)}\exp(-\alpha x/\varepsilon)), \tag{12}$$

for $0 \leq k \leq 4$, and $x \in \bar{\Omega}$.

Proof. Consider the BVP

$$\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) - c(x)y_1(x) = -f(x),$$
$$y_2'(0) = -q, \quad y_2(1) = s.$$

Rewrite this BVP as

$$\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) = -f(x) + c(x)y_1(x),$$
$$y_2'(0) = -q, \quad y_2(1) = s.$$

Then, $y_1 \in C^{(3)}(\bar{\Omega})$ and using the procedure adopted in [10] we have $|y_2^{(k)}(x)| \leq C(1 + \varepsilon^{-(k-1)} \exp(-\alpha x/\varepsilon))$, as required. □

4 Some analytical and numerical results for second order SPBVPs

We present some results for the following SPBVPs which are needed for the error analysis of the numerical method given in this section. Consider the auxiliary second order SPBVPs

$$Ly_2^\star \equiv -\varepsilon y_2^{\star''}(x) - a(x)y_2^{\star'}(x) + b(x)y_2^\star(x) = f(x) - c(x)u_{0_1}(x), \; x \in \Omega, \quad (13)$$
$$B_0 y_2^\star(0) \equiv -y_2^{\star'}(0) = q, \; B_1 y_2^\star(1) \equiv y_2^\star(1) = s, \quad (14)$$

where $u_{0_1}(x)$ is defined as in (9), $a(x)$, $b(x)$ and $f(x)$, are sufficiently smooth and $a(x) \geq \alpha$, $\alpha > 0$, $b(x) \geq 0$ and $0 \geq c(x) \geq -\gamma$, $\gamma > 0$.

4.1 Analytical results

Theorem 4. (Maximum Principle). *Consider the SPBVPs (13)-(14). Let $y_2^\star(x)$ be a smooth function satisfying $B_0 y_2^\star(0) \leq 0$, $B_1 y_2^\star(1) \geq 0$ and $Ly_2^\star(x) \geq 0$ for $x \in \Omega$. Then, $y_2^\star(x) \geq 0$, $\forall x \in \bar{\Omega}$.*

Proof. See [1]. □

Lemma 2. *If $y_2^\star(x)$ is the solution of the SPBVPs (13)-(14) then*

$$|y_2^\star(x)| \leq C \max\{|B_0 y_2^\star(0)|, \; |B_1 y_2^\star(1)|, \; \max_{x\in\Omega} |Ly_2^\star(x)|\}, \; \forall x \in \bar{\Omega}.$$

Proof. Define the barrier functions $\Psi^\pm(x)$ as $\Psi^\pm(x) = A'(2-x) \pm y_2^\star(x)$, $x \in \bar{\Omega}$, where $A' = C\max\{|B_0 y_2^\star(0)|, \; |B_1 y_2^\star(1)|, \; \max_{x\in\Omega} |Ly_2^\star(x)|\}$. It is easy to check that $B_0\Psi^\pm(0) \leq 0$, $B_1\Psi^\pm(1) \geq 0$ and $L\Psi^\pm(x) \geq 0$ for a proper choice of the constant C. Applying Theorem 4 to $\Psi^\pm(x)$, the required stability bound for $y_2^\star(x)$ is obtained. □

Theorem 5. *If $\bar{y}(x)$ and $y_2^\star(x)$ are solutions of the BVPs (7)-(8) and (13)-(14) respectively, then* $|y_2(x) - y_2^\star(x)| \leq C\varepsilon, \quad \forall x \in \bar{\Omega}$.

Proof. The second component y_2 of the solution $\bar{y}(x)$ of (7)-(8) satisfies the BVP

$$-\varepsilon {y_2}''(x) - a(x){y_2}'(x) + b(x)y_2(x) = f(x) - c(x)y_1(x), \quad x \in \Omega,$$

$$-y_2'(0) = q, \quad y_2(1) = s.$$

Further, the function $w(x) = y_2(x) - y_2^\star(x)$ satisfies the BVP

$$-\varepsilon w''(x) - a(x)w'(x) + b(x)w(x) = -c(x)[y_1(x) - u_{0_1}(x)], \quad x \in \Omega,$$

$$w'(0) = 0, \quad w(1) = 0.$$

From the stability result given in [1], we have $|w(x)| \leq C|y_1(x) - u_{0_1}(x)|$. From Theorem 2, we further have $|y_1(x) - y_{1as}(x)| \leq C\varepsilon$ or $|y_1(x) - u_{0_1}(x) - v_{0_1}(x)| \leq C\varepsilon$. Then $|y_1(x) - u_{0_1}(x)| - |v_{0_1}(x)| \leq |y_1(x) - u_{0_1}(x) - v_{0_1}(x)|$, implies that

$$|y_1(x) - u_{0_1}(x)| \leq |v_{0_1}(x)| + C\varepsilon \leq C\varepsilon.$$

Therefore, $|w(x)| \leq C\varepsilon$. Hence, $|y_2(x) - y_2^\star(x)| \leq C\varepsilon$. □

4.2 Description of the method

Step 1: An asymptotic approximation is derived for the solution of (7)-(8) which is given by (9)-(10).
Step 2: The first component of the solution $\bar{y}(x)$ of the BVPs (7)-(8), namely y_1 is approximated by the first component of the solution of the reduced problem namely u_{0_1} given by (9). Then replacing y_1 appearing in the second equation of (7) by u_{0_1} and taking the same boundary values, one gets the auxiliary SPBVPs (13)-(14). The solution of this problem is taken as an approximation to y_2 which is the second equation of (7) which has to be solved .
Step 3: In order to solve the auxiliary second order problem (13)-(14) numerically, we divide the interval $[0, 1]$ into two subintervals $[0, \tau]$ and $[\tau, 1]$ called inner and outer region respectively, where $\tau = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha} \ln N\}$. The inner region problem for (13)-(14) is given by

$$\begin{cases} -\varepsilon y_2''(x) - a(x)y_2'(x) + b(x)y_2(x) \\ \qquad = f(x) - c(x)u_{0_1}(x), \quad x \in (0, \tau), \\ -y_2'(0) = q, \; y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau). \end{cases} \tag{15}$$

The outer region problem for (13)-(14) is given by

$$\begin{cases} -\varepsilon y_2''(x) - a(x)y_2'(x) + b(x)y_2(x) \\ \qquad = f(x) - c(x)u_{0_1}(x), \quad x \in (\tau, 1), \\ y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau),\ y_2(1) = r. \end{cases} \tag{16}$$

Step 4: The inner region problem (15) is solved by the Shooting method using the initial conditions $\tilde{y}_2(0) = u_{0_2}(0) + v_{0_2}(0)$, $-\tilde{y}_2'(0) = q$. Here, Shooting method in the sense that BVP (15) is replaced by the IVP (17) on the interval $[0, \tau]$. **Step 5:** The outer region problem (16) subject to boundary conditions $y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau)$, $y_2(1) = r$ is solved by the standard FD scheme.
Step 6: After solving both the inner region and the outer region problems, we combine their solutions to obtain an approximate solution y_2 which is the derivative of the solution of the original problem (1)-(2) over the interval $\bar{\Omega}$.

4.3 Inner region problem

Using **Step 4** for the BVPs (15), we get the following IVPs

$$\begin{cases} -\varepsilon \tilde{y}_2{}''(x) - a(x)\tilde{y}_2{}'(x) + b(x)\tilde{y}_2(x) \\ \qquad = f(x) - c(x)u_{0_1}(x), \quad x \in (0, \tau], \\ \tilde{y}_2(0) = \bar{q} = u_{0_2}(0) + v_{0_2}(0), \quad -\tilde{y}_2'(0) = q. \end{cases} \tag{17}$$

This IVPs is equivalent to the system

$$\begin{cases} P_1^*\bar{y}^* = y_1^{*\prime}(x) + y_2^*(x) = 0, \\ P_2^*\bar{y}^* = \varepsilon y_2^{*\prime}(x) + a(x)y_2^*(x) + b(x)y_1^*(x) \\ \qquad = f^*(x), \quad x \in (0, \tau], \\ y_1^*(0) = \bar{q}, \quad y_2^*(0) = q, \end{cases} \tag{18}$$

where, $f^*(x) = f(x) - c(x)u_{0_1}(x)$, $\bar{y}^* = (y_1^*, y_2^*)^T$, $a(x) \geq \alpha$, $\alpha > 0$ and $b(x) \geq 0$.

Theorem 6. (Maximum Principle). *Consider the IVPs (18). Let* $y_1^*(0) \geq 0$, $y_2^*(0) \geq 0$ *and* $P_1^*\bar{y}^*(x) \geq 0$, $P_2^*\bar{y}^*(x) \geq 0$ *for* $x \in (0, \tau]$. *Then,* $\bar{y}^*(x) \geq 0$, $\forall x \in [0, \tau]$.

Proof. See [23]. □

Lemma 3. (Stability Result). *If $\bar{y}^*(x)$ is the solution of the IVPs (18), then*

$$||\bar{y}^*(x)|| \leq C \max\{|y_1^*(0)|, |y_2^*(0)|, \max_{x\in(0,\tau]} |P_1^*\bar{y}^*(x)|, \max_{x\in(0,\tau]} |P_2^*\bar{y}^*(x)|\}, \quad \forall x \in \bar{\Omega}.$$

Proof. Defining two barrier functions $\bar{\chi}^{\pm}(x) = (\chi_1^{\pm}(x), \chi_2^{\pm}(x))^T$ as

$$\chi_1^{\pm}(x) = M'(1+x) \pm y_1^*(x) \quad \text{and} \quad \chi_2^{\pm}(x) = M' \pm y_2^*(x),$$

where

$$M' = C \max\{|y_1^*(0)|, |y_2^*(0)|, \max_{x\in(0,\tau]} |P_1^*\bar{y}^*(x)|, \max_{x\in(0,\tau]} |P_2^*\bar{y}^*(x)|\}.$$

We have

$$\begin{aligned} P_1^*\bar{\chi}^{\pm}(x) &= \chi_1^{\pm\prime}(x) + \chi_2^{\pm}(x) = 2M' \pm P_1^*\bar{y}^*(x) \geq 0 \quad \text{and} \\ P_2^*\bar{\chi}^{\pm}(x) &= \varepsilon\chi_2^{\pm\prime}(x) + a(x)\chi_2^{\pm}(x) + b(x)\chi_1^{\pm}(x) \pm P_2^*\bar{y}^*(x) \\ &\geq M'\alpha \pm P_2^*\bar{y}^*(x) \geq 0, \end{aligned}$$

by a proper choice of C. Furthermore, we have $\chi_1^{\pm}(0) = M' \pm y_1^*(0) \geq 0$ and $\chi_2^{\pm}(0) = M' \pm y_2^*(0) \geq 0$, by a proper choice of C. Applying Theorem 6 to the barrier functions $\bar{\chi}^{\pm}(x)$, we get the desired result. □

Theorem 7. *Let $\bar{y}^*(x)$ be the solution of the IVPs (18). Then $y_1^*(x)$ and $y_2^*(x)$ satisfy*

$$|y_1^{*(k)}(x)| \leq C\varepsilon^{-(k-1)}, \quad |y_2^{*(k)}(x)| \leq C\varepsilon^{-k} \quad \textit{for} \quad 0 \leq k \leq 2,\ x \in (0,\tau].$$

Proof. For $k = 0$, the result follows from Lemma 3. From (18), it is evident that $|y_1^{*'}(x)| \leq C$ and $|y_2^{*'}(x)| \leq C\varepsilon^{-1}$. Differentiating the equations in (18) once and using the above estimates of $|y_1^{*'}(x)|$ and $|y_2^{*'}(x)|$, it is found that $|y_1^{*''}(x)| \leq C\varepsilon^{-1}$ and $|y_2^{*''}(x)| \leq C\varepsilon^{-2}$. □

4.4 Numerical schemes

Applying Euler's Finite Difference scheme for (18), we get

$$\begin{cases} P_1^{*N/2}\bar{y}_i^* = D^- y_{1,i}^* + y_{2,i}^* = 0, \\ P_2^{*N/2}\bar{y}_i^* = \varepsilon D^- y_{2,i}^* + a(x_i)y_{2,i}^* + b(x_i)y_{1,i}^* = f^*(x_i), \quad 1 \leq i \leq N/2, \\ y_{1,0}^* = \bar{q}, \quad y_{2,0}^* = q, \end{cases} \tag{19}$$

where, $D^- y_{j,i}^* = (y_{j,i}^* - y_{j,i-1}^*)/h_1$, $h_1 = \frac{2\tau}{N}$, $x_i = ih_1$, $j = 1,2$ and $1 \leq i \leq N/2$. This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Theorem 8. (Discrete Maximum Principle). *Consider the discrete IVP (19). Let* $y^*_{1,0} \geq 0$, $y^*_{2,0} \geq 0$. *Then* $P_1^{*N/2}\bar{y}_i{}^* \geq 0$, $P_2^{*N/2}\bar{y}_i{}^* \geq 0$ *for* $1 \leq i \leq N/2$, *implies that* $\bar{y}_i{}^* \geq 0$ *for* $0 \leq i \leq N/2$.

Proof. See [23]. □

Lemma 4. (Stability Result). *Consider the discrete IVP (19). If* $\bar{y}^*_i$ *is any mesh function, then*

$$||\bar{y}^*_i|| \leq C \max\{|y^*_{1,0}|,\ |y^*_{2,0}|,\ \max_{1\leq i\leq N/2} |P_1^{*N/2}\bar{y}^*_i|,\ \max_{1\leq i\leq N/2} |P_2^{*N/2}\bar{y}^*_i|\}$$

for $0 \leq i \leq N/2$.

Proof. Set

$$M' = C \max\{|y^*_{1,0}|,\ |y^*_{2,0}|,\ \max_{1\leq i\leq N/2} |P_1^{*N/2}\bar{y}^*_i|,\ \max_{1\leq i\leq N/2} |P_2^{*N/2}\bar{y}^*_i|\}.$$

Define the barrier functions $\bar{\chi}^\pm_i = (\chi^\pm_{1,i}, \chi^\pm_{2,i})^T$ by

$$\chi^\pm_{1,i} = M'\{1 + x_i\} \pm y^*_{1,i} \quad \text{and} \quad \chi^\pm_{2,i}(x) = M' \pm y^*_{2,i}\ \ for\ \ 0 \leq i \leq N/2.$$

Then for a proper selection of the constant C, applying Theorem 8 to the barrier functions $\bar{\chi}^\pm_i$, we can obtain the desired bound for $\bar{y}^*_i$. □

4.5 Outer region problem

The outer region problem for (13)-(14) is given by

$$\begin{cases} Ly_2(x) := -\varepsilon y_2^{''}(x) - a(x)y_2^{'}(x) + b(x)y_2(x) \\ \qquad\qquad = f(x) - c(x)u_{0_1}(x), \quad x \in (\tau, 1), \\ B_0 y_2(0) = y_2(\tau) = u_{0_2}(\tau) + v_{0_2}(\tau) = \bar{r}, \quad B_1 y_2(1) = y_2(1) = r, \end{cases} \tag{20}$$

where $u_{0_1}(x)$ is defined as in (9), $a(x)$, $b(x)$ and $f(x)$ are sufficiently smooth and $a(x) \geq \alpha$, $b(x) \geq 0$ and $0 \geq c(x) \geq -\gamma$, $\gamma > 0$.

4.5.1 Analytical results

Theorem 9. (Maximum Principle). *Consider the BVP (20). Let* $y_2(x)$ *be a smooth function satisfying* $B_0 y_2(0) \geq 0$, $B_1 y_2(1) \geq 0$ *and* $Ly_2(x) \geq 0$ *for* $x \in (\tau, 1)$. *Then,* $y_2(x) \geq 0$ *for* $x \in [\tau, 1]$.

Proof. See [1]. □

Lemma 5. (Stability result). *If $y_2(x)$ is the solution of the BVP (20) then*

$$|y_2(x)| \leq C \max\{|B_0 y_2(0)|, |B_1 y_2(1)|, \max_{x\in(\tau,1)} |Ly_2(x)|\} \quad for \quad x \in [\tau, 1].$$

Proof. Define the barrier functions $\Phi^{\pm}(x)$ as $\Phi^{\pm}(x) = M(2-x) \pm y_2(x)$, $x \in [\tau, 1]$, where

$$M = C \max\{|B_0 y_2(0)|, \ |B_1 y_2(1)|, \ \max_{x\in(0,\tau)} |Ly_2(x)|\}.$$

It is easy to check that $B_0\Phi^{\pm}(\tau) \geq 0$, $B_1\Phi^{\pm}(1) \geq 0$ and $L\Phi^{\pm}(x) \geq 0$ for a proper choice of the constant C. Applying Theorem 9 to $\Phi^{\pm}(x)$, the required stability bound for $y_2(x)$ is obtained. □

To solve this BVP (20), we apply the standard FD scheme defined by

$$\begin{cases} L^{N/2} y_{2,i} := -\varepsilon\delta^2 y_{2,i} - a(x_i) D^+ y_{2,i} + b(x_i) y_{2,i} \\ \qquad\qquad = f(x_i) - c(x_i) u_{0_1}(x_i), \quad 1 \leq i \leq N/2 - 1, \\ B_0^{N/2} y_{2,0} = y_{2,0} = \bar{r}, \quad B_1^{N/2} y_{2,N} = y_{2,N/2} = r, \end{cases} \tag{21}$$

where, $D^+ y_{2,i} = (y_{2,i+1} - y_{2,i})/h_2$, $\delta^2 y_{2,i} = (y_{2,i+1} - 2y_{2,i} + y_{2,i-1})/h_2^2$,

$$x_i = \tau + ih_2, \quad \text{and} \quad h_2 = \frac{2(1-\tau)}{N}, \quad 0 \leq i \leq N/2.$$

This fitted mesh is denoted by $\bar{\Omega}_\tau^{N/2}$.

Theorem 10. (Discrete Maximum Principle). *Consider the discrete BVP (21). If $B_0^{N/2} y_{2,0} \geq 0$, $B_1^{N/2} y_{2,N} \geq 0$ and $L^{N/2} y_{2,i} \geq 0$ for $1 \leq i \leq N/2 - 1$. Then $y_{2,i} \geq 0$ for $0 \leq i \leq N/2$.*

Proof. See [1]. □

Lemma 6. (Discrete Stability Result). *If $y_{2,i}$ is the solution of the BVP (21), then*

$$|y_{2,i}| \leq C \max\{|B_0^{N/2} y_{2,0}|, \quad |B_1^{N/2} y_{2,N}|, \quad \max_{1\leq i\leq N/2-1} |L^{N/2} y_{2,i}|\}$$

for $0 \leq i \leq N/2$.

Proof. Set

$$M'' = C \max\{|B_0^{N/2} y_{2,0}|, \quad |B_1^{N/2} y_{2,N}|, \quad \max_{1\leq i\leq N/2-1} |L^{N/2} y_{2,i}|\}.$$

Define the barrier function by $\phi_i^{\pm} = M''\{1 + x_i\} \pm y_{2,i}$ for $0 \leq i \leq N/2$.

Then for a proper selection of the constant C, applying Theorem 10 to the barrier functions $\bar{\phi}_i^{\pm}$, we can obtain the desired bound for $y_{2,i}$. □

5 Error estimates

In this section, we derive error estimates for the solution of (13)-(14).

5.1 Inner region problem

In order to derive error estimate for the solution of the inner region problem, we prove the following theorems.

Theorem 11. *Let* $\bar{y}^* = (y_1^*, y_2^*)^T$ *and* $\bar{y}_i^* = y_{1,i}^*, y_{2,i}^*)^T$ *be, respectively, the solutions of (18) and (19). Then*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N \quad \textit{for} \;\; 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Lemma 4.1 in [6] and Theorem 7 it is clear that for each i, the consistency errors due to $\bar{y}^*(x)$ with $P_1^{*N/2}$ and $P_2^{*N/2}$ are bounded as given below.

$$\begin{aligned} |P_1^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)| &= |(D^+ - D)y_1^*(x_i)| \\ &= \frac{h_1}{2}|y_1^{*''}(t)| = \frac{h_1}{2\varepsilon} \end{aligned} \tag{22}$$

and

$$\begin{aligned} |P_2^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)| &= \varepsilon|(D^+ - D)y_2^*(x_i)| \\ &= \frac{\varepsilon h_1}{2}|y_2^{*''}(t)| = \frac{h_1}{2\varepsilon}, \end{aligned} \tag{23}$$

for some point t satisfying $x_{i-1} \leq t \leq x_i$.

Since $\tau = \min\{\frac{1}{2}, \frac{\varepsilon}{\alpha}\ln N\}$, the argument is considered for two cases $\tau = \frac{1}{2}$ and $\tau = \frac{\varepsilon}{\alpha}\ln N$ separately.

Case 1: $\tau = \frac{1}{2}$. Note that $\frac{1}{2} \leq \frac{\varepsilon}{\alpha}\ln N$ implies $\varepsilon^{-1} \leq C\ln N$.

From (22) and (23) and using $h_1 \leq CN^{-1}$, we have

$$\begin{cases} |P_1^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)|, \leq CN^{-1}\ln N, \\ |P_2^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1}\ln N. \end{cases}$$

Case 2: $\tau = \frac{\varepsilon}{\alpha}\ln N$.

From (22) and (23), we have

$$\begin{cases} |P_1^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1}\ln N, \\ |P_2^{*N/2}(\bar{y}^*(x_i) - \bar{y}_i^*)| \leq CN^{-1}\ln N. \end{cases}$$

Since $y_1^*(0) = y_{1,0}^*$ and $y_2^*(0) = y_{2,0}^*$, by the discrete stability result given by Lemma 4 it follows that

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N,$$

which completes the proof. □

Theorem 12. *Let $\bar{y}^* = (y_1^*, y_2^*)^T$ and $\bar{y}^{*1} = (y_1^{*1}, y_2^{*1})^T$ be, respectively, the solutions of the IVPs*

$$\begin{cases} y_1^{*'} - y_2^* = 0, \\ \varepsilon y_2^{*'} + a(x)y_2^* - b(x)y_1^* = f(x) + c(x)u_{0_1}(x), \quad x \in \Omega, \\ y_1^*(0) = \alpha', \quad y_2^*(0) = \beta' \end{cases} \tag{24}$$

and

$$\begin{cases} y_1^{*1} - y_2^{*1} = 0, \\ \varepsilon y_2^{*1'} + a(x)y_2^{*1} - b(x)y_1^{*1} = f(x) + c(x)u_{0_1}(x), \quad x \in \Omega, \\ y_1^{*1}(0) = \alpha' + O(\varepsilon), \quad y_2^{*1}(0) = \beta', \end{cases} \tag{25}$$

then $||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon$.

Proof. Let $\bar{w} = \bar{y}^* - \bar{y}^{*1}$. Then $\bar{w}$ satisfies

$$\begin{cases} w_1' - w_2 = 0, \\ \varepsilon w_2' + a(x)w_2 - b(x)w_1 = 0, \quad x \in \Omega, \\ w_1(0) = O(\varepsilon), \quad w_2(0) = 0. \end{cases} \tag{26}$$

Using the maximum principle for the system (26) as in [1], we have

$$||\bar{y}^*(x) - \bar{y}^{*1}(x)|| \leq C\varepsilon, \quad x \in \Omega.$$

□

Theorem 13. *Let $\bar{y}^* = (y_1^*, y_2^*)^T$ be the solution of the IVP (24). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (25) after applying the Euler's Finite Difference scheme as given in (19). Then,*

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq C\varepsilon + CN^{-1}\ln N \quad \text{for} \quad 0 \leq i \leq N/2 \quad \text{and} \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 12, $||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| \leq C\varepsilon$. From Theorem 11, $||\bar{y}^{*1}(x_i) - \bar{y}_i^*|| \leq CN^{-1}\ln N$. Using these estimates in the inequality ,

$$||\bar{y}^*(x_i) - \bar{y}_i^*|| \leq ||\bar{y}^*(x_i) - \bar{y}^{*1}(x_i)|| + ||\bar{y}^{*1}(x_i) - \bar{y}_i^*||,$$

where $\bar{y}^{*1}(x)$ is the solution of the system (25), this theorem is proved. □

The BVP (13)-(14) is equivalent to the following IVP

$$\begin{cases} -\varepsilon y_2''(x) - a(x)y_2'(x) + b(x)y_2(x) = f^*(x), & x \in \Omega, \\ y_2(0) = q^*, \quad y_2'(0) = -q, \end{cases} \tag{27}$$

where q^* is the exact value of the solution of the BVP (13)-(14) at $x = 0$. Because of uniqueness of the solutions of the IVP (27) and the BVP (13)-(14), we have the following result on the error estimate for the inner region problem.

Theorem 14. *Let $y_2^\star(x_i)$ be the solution of the BVP (13)-(14). Further, let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (19). Then,*

$$|y_2^\star(x_i) - y_{1,i}^*| \le C\varepsilon + CN^{-1}\ln N \quad \textit{for} \quad 0 \le i \le N/2 \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. Consider the inequality ,

$$|y_2^\star(x_i) - y_{1,i}^*| \le |y_2^\star(x_i) - y_1^{*1}(x_i)| + |y_1^{*1}(x_i) - y_{1,i}^*|,$$

where $y_1^{*1}(x)$ is the solution of the system (25). The proof follows from Theorems 12 and 13. □

Theorem 15. *Let $\bar{y}(x)$ be the solution of the BVP (7)-(8) and let $\bar{y}_i^* = (y_{1,i}^*, y_{2,i}^*)^T$ be the numerical solution of the IVP (19). Then,*

$$|y_2(x_i) - y_{1,i}^*| \le C\varepsilon + CN^{-1}\ln N \quad \textit{for} \ \ 0 \le i \le N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. Consider the inequality,

$$|y_2(x_i) - y_{1,i}^*| \le |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{1,i}^*|,$$

where $y_2^\star(x)$ is the solution of the BVP (13)-(14). The proof follows from Theorems 5 and 14. □

Remark 2. In particular, Theorem 14 and Theorem 15 are true over the interval $[0, \tau]$, that is, for the inner region problem.

5.2 Outer regin problem

Adopting the method of analysis provided as in [2], the following theorem can be proved.

Theorem 16. *Let $y_2(x_i)$ be the solution of the BVP (20) and $y_{2,i}$ be its numerical solution given by (21). Then,*

$$|y_2(x_i)) - y_{2,i}| \le CN^{-1}\ln N \quad \textit{for} \quad 0 \le i \le N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. See [2]. □

Theorem 17. *Let $y_2^\star(x_i)$ be the solution of the BVP (13)-(14) and $y_{2,i}$ be the numerical solution of the BVP (20) after applying the standard FD scheme as given in (21). Then,*

$$|y_2^\star(x_i) - y_{2,i}| \leq C\varepsilon + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/2, \quad x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 5, $|y_2^\star(x_i) - y_2(x_i)| \leq C\varepsilon$. From Theorem 16, $|y_2(x_i) - y_{2,i}| \leq CN^{-1}\ln N$. Using these estimates in the inequality,

$$|y_2^\star(x_i) - y_{2,i}| \leq |y_2^\star(x_i) - y_2(x_i)| + |y_2(x_i) - y_{2,i}|,$$

where $y_2(x_i)$ is the solution of the BVP (20), this theorem is proved. □

Theorem 18. *Let $\bar{y}(x)$ be the solution of the BVP (7)-(8) and $y_{2,i}$ be the numerical approximation obtained for $y_2(x_i)$ for the BVP (20) after applying the standard FD scheme as given in (21). Then,*

$$|y_2(x_i) - y_{2,i}| \leq C\varepsilon + CN^{-1}\ln N \quad \textit{for} \quad 0 \leq i \leq N/2, \; x_i \in \bar{\Omega}_\tau^{N/2}.$$

Proof. From Theorem 5, $|y_2(x_i) - y_2^\star(x_i)| \leq C\varepsilon$. From Theorem 17, $|y_2^\star(x_i) - y_{2,i}| \leq CN^{-1}\ln N$. Using these estimates in the inequality,

$$|y_2(x_i) - y_{2,i}| \leq |y_2(x_i) - y_2^\star(x_i)| + |y_2^\star(x_i) - y_{2,i}|.$$

where $y_2^\star(x_i)$ is the solution of the BVP (13)-(14), this theorem is proved. □

6 Nonlinear problem

Consider the quasilinear BVP

$$\varepsilon y^{iv}(x) = F(x, y, y'', y'''), \quad x \in \Omega, \tag{28}$$
$$y(0) = p, \quad y'''(0) = q, \quad y(1) = r, \quad -y''(1) = s, \tag{29}$$

where $F(x, y, y'', y''')$ is a smooth function such that

$$\begin{cases} F_{y'''}(x, y, y'', y''') \geq \alpha, \;\; \alpha > 0, \\ F_{y''}(x, y, y'', y''') \geq 0, \\ 0 \geq F_y(x, y, y'', y''') \geq -\gamma, \;\; \gamma > 0, \;\; \alpha > 3\gamma. \end{cases} \tag{30}$$

Assume that the reduced problem

$$F(x, y, y'', y''') = 0, \quad y(0) = p, \quad y(1) = r, \quad -y''(1) = s$$

has a solution $y_0 \in C^4(\bar{\Omega})$. Then the BVP (28)-(29) has a unique solution and has a less severe boundary layer of width $O(\varepsilon)$ near $x = 0$ [12]. The analytical results such as existence, uniqueness and asymptotic behavior of the solution of (28)-(29) can be found in [12, 19, 29].

In order to obtain a numerical solution of (28)-(29), first Newton's method of quasi-linearization is applied [1] and then the problem is linearized. Consequently, we get a sequence $\{y^{[m]}\}_0^\infty$ of successive approximations with a proper choice of initial guess $y^{[0]}$.

We define $y^{[m+1]}$ for each fixed non-negative integer m, to be the solution of the following linear problem:

$$\begin{cases} \varepsilon(y^{iv})^{[m+1]} + a^m(x)(y''')^{[m+1]} - b^m(x)(y'')^{[m+1]} + c^m(x)y^{[m+1]} = F^{[m]}(x), \\ y^{[m+1]}(0) = p, \quad (y''')^{[m+1]}(0) = q, \quad y^{[m+1]}(1) = r, \quad -(y'')^{[m+1]}(1) = s, \end{cases} \tag{31}$$

where

$$\begin{cases} a^m(x) = F_{y'''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}), \\ b^m(x) = F_{y''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}), \\ c^m(x) = F_y(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}), \\ F^{[m]}(x) = F(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \\ +(y''')^{[m]}F_{y'''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \\ -(y'')^{[m]}F_{y''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \\ -(y)^{[m]}F_y(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}). \end{cases} \tag{32}$$

Remark 3. If the initial guess $y^{[0]}$ is sufficiently close to the solution $y(x)$ of (28)-(29), then, following the method of proof given in [1], one can prove that the sequence $\{y^{[m]}\}_0^\infty$ converges to $y(x)$. From (30), it follows that for each fixed m:

$$a^m(x) = F_{y'''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \geq \alpha, \quad \alpha > 0,$$

$$b^m(x) = F_{y''}(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \geq 0,$$

$$0 \geq c^m(x) = F_y(x, y^{[m]}, (y'')^{[m]}, (y''')^{[m]}) \geq -\gamma, \quad \gamma > 0, \quad \alpha > 3\gamma,$$

Remark 4. Problem (31)-(32) for each fixed m is a linear BVP of fourth order and is of the form (1)-(2). Hence it can be solved by numerical method discussed above.

Remark 5. *The solution of the reduced problem of (28)-(29) or a suitable approximation will be taken as the initial guess $y^{[0]}$ to generate the successive approximations $\{y^{[m]}\}_0^\infty$.*

7 Numerical illustrations

In this section, we present three examples to illustrate the method described in this paper. Let Y^N be a numerical approximation for the exact solution y on the mesh Ω^N and N is the number of mesh points. We compute the maximum point-wise errors using

$$E_\varepsilon^N = \max_{x \in \bar{\Omega}^N} \mid Y^N(x_j) - y(x_j) \mid .$$

The computed maximum pointwise errors E_ε^N for various values of ε and N are tabulated in Table 1, Table 2 and Table 3.

Example 1. Consider the BVP

$$\begin{aligned} &\varepsilon y^{iv}(x) + y'''(x) - y''(x) = 0, \\ &y(0) = 1, \quad y'''(0) = 1, \quad y(1) = 0, \quad -y''(1) = 0. \end{aligned}$$

The numerical results are presented in Table 1. The exact solution of auxiliary second order SPBVPs for this problem is

$$y_2^\star(x) = [m_1 e^{m_1 x + m_2} - e^{m_1 + m_2 x}]/(m_2 e^{m_1} - m_1 e^{m_2}),$$

where

$$m_1 = (-1 + \sqrt{1 + 4\varepsilon})/(2\varepsilon), \quad m_2 = (-1 - \sqrt{1 + 4\varepsilon})/(2\varepsilon).$$

The graph for this exact solution is given in Figure 1.

Example 2. Consider the BVP

$$\begin{aligned} &\varepsilon y^{iv}(x) + 4(x+2)y'''(x) - (x+2)y''(x) - y(x) = \sqrt{\varepsilon}(sinhx), \\ &y(0) = 1, \quad y'''(0) = 0, \quad y(1) = 0, \quad -y''(1) = 0. \end{aligned}$$

The numerical results are presented in Table 2.

Example 3. Consider the quasilinear BVP

$$\begin{aligned} &\varepsilon y^{iv}(x) + 6y'''(x) - 8y''(x) - (1/2)y^2(x) = \varepsilon e^{-3x} + 1/2, \\ &y(0) = 0, \quad y'''(0) = 0, \quad y(1) = 0, \quad -y''(1) = 0. \end{aligned}$$

The numerical results are presented in Table 3. This SPBVP is linearized using the Newton's Method of quasi-linearization. The initial approximation for y_1 is taken to be $y^0(x) = x$. In the Tables 1, 2 and 3, the numerical results appearing in the rows 1-8 correspond to the boundary layer region. The rest of the rows namely 9-16 correspond to the outer region.

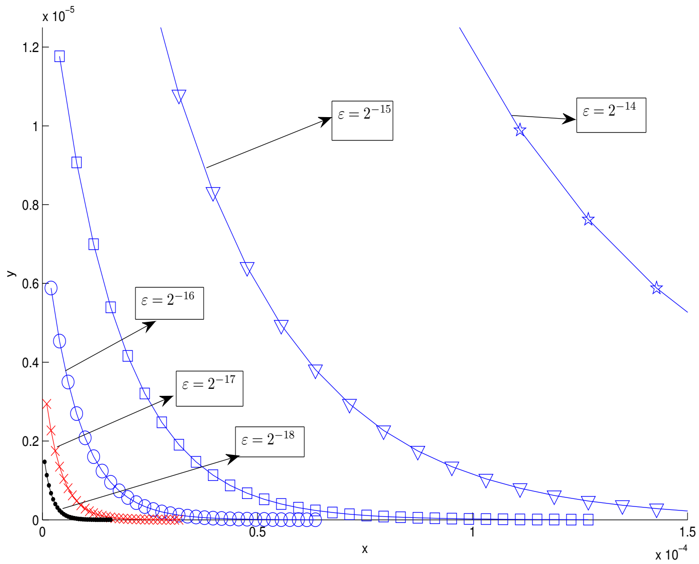

Figure 1: Graph of exact solution of auxiliary problem with in the layer region.

Table 1: *Comparison of Numerical and Exact solutions for $y_2^\star$ of Example1*

Number of mesh point N=64			
ε	Approx.	Exact	Relative Error
2^{-13}	2.180034141639589e-006	2.170734141639589e-006	4.284264858420493e-003
2^{-14}	1.088000198318126e-006	1.085700198318126e-006	2.118448539995594e-003
2^{-15}	5.431204085870404e-007	5.429504085870399e-007	3.131041017960758e-004
2^{-16}	2.715075350959110e-007	2.714875350959110e-007	7.366820724552635e-005
2^{-17}	1.357500756793042e-007	1.357489756793042e-007	8.103191898982985e-006
2^{-18}	6.787608981531353e-008	6.787578981531353e-008	4.419838072150196e-006
2^{-19}	3.393834049157181e-008	3.393822049157181e-008	3.535836536519309e-006
2^{-20}	1.696928162872805e-008	1.696919162872805e-008	5.303729368377519e-006
2^{-13}	9.700859856710905e-001	9.693859856710905e-001	7.221065812246451e-004
2^{-14}	9.700006136435082e-001	9.693096136435082e-001	7.128785171155152e-004
2^{-15}	9.694714249518869e-001	9.692714249518869e-001	2.063405511102380e-004
2^{-16}	9.694052339937300e-001	9.692523299373004e-001	1.577546441797454e-004
2^{-17}	9.693428822626144e-001	9.692427822626144e-001	1.032764977278165e-004
2^{-18}	9.692492083834264e-001	9.692380083834264e-001	1.155546924812541e-005
2^{-19}	9.692456224406448e-001	9.692356214406448e-001	1.031844040685589e-005
2^{-20}	9.692354279575486e-001	9.692344279575480e-001	1.031742137648939e-006

8 Conclusions

In this paper, we presented a numerical method to solve fourth order SPBVPs for ODEs subject to particu-

Table 2: *Comparison of Numerical and Exact solutions for $y_2^\star$ of Example*2

Number of mesh point N=16			
ε	Approx.	Exact	Relative Error
2^{-13}	5.949767218381516e-008	5.949643407515523e-008	2.915640149758703e-003
2^{-14}	2.103595173563539e-008	2.103534231337541e-008	2.916087350004491e-003
2^{-15}	7.437393581949865e-009	7.437147767220178e-009	2.916310952422627e-003
2^{-16}	2.629526593874225e-009	2.629434319680751e-009	2.916422754205524e-003
2^{-17}	9.296799655283715e-010	9.296463931558273e-010	2.916478655240370e-003
2^{-18}	3.286918438535174e-010	3.286798065503170e-010	2.916506605793629e-003
2^{-19}	1.162101759367909e-010	1.162058904659394e-010	2.916520581079230e-003
2^{-20}	4.108651234500969e-011	4.108499196232649e-011	2.916527568724325e-003
2^{-13}	9.895488206428230e-006	4.534376803447920e-007	2.082325959083009e+001
2^{-14}	7.116668821305247e-006	3.173894284766306e-007	2.142251374112559e+001
2^{-15}	5.075514879970357e-006	2.232725180728062e-007	2.173237621799607e+001
2^{-16}	3.604414608041251e-006	1.574670599112803e-007	2.188995939895011e+001
2^{-17}	2.554213519843517e-006	1.112005809470889e-007	2.196942604156767e+001
2^{-18}	1.808054813005443e-006	7.857920415580200e-008	2.200932966208901e+001
2^{-19}	1.279179397779354e-006	5.554567665102278e-008	2.202932424094973e+001
2^{-20}	9.047611223334508e-007	3.927028407937256e-008	2.203933224686534e+001

lar type of boundary conditions by adopting the techniques of [5, 14, 28] and [9, 19] who used to solve second and fourth order SPBVPs for ODEs. The boundary conditions help us to reduce the given fourth order ordinary differential equation into a weakly coupled system of two second order equations subject to suitable boundary conditions. Of course, an approximate solution can be improved by taking better approximate initial condition as said in Section 4. This is the reason for taking the solution of the IVPs only in the intervals $[0, \tau]$. In [19], both inner and outer region problems are BVPs, whereas in our case the inner region problem is an IVP and the outer region problem is a BVP. Though the present method yields almost the same order of convergence as given in [19], it produces very good reduction on the maximum-pointwise error especially in the inner region compared with [19]. This is the contribution of the present method used with IVP in the inner region. Naturally IVPs can be treated more easily compared with BVPs. Error estimates derived in Section 5 show first order convergence. Our numerical experiments show that this method gives good approximate solutions especially in layer region. This can be seen from the numerical results presented in Table 1, Table 2 and Table 3.

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Table 3: *Comparison of Numerical and Exact solutions for $y_2^\star$ of Example3*

Number of mesh point N=16			
ε	Approx.	Exact	Relative Error
2^{-13}	-1.158203738097887e-012	-5.845056470929231e-013	9.815099201492548e-001
2^{-14}	-2.047461672290371e-013	-1.033283494779239e-013	9.815100915047629e-001
2^{-15}	-3.619459282439118e-014	-1.826616549380359e-014	9.815101771999942e-001
2^{-16}	-6.398381897969334e-015	-3.229043097139054e-015	9.815102200519804e-001
2^{-17}	-1.131086697884004e-015	-5.708205156889433e-016	9.815102414790670e-001
2^{-18}	-1.999499356636325e-016	-1.009078481639718e-016	9.815102521928787e-001
2^{-19}	-3.534650362267373e-017	-1.783816333425386e-017	9.815102575498547e-001
2^{-20}	-6.248439406317065e-018	-3.153372219024979e-018	9.815102602283594e-001
2^{-13}	3.030517556316652e-005	3.420784366574728e-005	1.140869369234327e-001
2^{-14}	2.260102572266581e-005	2.550824458668147e-005	1.139717338892697e-001
2^{-15}	1.643335892125253e-005	1.854588298612394e-005	1.139079798169760e-001
2^{-16}	1.178732185739077e-005	1.330209157045507e-005	1.138745516102682e-001
2^{-17}	8.395376207129597e-006	9.474069593961251e-006	1.138574480727069e-001
2^{-18}	5.958060028452615e-006	6.723525308070145e-006	1.138487987393688e-001
2^{-19}	4.220677595262279e-006	4.762908265510224e-006	1.138444496557733e-001
2^{-20}	2.987197567137042e-006	3.370954698762453e-006	1.138422690065506e-001

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