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A model for the dynamical study of food–chain system considering interference of top predator in a polluted environment

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Abstract. The modeling investigation in this paper discusses the system level effects of a toxicant on a three species food chain system. In the models, we have assumed that the presence of top predator reduces the predatory ability of the intermediate predator. The stability analysis of the models is carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant are obtained. Further, it is also found that the predation rate of the intermediate predator is a bifurcating parameter and Hopf-bifurcation occurs at some critical value of this parameter. Finally, numerical simulation is carried out to support the analytical results.

Keywords: Stability, Bifurcation, Interference, Lyapunov function.
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1 Introduction

Species are regularly exposed to many natural and synthetic chemicals which are adversely affecting their growth rate directly or indirectly. The direct effects of toxicant on the species are alterations in their mortality and reproductive rates. The indirect effects are observed either through

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the food chain or through the reduction in the carrying capacity of the environment due to the degradation of the habitat. It is generally observed in nature that the toxicants decrease the growth rate of species and also their carrying capacity. The presence of toxicants in the environment affects not only the species but their resources also. These toxicants have very pronounced effects on the species if the availability of the resource is limited. There are many instances where the toxicants have been the main cause of extinction of many species and depletion of resources such as forestry, fertile crop and wild life.

Ecologists and mathematicians have often used food chain systems to describe the feeding relationships between species within ecosystems and there has been considerable interest in the predator-prey models, especially for systems of three species [5, 7, 10, 12, 16]. However, the ecological communities in nature are observed to exhibit very complex dynamical behaviors and three species continuous time models are reported to have more complicated patterns. In [14], Lv and Zhao have proposed and examined the dynamic complexities of a three species food chain model and found different forms of complexities in their model. In [22], Sun and Loreau proposed a three-species food chain model with dynamically variable adaptive traits in the intermediate consumer and from the stability analysis they have shown that the positive equilibrium is globally stable under specific conditions. However, recently in [5], Gomes et al., have considered the classical fishpond management for tilapia fish culture model and studied three levels consisting of young tilapia ($prey$), developed tilapia ($predator$) and tucunare fish ($top - predator$) in order to describe the dynamical behavior of a three-species food chain system. It may be noted here that these studies have not incorporated the effects of toxicants on the survival or extinction of prey populations in the food chain systems.

In a food–chain system with prey–predator relationship, it is observed that the predator interference occur naturally in the presence of top predator on intermediate predator. Mathematical studies related to the effect of interference on the dynamics of prey–predator population have been carried out by several researchers [3, 4, 9, 20].

The study of the effects of toxic substances on ecological communities is of great interest, both from environmental and conservational points of view. Species exposed to polluted environment become vulnerable to several stresses due to which their existence may be threatened in long run. In the experiment study of [21], the authors have observed that the fish from the polluted environment suffered significantly greater mortality in the presence of a predator, the blue crab *Callinectes sapidus* Rathbun, than fish from the unpolluted environment. In the study of [17], the authors evaluated that the during exposure to sublethal concentrations of LC (lambda-cyhalothrin) the predator-prey interactions between G. *pulex* and L. *nigra* were significantly altered. The relative frequency of successful predation by G. *pulex* on L. *nigra* decreased from nearly 100 percent in the control and the < 1 ng L^{-1} treatments to approximately 50 percent in the 6.6 ng L^{-1} treatment, and no predation was observed in the 62.1 ng L^{-1} treatment during the 60 min observation period. These findings probably reflect an increased stress response of G. *pulex* to increasing concentrations of LC prompting behavioural hyperactivity that overrules the natural instinct of catching the prey. So, inorder to use and regulate toxic substances wisely, we must asses the risk of the populations exposed to toxicants. Some investigators have studied the effects of toxicant on one and two interacting species systems using mathematical models [1, 6, 8, 11, 18, 19]. Previously, some research have been done on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species in the system [2, 6, 15].

In this paper therefore we have studied the dynamical behaviour of a three-species food-chain system under toxicant stress considering modified smith model for prey species [6] and predatory interference by top predator using mathematical model.

2 Mathematical Model

The model formulation has been carried out in the light of the research papers of [14] and [6]. In the model, the underlying food chain system consists of a prey population, an intermediate predator population and a top predator population with Holling type-II functional responses. It is assumed in the model that the presence of the top predator reduces the predatory ability of the intermediate predator [14]. In the model, the growth equation for the prey population in the absence of predator is assumed to be governed by a modified smith-type differential equation [6]. The state variables of the model are $x(t)$, the density of the prey population; $y(t)$, the density of the intermediate predator population; $z(t)$, the density of the top predator population; $c_o(t)$, the organism toxicant concentration in the prey population; and $c_e(t)$, the environmental toxicant concentration.

Taking these as state variables, we formulate the mathematical model using following system of nonlinear ordinary differential equations in order to study the effect of toxicant on a three-species food chain system:

$Model$ **A:** (*with toxicant*)

$$
\begin{aligned}
\frac{dx}{dt} &= x\left(\frac{r(c_o)B(c+r_0-a)-acx}{B(c+r_0-a)+ax}\right) - \left(\frac{A_1x}{B_1+x}\right)\frac{y}{1+z},\\
\frac{dy}{dt} &= \beta_{10}\left(\frac{A_1x}{B_1+x}\right)\frac{y}{1+z} - \left(\frac{A_2y}{B_2+y}\right)z - D_1y,\\
\frac{dz}{dt} &= \beta_{20}\left(\frac{A_2y}{B_2+y}\right)z - D_2z, \qquad (1)\\
\frac{dc_o}{dt} &= a_1c_e + \frac{d_1}{a_1}\theta\beta - (l_1+l_2)c_o,\\
\frac{dc_e}{dt} &= g_0 + k_1l_1c_ox - k_1a_1c_ex - k_2c_e.
\end{aligned}
$$

The initial conditions are

$$x(0)=x_0>0,\ y(0)=y_0>0,\ z(0)=z_0>0,\ c_o(0)=0,\ c_e(0)=c_{e0}>0.$$

In $Model$ A, B is the population carrying capacity; r_0 is the intrinsic growth rate of the population; a is a measure of the population response to stress effects; D_1 and D_2 are the death rates of y and z respectively.

We assume that the toxicant concentration θ in the population is a constant; β is the average rate of food intake per unit organismal mass; l_1 and l_2 are egestion and depuration rates respectively; $k_1l_1c_ox$ is the total toxicant ingested; $k_1a_1c_ex$ is the total toxicant uptake from the environment; k_2c_e is the term which describes the loss due to detoxifying process such as hydrolysis, volatilization, etc.; The exogenous input to the body burden c_o is assumed to be from the environment at a rate proportional to the environmental concentration: a_1c_e, where a_1 is the population rate of toxicant uptake per unit mass; g_0 represents exogenous input of toxicant into the environment; d_1 is a constant numerically less than or equal to the numerical value of a_1; c is the rate of replacement of mass in the population at saturation.

In the model $A_iu/(B_i+u)$, (i=1,2; $u=x$ and y), account for the interactions between two different species, representing the Holling type-II functional response. This functional response is parameterized by the constants A_i and B_i ($i=1,2$), and we verify that B_i is the value of the prey population level when the predation rate per unit prey is half their maximum value [14].

Exposure to toxicant may lead to changes in fecundity and mortality rates of a population. This stress can be modelled by assuming that the growth rate of the population is a function of the body burden $r(c_o) = r_0 - H(c_o)$. Here H is a non-decreasing function of c_o with $H(0)=0$ and

r_0 is the intrinsic growth rate of the population. $H(c_o)$ is a dose-response function, which is assumed to be linear and taken as $H(c_o) = r_1 c_o$ [6].

We can reduce the number of parameters in the above system by the following scaling transformations, even if, for our analytical and numerical tests, we will continue to use the original system:

$$x \to \frac{x}{B_1},\ y \to \frac{A_1 y}{B_1},\ z \to \frac{A_1 z}{A_2 B_1},\ c_o \to \frac{r_0 c_o}{k_1},\ c_e \to \frac{r_0 c_e}{a_1},\ t \to D_1 t.$$

Thus, system (1) after re-scaling becomes as follows:

$Model$ **1:**

$$\frac{dx}{dt} = x e_1 \left(\frac{e_0(c_o) e_2 - x}{e_4 + x} \right) - \frac{u_1 x y}{(1+x)(1+e_5 z)}, \tag{2}$$

$$\frac{dy}{dt} = \frac{o_6 x y}{(1+x)(1+e_5 z)} - \frac{u_2 y z}{e_8 + y} - y, \tag{3}$$

$$\frac{dz}{dt} = \frac{o_7 y z}{e_8 + y} - u_3 z, \tag{4}$$

$$\frac{dc_o}{dt} = o_2 c_e + o_3 - u_4 c_o, \tag{5}$$

$$\frac{dc_e}{dt} = u_0 + o_4 c_o x - u_5 c_e x - u_6 c_e. \tag{6}$$

The initial conditions are

$$x(0) = x_0 > 0,\ y(0) = y_0 > 0,\ z(0) = z_0 > 0,\ c_o(0) = 0,\ c_e(0) = c_{e0} > 0.$$

Here,

$$e_1 = \tfrac{c}{D_1},\quad e_2 = \tfrac{T_0 r_0 r_1}{a c k_1 B_1},\quad e_3 = \tfrac{k_1}{r_1},\quad e_4 = \tfrac{T_0}{a B_1},\quad e_5 = \tfrac{A_1}{A_2 B_1},$$
$$e_8 = \tfrac{A_1 B_2}{B_1},\quad u_0 = \tfrac{g_0 a_1}{r_0 D_1},\quad u_1 = \tfrac{1}{D_1 B_1^2},\quad u_2 = \tfrac{1}{D_1},\quad u_3 = \tfrac{D_2}{D_1},$$
$$u_4 = \tfrac{(l_1 + l_2)}{D_1},\quad u_5 = \tfrac{a_1 k_1 B_1}{D_1},\quad u_6 = \tfrac{k_2}{D_1},\quad o_2 = \tfrac{k_1}{D_1},\quad o_3 = \tfrac{k_1 d_1 \theta \beta}{r_0 a_1 D_1},$$
$$o_4 = \tfrac{l_1 a_1 B_1}{D_1},\quad o_6 = \tfrac{\beta_{10}}{A_1 D_1},\quad o_7 = \tfrac{A_2 \beta_{20}}{D_1},$$

$T_0 = B(c + r_0 - a), e_0(c_o) = e_3 - c_o$. All these parameters, of course, assume only positive values.

Now, if the effect of toxicant is not considered in the above $Model$ 1, then we have the following $Model$ 2 for three species food chain system:

$Model$ **B:** (*without toxicant*)

$$\begin{aligned}
\frac{dx}{dt} &= x\left(\frac{r_0B(c+r_0-a)-acx}{B(c+r_0-a)+ax}\right)-\left(\frac{A_1x}{B_1+x}\right)\frac{y}{1+z},\\
\frac{dy}{dt} &= \beta_{10}\left(\frac{A_1x}{B_1+x}\right)\frac{y}{1+z}-\left(\frac{A_2y}{B_2+y}\right)z-D_1y,\\
\frac{dz}{dt} &= \beta_{20}\left(\frac{A_2y}{B_2+y}\right)z-D_2z,
\end{aligned} \tag{7}$$

with the initial conditions as

$$x(0)=x_0>0,\ y(0)=y_0>0,\ z(0)=z_0>0,$$

where, the state variables and parameters are the same as defined for the $Model$ 1.

We can reduce the number of parameters in the above system, even if, for our analytical and numerical tests, we will continue to use the original system. Here, $o_5 = e_2e_3$ and rest of the parameters are the same as defined for the $Model$ 1. All these parameters, of course, assume only positive values. Thus, system (7) after re-scaling becomes as follows:

$Model$ **2:**

$$\frac{dx}{dt} = xe_1\left(\frac{o_5-x}{e_4+x}\right)-\frac{u_1xy}{(1+x)(1+e_5z)}, \tag{8}$$

$$\frac{dy}{dt} = \frac{o_6xy}{(1+x)(1+e_5z)}-\frac{u_2yz}{e_8+y}-y, \tag{9}$$

$$\frac{dz}{dt} = \frac{o_7yz}{e_8+y}-u_3z. \tag{10}$$

The initial conditions are

$$x(0)=x_0>0,\ y(0)=y_0>0,\ z(0)=z_0>0.$$

3 Boundedness of the Models:

To analyze the $Models$ 1 and 2, in this section we need the bounds of dependent variables involved. First we see the boudedness of $Model$ 2 and $Model$ 1. So here, we find the region of attraction for all the $Models$ in the following lemma.

Lemma 1. *The set*

$$\begin{aligned}\Omega &= \{(x,y,z,c_o,c_e)\in R_+^6 : 0\le x(t)\le o_5, 0\le o_6x(t)+u_1y(t)+\frac{u_1u_2}{o_7}z(t)\le\theta_1\}\\ &\quad 0\le c_o(t)+c_e(t)\le\theta_2, o_6x(t)+u_1y(t)+\frac{u_1u_2}{o_7}z(t)+c_e(t)\ge\theta_3\}\end{aligned}$$

is a region of attraction for all solutions initiating in the interior of the positive region, where

$$\begin{aligned}\theta_1 &= (1+e_1o_5)o_5o_6/\Phi_1, \theta_2=u_0/\Phi_2, \theta_3=u_0/\Phi_3, \Phi_1=\min\{1,1,u_3\},\\ \Phi_2 &= \min\{u_4-o_4o_5, u_6-o_2\}, \Phi_3=\max\{u_5\theta_2,1,u_3,u_6\}.\end{aligned}$$

Proof. From equation (8) we obtain $dx/dt \le xe_1(o_5-x)$. Then, by the usual comparison theorem, we get

$$x\le o_5.$$

Now, let us consider the following function

$$\psi_1(t)=o_6x(t)+u_1y(t)+\frac{u_1u_2}{o_7}z(t).$$

By using equations (8)-(10), we conclude

$$d\psi_1/dt+\Phi_1\psi_1\le(1+e_1o_5)o_5o_6,$$

where $\Phi_1=\min\{1,1,u_3\}$ and then by the usual comparison theorem, we get $\psi_1(t)\le(1+e_1o_5)o_5o_6$ as $t\to\infty$, and hence,

$$o_6x(t)+u_1y(t)+\frac{u_1u_2}{o_7}z(t)\le\theta_1,$$

where $\theta_1=(1+e_1o_5)o_5o_6/\Phi_1$.
Now, let us consider the function: $\psi_2(t)=c_o(t)+c_e(t)$, by using equations (5) - (6), we get

$$d\psi_2/dt+\Phi_2\phi_2\le u_0,$$

where $\Phi_2=\min\{u_4-o_4o_5, u_6-o_2\}$ and then by the usual comparison theorem, we get as $t\to\infty$, $\psi_2(t)\le u_0/\phi_2$ and hence,

$$c_o(t)+c_e(t)\le\theta_2,$$

where $\theta_2=u_0/\Phi_2$. Again let us consider the function

$$\psi_3(t)=o_6x(t)+u_1y(t)+\frac{u_1u_2}{o_7}z(t)+c_e(t).$$

By using the equations (2)–(6), we get $d\psi_3/dt + \Phi_3\psi_3 \geq u_0$, where, $\Phi_3 = \max\{u_5\theta_2, 1, u_3, u_6\}$ and then by the usual comparison theorem, we get as $t \to \infty$, $\psi \geq u_0/\Phi_3$ and hence

$$o_6 x(t) + u_1 y(t) + \frac{u_1 u_2}{o_7} z(t) + c_e(t) \geq \theta_3,$$

where $\theta_3 = u_0/\Phi_3$. Hence the solution of the $Models$ 1 and 2 are bounded in Ω. □

4 Analysis of $Model$ 2

4.1 Equilibria of $Model$ 2:

The $Model$ 2 has following four non-negative equilibria in x, y and z space namely, $E_{20} = (0,0,0)$, $\tilde{\tilde{E}}_{21} = (\tilde{\tilde{x}}, 0, 0)$, $\hat{\hat{E}}_{22} = (\hat{\hat{x}}, \hat{\hat{y}}, 0)$ and $\bar{\bar{E}}_{23} = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$. The existence of E_{20} is obvious. We prove the existence of $\tilde{\tilde{E}}_{21}$, $\hat{\hat{E}}_{22}$ and $\bar{\bar{E}}_{23}$ as follows:

1. **Existence of** $\tilde{\tilde{E}}_{21} = (\tilde{\tilde{x}}, 0, 0)$,
   from (8),
$$\tilde{\tilde{x}} = o_5. \tag{11}$$

2. **Existence of** $\hat{\hat{E}}_{22} = (\hat{\hat{x}}, \hat{\hat{y}}, 0)$,
   from (9),
$$\hat{\hat{x}} = o_6 - 1 > 0, \tag{12}$$
   $\hat{\hat{x}} > 0$ if $o_6 > 1$,
   from (8),
$$\hat{\hat{y}} = \frac{e_1 o_6}{u_1}\left(\frac{o_5 - (o_6 - 1)}{e_4 + (o_6 - 1)}\right) > 0, \tag{13}$$
   $\hat{\hat{y}} > 0$ if $o_5 > (o_6 - 1)$.

3. **Existence of** $\bar{\bar{E}}_{23} = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$,
   from (10),
$$\bar{\bar{y}} = \frac{u_3 e_8}{o_7 - u_3}, \tag{14}$$
   $\bar{\bar{y}} > 0$ if $o_7 > u_3$,
   from (8) and (9),
$$\bar{\bar{z}} = \frac{u_3 + (o_7 - u_3)}{u_1 u_2 u_3}\left(\bar{\bar{x}} e_1 o_6 \frac{(o_5 - \bar{\bar{x}})}{(e_4 + \bar{\bar{x}})} - \frac{u_1 u_3 e_8}{o_7 - u_3}\right), \tag{15}$$

and $\bar{\bar{x}}$ is given by

$$\bar{\bar{x}}^2 e_1 o_6 (o_7 - u_3) - \bar{\bar{x}}[e_1 o_5 o_6 (o_7 - u_3) - u_1 u_3 e_8] + u_1 u_3 e_4 e_8 = 0.$$

$\bar{\bar{x}}, \bar{\bar{y}}$ and $\bar{\bar{z}}$ are positive provided $o_7 > u_3$; $e_1 o_5 o_6 (o_7 - u_3) > u_1 u_3 e_8$; $\varpi_2 > 0$; and

$$\frac{\varpi_1 - \sqrt{\varpi_2}}{2e_1 o_6 (o_7 - u_3)} < \bar{\bar{x}} < \frac{\varpi_1 + \sqrt{\varpi_2}}{2e_1 o_6 (o_7 - u_3)}, \tag{16}$$

where $\varpi_1 = e_1 o_5 o_6 (o_7 - u_3) - u_1 u_3 e_8$ and $\varpi_2 = \varpi_1^2 - 4e_1 u_1 u_3 e_4 o_6 e_8 (o_7 - u_3)$.

4.2 Dynamical behaviour of *Model* 2:

The general variational matrix corresponding to the *Model* 2 is

$$J_2(x, y, z) = \begin{bmatrix} -m_{11} & -m_{12} & m_{13} \\ m_{21} & m_{22} & -m_{23} \\ 0 & m_{32} & m_{33} \end{bmatrix},$$

where,

$$\begin{aligned}
m_{11} &= \frac{u_1 y}{(1+x)^2(1+e_5 z)} + \frac{e_1(x(o_5+e_5) - (x+e_4)(o_5 - x))}{(e_4+x)^2}, \\
m_{12} &= \frac{u_1 x}{(1+x)(1+e_5 z)}, \quad m_{13} = \frac{u_1 e_5 x y}{(1+x)(1+e_5 z)^2}, \\
m_{21} &= \frac{o_6 y}{(1+x)^2(1+e_5 z)}, \quad m_{22} = \frac{o_6 x}{(1+x)(1+e_5 z)} - \frac{u_2 e_8 z}{(e_8+y)^2} - 1, \\
m_{23} &= y\left(\frac{e_5 o_6 x}{(1+x)(1+e_5 z)^2} + \frac{u_2}{(e_8+y)}\right), \quad m_{32} = \frac{o_7 e_8 z}{(e_8+y)^2}, \\
m_{33} &= \frac{o_7 y}{e_8 + y} - u_3.
\end{aligned}$$

1. At E_{20}, the eigenvalues of the characteristic equation are $e_1 o_5 / e_4, -1$ and $-u_3$, which shows that E_{20} is unstable.

2. At $\tilde{\tilde{E}}_{21}$, the eigen values of the characteristic equation are $o_5 o_6/(1 + o_5) - 1$ $-e_1 o_5 (o_5 + e_4)/(e_4 + o_5)^2$ and $-u_3$, which shows $\tilde{\tilde{E}}_{21}$ is locally asymptotically stable if

$$o_5 o_6 < (1 + o_5), \tag{17}$$

holds good.

3. At $\hat{\hat{E}}_{22}$, one of the eigenvalue of the characteristic equation is $o_7 y/(e_8 + y) - u_3$ and the other two eigenvalues are given by the roots of the following quadratic equation

$$\lambda^2 + \lambda\hat{\hat{x}}[\frac{e_1(o_5 + e_4)}{(e_4 + \hat{\hat{x}})^2} - \frac{u_1\hat{\hat{y}}}{(1 + \hat{\hat{x}})^2}] + \frac{u_1 o_6 \hat{\hat{x}}\hat{\hat{y}}}{(1 + \hat{\hat{x}})^3} = 0. \tag{18}$$

From the Routh-Hurwitz criteria it is found that $\hat{\hat{E}}_{22}$ is locally asymptotically stable if the following condition hold good.

$$u_1\hat{\hat{y}}(e_4 + \hat{\hat{x}})^2 < e_1(o_5 + e_4)(1 + \hat{\hat{x}})^2. \tag{19}$$

4. The characteristic equation about $\bar{\bar{E}}_{23}$ is given by

$$\lambda^3 + F_1\lambda^2 + F_2\lambda + F_3 = 0, \tag{20}$$

where,

$$\begin{aligned} F_1 &= N_{11} - N_{22}, \\ F_2 &= N_{12}N_{21} + N_{23}N_{32} - N_{11}N_{22}, \\ F_3 &= (N_{11}N_{23} - N_{13}N_{21})N_{32}, \end{aligned}$$

and

$$\begin{aligned} N_{11} &= \frac{e_1(o_5 + e_5)}{(e_4 + \bar{\bar{x}})^2} - \frac{u_1\bar{\bar{y}}}{(1 + \bar{\bar{x}})^2(1 + e_5\bar{\bar{z}})}, \\ N_{12} &= \frac{u_1\bar{\bar{x}}}{(1 + \bar{\bar{x}})(1 + e_5\bar{\bar{z}})}, \\ N_{13} &= \frac{u_1 e_5\bar{\bar{x}}\bar{\bar{y}}}{(1 + \bar{\bar{x}})(1 + e_5\bar{\bar{z}})^2}, \\ N_{21} &= \frac{o_6\bar{\bar{y}}}{(1 + \bar{\bar{x}})^2(1 + e_5\bar{\bar{z}})}, \\ N_{22} &= \frac{u_2\bar{\bar{y}}\bar{\bar{z}}}{(e_8 + \bar{\bar{y}})^2}, \\ N_{23} &= \bar{\bar{y}}(\frac{e_5 o_6\bar{\bar{x}}}{(1 + \bar{\bar{x}})(1 + e_5\bar{\bar{z}})^2} + \frac{u_2}{e_8 + \bar{\bar{y}}}), \\ N_{32} &= \frac{o_7 e_8\bar{\bar{z}}}{(e_8 + \bar{\bar{y}})^2}. \end{aligned}$$

According to Routh-Hurwitz criteria $\bar{\bar{E}}_{23}$ is locally asymptotically stable if $F_1 > 0$, $F_2 > 0$, $F_3 > 0$ and $F_1F_2 > F_3$. It is difficult to interpret the results in ecological terms, from these complicated expressions, however, numerical examples are taken and graphs are plotted to illustrate the dynamical behaviour of the system about equilibrium $\bar{\bar{E}}_{23}$.

We are now in a position to make an attempt to find out the conditions under which the system undergoes Hopf-bifurcation. For this purpose, we choose the parameter A_1 in the system (7) as bifurcation parameter as it plays a crucial role in Holling type II functional response which describes the predation of intermediate consumer. We shall now apply the Lius criteria, [13] to obtain the conditions for small amplitude periodic solution arising from Hopf-bifurcation.

As the equilibrium population densities are functions of A_1, the coefficients of the characteristic equation (20) are functions of the parameter A_1 and hence we can use the notation $F_i = F_i(A_1)$ for $i = 1, 2, 3$. Noting that the quantities $F_i s$ are smooth functions of the parameter A_1, we first state in our case, the definition of a simple Hopf-Bifurcation.

If a critical value $\bar{\bar{A}}_1$ of parameter A_1 be found such that (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_1(A_1) = u(A_1) + iv(A_1)$, $\lambda_2(A_1) = u(A_1) - iv(A_1) = \overline{\lambda_1}(A_1)$. These eigen values will become purely imaginary at $A_1 = \bar{\bar{A}}_1$, *i.e.*, $\lambda_1(\bar{\bar{A}}_1) = iv_0$, $\lambda_2(\bar{\bar{A}}_1) = -iv_0$, with $v(\bar{\bar{A}}_1) = v_0 > 0$, and the other eigenvalue remains real and negative; and (ii) the transversality condition,

$$dRe\lambda_i(\bar{\bar{A}}_1)/dA_1 \mid_{A_1=\bar{\bar{A}}_1} = du(A_1)/dA_1 \mid_{A_1=\bar{\bar{A}}_1} \neq 0,$$

is satisfied. Then we find at $A_1 = \bar{\bar{A}}_1$, a simple Hopf-bifurcation. Without knowing eigenvalues, [13] proved that (referring the result to the current case): if $F_1(A_1), F_3(A_1), \Delta(A_1) = F_1(A_1)F_2(A_1) - F_3(A_1)$ are smooth functions of the parameter 'A_1' in an open interval containing $\bar{\bar{A}}_1 \epsilon \Re^+$ such that following conditions hold:
(i_*) $F_1(\bar{\bar{A}}_1) > 0$, $\Delta(\bar{\bar{A}}_1) = 0$, $F_3(\bar{\bar{A}}_1) > 0$;
(ii_*) $d\Delta(A_1)/dA_1 \mid_{A_1=\bar{\bar{A}}_1} \neq 0$
then (i_*) and (ii_*) are equivalent to conditions (i) and (ii) for the occurrence of a simple Hopf-bifurcation at $A_1 = \bar{\bar{A}}_1$. Hence we can propose the following theorem.

Theorem 1. *If a critical value $\bar{\bar{A}}_1$ of parameter A_1 be found such that $F_1(\bar{\bar{A}}_1) > 0$, $F_3(\bar{\bar{A}}_1) > 0$ and $\Delta(\bar{\bar{A}}_1) = 0$ and further $\Delta' \neq 0$ (where prime denotes differentiation with respect to A_1) then system (7) undergoes Hopf-bifurcation around $\bar{\bar{E}}_{23}$.*

Theorem 2. *Let the following inequalities hold in the region Ω.*

$$\frac{u_1\bar{\bar{y}}}{\sigma_2}(1+e_5\theta_3) < \frac{e_1(e_4+o_5)}{\sigma_1}, \quad (21)$$

$$\frac{\theta_3 o_6}{\sigma_2}(1+\bar{\bar{x}})(1+e_5\bar{\bar{z}}) < (1+\frac{u_2e_8\theta_1}{\sigma_3}), \quad (22)$$

$$o_7\theta_3(e_8+\bar{\bar{y}}) < u_2\sigma_3, \quad (23)$$

$$2[(1+\bar{\bar{x}})\frac{u_1e_5\bar{\bar{y}}}{\sigma_2}]^2 < G_{11}(u_2-\frac{o_7\theta_3}{\sigma_3}(e_8+\bar{\bar{y}})), \quad (24)$$

$$2G_{23}^2 < G_{22}(u_2-\frac{o_7\theta_3}{\sigma_3}(e_8+\bar{\bar{y}})), \quad (25)$$

where,

$$G_1 > \frac{u_1(1+\bar{\bar{x}})}{o_6\bar{\bar{y}}}, \quad (26)$$

$$G_{11} = \frac{e_1(e_4+o_5)}{\sigma_1} - \frac{u_1\bar{\bar{y}}}{\sigma_2}(1+e_5\theta_3),$$

$$G_{22} = G_1\left((1+\frac{u_2e_8\theta_1}{\sigma_3}) - \frac{\theta_3o_6}{\sigma_2}(1+\bar{\bar{x}})(1+e_5\bar{\bar{z}})\right),$$

$$G_{23} = G_1\left((1+\bar{\bar{x}})\frac{e_5o_6\bar{\bar{x}}\bar{\bar{y}}}{\sigma_2} + (e_8+y)\frac{u_2\bar{\bar{y}}}{\sigma_3}\right) - \frac{o_7e_8\bar{\bar{z}}}{\sigma_3},$$

$$\sigma_1 = (e_4+\theta_1)(e_4+\bar{\bar{x}}), \quad \sigma_2 = (1+\theta_1)(1+\bar{\bar{x}})(1+e_5\theta_1)(1+e_5\bar{\bar{z}}),$$

$$\sigma_3 = (e_8+\theta_1)(e_8+\bar{\bar{y}}),$$

then the positive equilibrium $\bar{\bar{E}}_{23}$ is globally asymptotically stable with respect to all solutions initiating in the interior of positive region Ω.

Proof. We consider the following positive definite function about $\bar{\bar{E}}_{23}$:

$$V_2 = (x-\bar{\bar{x}}-\bar{\bar{x}}ln(x/\bar{\bar{x}})) + (G_1/2)(y-\bar{\bar{y}})^2 + (1/2)(z-\bar{\bar{z}})^2.$$

Differentiating V_2 with respect to time t, we get

$$\dot{V}_2 = (x-\bar{\bar{x}}/x)(dx/dt) + G_1(y-\bar{\bar{y}})(dy/dt) + (z-\bar{\bar{z}})(dz/dt).$$

Using system of equations (8)-(10), we get after some algebraic manipulations

$$\begin{aligned}
\dot{V_2} &= -(x-\bar{\bar{x}})^2\left(\frac{e_1(e_4+o_5)}{\sigma_1}-\frac{u_1\bar{\bar{y}}}{\sigma_2}(1+e_5z)\right)\\
&\quad -(y-\bar{\bar{y}})^2G_1\left((1+\frac{u_2e_8z}{\sigma_3})-\frac{xo_6}{\sigma_2}(1+\bar{\bar{x}})(1+e_5\bar{\bar{z}})\right)\\
&\quad -(z-\bar{\bar{z}})^2\left(u_2-\frac{o_7y}{\sigma_3}(e_8+\bar{\bar{y}})\right)+(x-\bar{\bar{x}})(z-\bar{\bar{z}})(1+\bar{\bar{x}})\frac{u_1e_5\bar{\bar{y}}}{\sigma_2}\\
&\quad +(x-\bar{\bar{x}})(y-\bar{\bar{y}})\left(\frac{1+e_5\bar{\bar{z}}}{\sigma_2}\right)(G_1o_6\bar{\bar{y}}-u_1(1+\bar{\bar{x}}))\\
&\quad -(y-\bar{\bar{y}})(z-\bar{\bar{z}})\left(G_1\left((1+\bar{\bar{x}})\frac{e_5o_6\bar{\bar{x}}\bar{\bar{y}}}{\sigma_2}+(e_8+y)\frac{u_2\bar{\bar{y}}}{\sigma_3}\right)-\frac{o_7e_8\bar{\bar{z}}}{\sigma_3}\right).
\end{aligned}$$

Now, $\dot{V_2}$ can further be written as sum of the quadratic forms as

$$\begin{aligned}
\dot{V_2} &\leq -[((a_{11}/2)(x-\bar{\bar{x}})^2-a_{13}(x-\bar{\bar{x}})(z-\bar{\bar{z}})+(a_{22}/2)(z-\bar{\bar{z}})^2)\\
&\quad +((a_{22}/2)(y-\bar{\bar{y}})^2+a_{23}(y-\bar{\bar{y}})(z-\bar{\bar{z}})+(a_{33}/2)(z-\bar{\bar{z}})^2)],
\end{aligned}$$

where,

$$\begin{aligned}
a_{11} &= \frac{e_1(e_4+o_5)}{\sigma_1}-\frac{u_1\bar{\bar{y}}}{\sigma_2}(1+e_5z),\\
a_{22} &= G_1\left((1+\frac{u_2e_8z}{\sigma_3})-\frac{xo_6}{\sigma_2}(1+\bar{\bar{x}})(1+e_5\bar{\bar{z}})\right),\\
a_{13} &= (1+\bar{\bar{x}})\frac{u_1e_5\bar{\bar{y}}}{\sigma_2}, a_{23}=G_1\left((1+\bar{\bar{x}})\frac{e_5o_6\bar{\bar{x}}\bar{\bar{y}}}{\sigma_2}+(e_8+y)\frac{u_2\bar{\bar{y}}}{\sigma_3}\right)-\frac{o_7e_8\bar{\bar{z}}}{\sigma_3},\\
a_{33} &= \left(u_2-\frac{o_7y}{\sigma_3}(e_8+\bar{\bar{y}})\right)/2, \sigma_1=(e_4+x)(e_4+\bar{\bar{x}}),\\
\sigma_2 &= (1+x)(1+\bar{\bar{x}})(1+e_5z)(1+e_5\bar{\bar{z}}), \sigma_3=(e_8+y)(e_8+\bar{\bar{y}}).
\end{aligned}$$

Now, by using Sylvesters criteria and by choosing $G_1=\frac{u_1(1+\bar{\bar{x}})}{o_6\bar{\bar{y}}}>0$, we get that $\dot{V_2}$ is negative definite under the following conditions:

$$a_{11}>0, \tag{27}$$
$$a_{22}>0, \tag{28}$$
$$a_{33}>0, \tag{29}$$
$$a_{11}a_{22}>a_{12}^2, \tag{30}$$
$$a_{11}a_{33}>a_{13}^2, \tag{31}$$
$$a_{22}a_{33}>a_{23}^2. \tag{32}$$

We note that the fourth inequality, *i.e.*, $a_{11}a_{22} > a_{12}^2$ is satisfied due to the proper choice of G_1, and other inequalities, (21) $\Rightarrow$ (27), (22) $\Rightarrow$ (28), (23) $\Rightarrow$ (29), (24) $\Rightarrow$ (31) and (25) $\Rightarrow$ (32). Hence V_2 is a Lyapunov function with respect to $\bar{\bar{E}}_{23}$, whose domain contains the region of attraction Ω, proving the theorem. □

5 Analysis of Model 1

5.1 Equilibria of Model 1:

The *Model* 1 has following four non-negative equilibria in x, y, z, c_o and c_e space namely, $E_{10}(0,0,0,0,u_0)$, $\tilde{E_{11}}(\tilde{x},0,0,\tilde{c_o},\tilde{c_e})$, $\hat{E_{12}}(\hat{x},\hat{y},0,\hat{c_o},\hat{c_e})$ and $\bar{E_{13}}(\bar{x},\bar{y},\bar{z},\bar{c_o},\bar{c_e})$. The existence of E_{10} is obvious. We prove the existence of $\tilde{E_{11}}$, $\hat{E_{12}}$ and $\bar{E_{13}}$ as follows:

1. **Existence of** $\tilde{E_{11}}(\tilde{x},0,0,\tilde{c_o},\tilde{c_e})$,
   from (2),
   $$\tilde{x} = e_2e_3 > 0, \tag{33}$$
   from (5) and (6),
   $$\tilde{c_e} = \frac{u_0u_4 + e_2e_3o_3o_4}{(e_2e_3u_5 + u_6)u_4 - o_2e_2e_3o_4}, \tag{34}$$
   $\tilde{c_e} > 0$ if $(e_2e_3u_5 + u_6)u_4 > o_2e_2e_3o_4$,
   $$\tilde{c_o} = \frac{o_2}{u_4}\left(\frac{u_0u_4 + e_2e_3o_3o_4}{(e_2e_3u_5 + u_6)u_4 - o_2e_2e_3o_4}\right) + \frac{o_3}{u_4} > 0. \tag{35}$$

2. **Existence of** $\hat{E_{12}}(\hat{x},\hat{y},0,\hat{c_o},\hat{c_e})$,
   from (3),
   $$\hat{x} = \frac{1}{o_6 - 1}, \tag{36}$$
   $\hat{x} > 0$ if $o_6 > 1$,
   from (5) and (6),
   $$\hat{c_e} = \frac{u_0u_4 + \hat{x}o_3o_4}{(\hat{x}u_5 + u_6)u_4 - o_2o_4\hat{x}}, \tag{37}$$
   $\hat{c_e} > 0$ if $\hat{x}u_5 + u_6)u_4 > o_2o_4\hat{x}$,
   $$\hat{c_o} = \frac{o_2}{u_4}\left(\frac{u_0u_4 + \hat{x}o_3o_4}{(\hat{x}u_5 + u_6)u_4 - o_2o_4\hat{x}}\right) + \frac{o_3}{u_4} > 0, \tag{38}$$

from (2),

$$\hat{y} = (1+\hat{x})\frac{e_1}{u_1}\left(\frac{e_2(e_3-\hat{c_0})-\hat{x}}{e_4+\hat{x}}\right), \tag{39}$$

$\hat{y} > 0$ if $e_3 > \hat{c_0}$ and $e_2(e_3-\hat{c_0}) > \hat{x}$.

3. **Existence of** $\bar{E_{13}}(\bar{x},\bar{y},\bar{z},\bar{c_o},\bar{c_e})$.
Now, we show the existence of $\bar{E_{13}}$ as follows:
Here $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{c_0}$ and $\bar{c_e}$ are the positive solutions of the system of algebraic equations given below:
from (4),

$$\bar{y} = \frac{u_3 e_8}{o_7 - u_3}, \tag{40}$$

$\bar{y} > 0$ if $o_7 > u_3$,
from (3),

$$x = \frac{(1+e_5 z)(1+\frac{u_2 z}{e_8+\bar{y}})}{o_6-(1+e_5 z)(1+\frac{u_2 z}{e_8+\bar{y}})} = g_1(z), \tag{41}$$

from (2),

$$c_o = e_3 - \frac{1}{e_2}\left(g_1(z) + \frac{u_1\bar{y}(e_4+g_1(z))}{e_1(1+g_1(z))(1+e_5 z)}\right) = g_2(z), \tag{42}$$

from (5),

$$c_e = \frac{1}{o_2}[u_4 g_2(z) - o_3] = g_3(z). \tag{43}$$

Let

$$Q(z) = u_0 + o_4 g_1(z) g_2(z) - (u_6 + u_5 g_1(z)) g_3(z). \tag{44}$$

To show the existence of $\bar{E_{13}}$, it suffices to show that equation (44) has a unique positive solution for this we may note that

$$Q(0) = u_0 + o_4 g_1(0) g_2(0) - (u_6 + u_5 g_1(0)) g_3(0) > 0, \tag{45}$$

$$Q(k_0) = u_0 + o_4 g_1(k_0) g_2(k_0) - (u_6 + u_5 g_1(k_0)) g_3(k_0) < 0. \tag{46}$$

This guarantees the existence of a root of $Q(z) = 0$ for $0 < z < k_0$, say $\bar{z}$. Further, this root will be unique provided

$$\begin{aligned} Q'(z) &= o_4[g_1(z)g_2'(z) + g_1'(z)g_2(z)] \\ &\quad -[(u_6+u_5 g_1(z))g_3'(z) + u_5 g_1'(z) g_3(z)] < 0. \end{aligned}$$

Knowing the value of $\bar{z}$, the values of $\bar{x}$, $\bar{c_0}$ and $\bar{c_e}$ can be computed from equations (41) to (43) respectively.

5.2 Dynamical behaviour of Model 1:

The general variational matrix corresponding to the $Model$ 1 is as follows:

$$J_1(x,y,z,c_0,c_e) = \begin{bmatrix} -v_{11} & -v_{12} & v_{13} & -v_{14} & 0 \\ v_{21} & v_{22} & -v_{23} & v_{24} & 0 \\ 0 & v_{32} & v_{33} & v_{34} & 0 \\ 0 & 0 & 0 & -v_{44} & v_{45} \\ -v_{51} & 0 & 0 & v_{54} & -v_{55} \end{bmatrix},$$

where,

$$\begin{aligned} v_{11} &= \frac{u_1 y}{(1+x)^2(1+e_5 z)} + \frac{e_1 x}{(e_4+x)^2}(e_2 e_0(c_0)+e_4) + e_1\left(\frac{r_1(C_0)e_2 - x}{e_4+x}\right), \\ v_{12} &= \frac{u_1 x}{(1+x)(1+e_5 z)}, \quad v_{13} = \frac{u_1 e_5 x y}{(1+x)^2(1+e_5 z)^2}, \quad v_{14} = \frac{x e_1 e_2}{e_4+x}, \\ v_{21} &= \frac{o_6 y}{(1+x)^2(1+e_5 z)}, \quad v_{22} = \frac{o_6 y}{(1+x)^2(1+e_5 z)} - \frac{e_8 u_2 z}{(e_8+y)^2} - 1, \\ v_{23} &= \frac{u_2 y}{(e_8+y)}, \quad v_{32} = \frac{o_7 e_8 z}{(e_8+y)^2}, \quad v_{33} = \frac{o_7 y}{e_8+y} - u_3, \quad v_{44} = u_4, v_{45} = o_2, \\ v_{51} &= (u_5 c_e - o_4 c_o), \quad v_{54} = o_4 x, \quad v_{55} = (u_5 x + u_6). \end{aligned}$$

1. At E_{10}, the eigenvalues of the characteristic equation are $e_1e_2e_3, -1$, $-u_3$, $-u_4$ and $-u_6$, showing the instability of E_{10}.

2. At $\tilde{E_{11}}$, two eigenvalues of the characteristic equation are, $o_6\tilde{x}/(1+\tilde{x}) - 1$, $-u_3$ and the other three eigenvalues are given by the roots of the following cubic equation

$$\lambda^3 + \lambda^2 S_1 + \lambda S_2 + S_3 = 0, \tag{47}$$

where,

$$\begin{aligned} S_1 &= u_4 + u_5\tilde{x} + u_6 + \frac{\tilde{x}e_1(e_2e_0(\tilde{c_0})+e_4)}{(e_4+\tilde{x})^2}, \\ S_2 &= \frac{\tilde{x}e_1(e_2e_0(\tilde{c_0})+e_4)}{(e_4+\tilde{x})^2}(u_4+u_5\tilde{x}+u_6) + u_4(u_5\tilde{x}+u_6) - o_2o_4\tilde{x}, \\ S_3 &= u_4(u_5\tilde{x}+u_6)\frac{\tilde{x}e_1(e_2e_0(\tilde{c_0})+e_4)}{(e_4+\tilde{x})^2} - o_2((u_5\tilde{c_e} - o_4\tilde{c_o})\frac{\tilde{x}e_1e_2}{e_4+\tilde{x}} \\ &\quad + o_2\tilde{x}^2\frac{e_1(e_2e_0(\tilde{c_0})+e_4)}{(e_4+\tilde{x})^2}). \end{aligned}$$

According to Routh-Hurwitz criteria $\tilde{E_{11}}$ is locally asymptotically stable if

$$o_6e_2e_3 < (1+e_2e_3), o_2o_4 < u_2u_5,$$

and $S_1S_2 - S_3 > 0$.

3. At $\hat{E_{12}}$, one of the eigenvalues of the characteristic equation is $(o_7\hat{y})/(e_8+\hat{y}) - u_3$ and the other four eigenvalues are given by the roots of the following equation is

$$\lambda^4 + \lambda^3 T_1 + \lambda^2 T_2 + \lambda T_3 + T_4 = 0, \tag{48}$$

where,

$$\begin{aligned}
T_1 &= u_4 + u_5\hat{x} + u_6 + \frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_0}) + e_4) - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2},\\
T_2 &= \frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3} + (u_4 + u_5\hat{x} + u_6)(\frac{e_1\hat{x}(e_2e_0(\hat{c_0}) + e_4)}{(e_4+\hat{x})^2} - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2})\\
&\quad + u_4(u_5\hat{x} + u_6) - o_2o_4\hat{x},\\
T_3 &= (u_4 + u_5\hat{x} + u_6)\frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3} + u_4(u_5\hat{x} + u_6)(\frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_0}) + e_4)\\
&\quad - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2}) - o_2[(u_5\hat{c_e} - o_4\hat{c_o})\frac{\hat{x}e_1e_2}{e_4+\hat{x}}\\
&\quad + o_4\hat{x}(\frac{e_1\hat{x}}{(e_4+\hat{x})^2}(e_2e_0(\hat{c_0}) + e_4) - \frac{u_1\hat{x}\hat{y}}{(1+\hat{x})^2})],\\
T_4 &= (u_4(u_5\hat{x} + u_6) - o_2o_4\hat{x})\frac{u_1o_6\hat{x}\hat{y}}{(1+\hat{x})^3}.
\end{aligned}$$

From the Routh-Hurwitz criteria it is found that $\hat{E_{12}}$ is locally asymptotically stable if the following conditions hold good.

$$\hat{y} < \frac{u_3e_8}{o_7 - u_3}, o_2o_4 < u_4u_5,$$

$T_i > 0$, $i = 1, 2, 3, 4$, $T_1T_2 > T_3$ and $T_1T_2T_3 > (T_3^2 + T_1^2T_4)$.

4. The characteristic equation of $\bar{E_{13}}$ is as follows:

$$\lambda^5 + \lambda^4 W_1 + \lambda^3 W_2 + \lambda^2 W_3 + \lambda W_4 + W_5 = 0, \tag{49}$$

where,

$$
\begin{aligned}
W_1 &= w_1 + w_9 + w_{13} - w_6, \\
W_2 &= w_2w_5 - w_1w_6 + w_7w_8 + w_1w_9 - w_6w_9 - w_{10}w_{12} + w_1w_{13} \\
&\quad -w_6w_{13} + w_9w_{13}, \\
W_3 &= w_1w_7w_8 - w_3w_5w_8 + w_2w_5w_9 - w_1w_6w_9 + w_7w_8w_9 \\
&\quad -w_4w_{10}w_{11} - w_1w_{10}w_{12} + w_6w_{10}w_{12} + w_2w_5w_{13} - w_1w_6w_{13} \\
&\quad +w_7w_8w_{13} + w_1w_9w_{13} - w_6w_9w_{13}, \\
W_4 &= w_1w_7w_8w_9 - w_3w_5w_8w_9 + w_4w_6w_{10}w_{11} - w_2w_5w_{10}w_{12} \\
&\quad +w_1w_6w_{10}w_{12} - w_7w_8w_{10}w_{12} - w_3w_5w_8w_{13} + w_1w_7w_8w_{13} \\
&\quad +w_2w_5w_9w_{13} - w_1w_6w_9w_{13} + w_7w_8w_9w_{13}, \\
W_5 &= w_3w_5w_8w_{10}w_{12} - w_4w_7w_8w_{10}w_{11} - w_1w_7w_8w_{10}w_{12} \\
&\quad -w_3w_5w_8w_9w_{13} + w_1w_7w_8w_9w_{13},
\end{aligned}
$$

and

$$
\begin{aligned}
w_1 &= \frac{e_1\bar{x}}{(e_4+\bar{x})^2}(e_2e_0(\bar{c_0}) + e_4) - \frac{u_1\bar{x}\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})}, \\
w_2 &= \frac{u_1\bar{x}}{(1+\bar{x})(1+e_5\bar{z})}, \quad w_3 = \frac{u_1e_5\bar{x}\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})^2}, \quad w_4 = \frac{\bar{x}e_1e_2}{e_4+\bar{x}}, \\
w_5 &= \frac{o_6\bar{y}}{(1+\bar{x})^2(1+e_5\bar{z})}, \quad w_6 = \frac{u_2\bar{y}\bar{z}}{(e_8+\bar{y})^2}, \quad w_7 = \frac{u_2\bar{y}}{(e_8+\bar{y})}, \\
w_8 &= \frac{o_7e_8\bar{z}}{(e_8+\bar{y})^2}, \quad w_9 = u_4, \quad w_{10} = o_2, \quad w_{11} = u_5\bar{c_e} - o_4\bar{c_o}, \\
w_{12} &= o_4\bar{x}, \quad w_{13} = u_5\bar{x} + u_6.
\end{aligned}
$$

According to Routh-Hurwitz criteria, the equilibrium point $\bar{E_{13}}$ is locally asymptotically stable if
$W_i > 0 (i = 1,2,3,4,5)$, $W_1W_2 > W_3$, $W_1W_2W_3 > (W_3^2 + W_1^2W_4)$ and $(W_3W_4 - W_2W_5)(W_1W_2 - W_3) > (W_1W_4 - W_5)^2$.
It is difficult to interpret the results in ecological terms from these complicated expressions, however, numerical examples are taken and graphs are ploted to illustrate the dynamical behaviour of the system about equilibrium $\bar{E_{13}}$.

Again, in the similar way the equilibrium population densities are functions of A_1 and the coefficients of the characteristic equation (49) are functions of the parameter A_1. Now we can use the notation $W_i = W_i(A_1)$ for $i = 1,2,3,4,5$. Now noting that the quantities W_is are smooth functions of the parameter A_1. As we have explained the definition of Hopf-bifurcation in previous section. Without knowing eigenvalues, [13] proved that (referring the result to the current case): if $W_i(A_1)$,

$$\begin{aligned}
\Delta_1(A_1) &= W_1(A_1)W_2(A_1) - W_3(A_1),\\
\Delta_2(A_1) &= W_1(A_1)W_2(A_1)W_3(A_1) - (W_3^2(A_1) + W_1^2(A_1)W_4(A_1)),\\
\Delta_3(A_1) &= [W_3(A_1)W_4(A_1) - W_2(A_1)W_5(A_1)][W_1(A_1)W_2(A_1) - W_3(A_1)]\\
&\quad -[W_1(A_1)W_4(A_1) - W_5(A_1)]^2
\end{aligned}$$

are smooth functions of the parameter 'A_1' in an open interval containing $\bar{A}_1 \epsilon \Re^+$ such that following conditions hold:

$$\begin{aligned}
&(iii_*) \quad W_1(\bar{A}_1) > 0, \Delta_1(\bar{A}_1) > 0, \Delta_2(\bar{A}_1) > 0 and \Delta_3(\bar{A}_1) = 0;\\
&(iv_*) \quad d\Delta_3(A_1)/dA_1 \mid_{A_1=\bar{A}_1} \neq 0
\end{aligned}$$

then (iii_*) and (iv_*) are equivalent to conditions (i) and (ii) mentioned in *section* 4.2, for the occurrence of a simple Hopf-bifurcation at $A_1 = \bar{A}_1$. Hence, in the similar way, we can propose the following theorem:

Theorem 3. *If a critical value $\bar{A}_1$ of parameter A_1 be found such that $W_i(\bar{A}_1) > 0$, $\Delta_1(\bar{A}_1) > 0$, $\Delta_2(\bar{A}_1) > 0$, $\Delta_3(\bar{A}_1) = 0$ and further $\Delta_3' \neq 0$ (where primes denotes differentiation with respect to A_1) then system (1) undergoes Hopf-bifurcation around $\bar{E_{13}}$.*

Theorem 4. *Let the following inequalities hold in the region Ω:*

$$\frac{u_1\bar{y}}{\sigma_{12}}(1 + e_5\theta_3) < \frac{e_1}{\sigma_{11}}(e_4 + e_2e_0(\bar{c_o})), \tag{50}$$

$$\frac{\theta_3 o_6}{\sigma_{12}}(1 + \bar{x})(1 + e_5\bar{z}) < (1 + \frac{u_2e_8\theta_1}{\sigma_{13}}), \tag{51}$$

$$o_7\theta_3(e_8 + \bar{y}) < u_2\sigma_{13}, \tag{52}$$

$$6[\frac{u_1e_5\bar{y}}{\sigma_{11}}(1 + \bar{x})]^2 < P_{12}\left(u_2 - \frac{o_7\theta_3}{\sigma_{13}}(e_8 + \bar{y})\right), \tag{53}$$

$$6[\frac{e_1e_2}{\sigma_{11}}(e_4 + \bar{x})]^2 < P_{12}u_4, \tag{54}$$

$$6(u_5\bar{c_e} - o_4\theta_2)^2 < P_{12}(u_6 + \theta_1u_5), \tag{55}$$

$$2P_{11}^2 < P_{13}\left(u_2 - \frac{o_7\theta_3}{\sigma_{13}}(e_8 + \bar{y})\right), \tag{56}$$

$$4[o_2 + o_4\bar{x}]^2 < u_4(u_6 + \theta_1u_5), \tag{57}$$

where,

$$K_1 > \frac{u_1(1 + \bar{x})}{o_6\bar{y}}, \tag{58}$$

$$
\begin{aligned}
P_{11} &= K_1(\frac{e_5 o_6 \bar{x}\bar{y}}{\sigma_{12}}(1+\theta_3) + \frac{u_2\bar{y}}{\sigma_{13}}(e_8+\theta_3)) - \frac{o_7 e_8 \bar{z}}{\sigma_{13}},\\
P_{12} &= \frac{e_1}{\sigma_{11}}(e_4 + e_2 e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1+e_5\theta_3),\\
P_{13} &= K_1\left((1+\frac{u_2 e_8 \theta_1}{\sigma_{13}}) - \frac{\theta_3 o_6}{\sigma_{12}}(1+\bar{x})(1+e_5\bar{z})\right),\\
\sigma_{11} &= (e_4+\theta_1)(e_4+\bar{x}), \sigma_{12} = (1+\theta_1)(1+\bar{x})(1+e_5\theta_1)(1+e_5\bar{z}),\\
\sigma_{13} &= (e_8+\theta_1)(e_8+\bar{y}),
\end{aligned}
$$

then the positive equilibrium $\bar{E_{13}}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive region Ω.

Proof. We consider the following positive definite function about $\bar{E_{13}}$:

$$
\begin{aligned}
V_1 &= (x-\bar{x}-\bar{x}ln(x/\bar{x})) + (K_1/2)(y-\bar{y})^2 + (K_2/2)(z-\bar{z})^2\\
&\quad +(K_3/2)(c_0-\bar{c_0})^2 + (K_4/2)(c_e-\bar{c_e})^2.\\
\dot{V_1} &= (x-x/x)(dx/dt) + K_1(y-y)(dy/dt) + K_2(z-z)(dz/dt)\\
&\quad +K_3(C_0-C_0)(dC_0/dt) + K_4(C_E-C_E)(dC_E/dt).
\end{aligned}
$$

Using system of equations (2)-(6), we get after some algebraic manipulations

$$
\begin{aligned}
\dot{V_1} &= -(x-\bar{x})^2\left(\frac{e_1}{\sigma_{11}}(e_4+e_2e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1+e_5 z)\right)\\
&\quad -(y-\bar{y})^2 K_1\left((1+\frac{u_2 e_8 z}{\sigma_{13}}) - \frac{x o_6}{\sigma_{12}}(1+\bar{x})(1+e_5\bar{z})\right)\\
&\quad -(z-\bar{z})^2 K_2\left(u_2 - \frac{o_7 y}{\sigma_{13}}(e_8+\bar{y})\right) - (c_e-\bar{c_e})^2 K_4(u_6+xu_5)\\
&\quad -(c_o-\bar{c_o})^2 K_3 u_4 - (x-\bar{x})(y-\bar{y})\frac{(1+e_5\bar{z})}{\sigma_{12}}(u_1(1+\bar{x}) - K_1 o_6\bar{y})\\
&\quad +(x-\bar{x})(z-\bar{z})\frac{u_1 e_5\bar{y}}{\sigma_{11}}(1+\bar{x}) - (x-\bar{x})(c_o-\bar{c_o})\frac{e_1e_2}{\sigma_{11}}(e_4+\bar{x})\\
&\quad -(y-\bar{y})(z-\bar{z})\left(K_1(\frac{e_5 o_6\bar{x}\bar{y}}{\sigma_{12}}(1+x) + \frac{u_2\bar{y}}{\sigma_{13}}(e_8+y)) - K_2\frac{o_7 e_8\bar{z}}{\sigma_{13}}\right)\\
&\quad -(x-\bar{x})(c_e-\bar{c_e})K_4(u_5\bar{c_e} - o_4 c_o) + (c_o-\bar{c_o})(c_e-\bar{c_e})(o_2K_3 + K_4 o_4\bar{x}).
\end{aligned}
$$

Now, $\dot{V_1}$ can further be written as sum of the quadratic forms:

$$
\begin{aligned}
\dot{V_1} \leq &-[\big((b_{11}/2)(x-\bar{x})^2 - b_{13}(x-\bar{x})(z-\bar{z}) + (b_{33}/2)(z-\bar{z})^2\big)\\
&+\big((b_{11}/2)(x-\bar{x})^2 + b_{14}(x-\bar{x})(c_o-\bar{c_o}) + (b_{44}/2)(c_o-\bar{c_o})^2\big)\\
&+\big((b_{11}/2)(x-\bar{x})^2 + b_{15}(x-\bar{x})(c_e-\bar{c_e}) + (b_{55}/2)(c_e-\bar{c_e})^2\big)\\
&+\big((b_{22}/2)(y-\bar{y})^2 + b_{23}(y-\bar{y})(z-\bar{z}) + (b_{33}/2)(z-\bar{z})^2\big)\\
&+\big((b_{44}/2)(c_o-\bar{c_o})^2 - b_{45}(c_o-\bar{c_o})(c_e-\bar{c_e}) + (b_{55}/2)(c_e-\bar{c_e})^2\big)],
\end{aligned}
$$

where,

$$
\begin{aligned}
b_{11} &= \frac{1}{3}\left(\frac{e_1}{\sigma_{11}}(e_4 + e_2 e_0(\bar{c_o})) - \frac{u_1\bar{y}}{\sigma_{12}}(1+e_5 z)\right), b_{13} = \frac{u_1 e_5 \bar{y}}{\sigma_{11}}(1+\bar{x}),\\
b_{14} &= \frac{e_1 e_2}{\sigma_{11}}(e_4+\bar{x}), b_{15} = (u_5\bar{c_e} - o_4 c_o),\\
b_{22} &= K_1\left((1+\frac{u_2 e_8 z}{\sigma_{13}}) - \frac{x o_6}{\sigma_{12}}(1+\bar{x})(1+e_5\bar{z})\right),\\
b_{23} &= K_1(\frac{e_5 o_6 \bar{x}\bar{y}}{\sigma_{12}}(1+x) + \frac{u_2\bar{y}}{\sigma_{13}}(e_8+y)) - \frac{o_7 e_8 \bar{z}}{\sigma_{13}},\\
b_{33} &= \frac{1}{2}\left(u_2 - \frac{o_7 y}{\sigma_{13}}(e_8+\bar{y})\right), b_{44} = \frac{u_4}{2}, b_{45} = o_2 + o_4\bar{x}, b_{55} = \frac{1}{2}(u_6 + x u_5),\\
\sigma_{11} &= (e_4+x)(e_4+\bar{x}), \sigma_{12} = (1+x)(1+\bar{x})(1+e_5 z)(1+e_5\bar{z}),\\
\sigma_{13} &= (e_8+y)(e_8+\bar{y}), e_0(\bar{c_o}) = e_3 - \bar{c_o}.
\end{aligned}
$$

Now, by using Sylvesters criteria and by choosing $K_1 = \frac{u_1(1+\bar{x})}{o_6\bar{y}} > 0$ and $K_2 = K_3 = K_4 = 1$, we get that $\dot{V_1}$ is negative definite under the following conditions:

$$b_{11} > 0, \tag{59}$$
$$b_{22} > 0, \tag{60}$$
$$b_{33} > 0, \tag{61}$$
$$b_{11}b_{22} > b_{12}^2, \tag{62}$$
$$b_{11}b_{33} > b_{13}^2, \tag{63}$$
$$b_{11}b_{44} > b_{14}^2, \tag{64}$$
$$b_{11}b_{55} > b_{15}^2, \tag{65}$$
$$b_{22}b_{33} > b_{23}^2, \tag{66}$$
$$b_{44}b_{55} > b_{45}^2. \tag{67}$$

We note that the fourth inequality, *i.e.*, $b_{11}b_{22} > b_{12}^2$ is satisfied due to the proper choice of K_1 and other inequalities, (50) ⇒ (59), (51) ⇒ (60), (52) ⇒ (61), (53) ⇒ (63), (54) ⇒ (64), (55) ⇒ (65), (56) ⇒ (66) and (57) ⇒ (67). Hence V_1 is a Lyapunov function with respect to $\bar{E_{13}}$, whose domain contains the region of attraction Ω, proving the theorem. □

6 Numerical Example

In this section, we demonstrate the dynamical behavior of a three species food chain system with toxicant and without toxicant with the help of numerical examples.

6.1 Numerical Example for Model 2:

We choose the following values of parameters for $\tilde{\tilde{E}}_{21}$:

$$r_0 = 0.41, \quad c = 0.1, \quad A_1 = 1.5, \quad B_1 = 43.1, \quad \beta_{10} = 0.6, \quad D_1 = 0.9,$$
$$B = 153.3, \quad a = 0.5, \quad A_2 = 0.5, \quad B_2 = 8.9, \quad \beta_{20} = 0.1, \quad D_2 = 0.01.$$

It is found that under the above set of parameters, the equilibrium point $\tilde{\tilde{E}}_{21} = (12.5706, 0, 0)$ is locally asymptotically stable (see Fig. 1).

We choose the following values of parameters for $\hat{\hat{E}}_{22}$:

$$B = 130.3, \quad A_1 = 0.8, \quad B_1 = 5.1, \quad \beta_{10} = 0.003,$$
$$D_1 = 0.001, \quad A_2 = 0.003, \quad B_2 = 6.5, \quad \beta_{20} = 0.02.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{\tilde{E}}_{21}$, it is found under the above set of parameters that the equilibrium $\hat{\hat{E}}_{22} = (3.6744, 1.2250, 0)$ is locally asymptotically stable (see Fig. 2).

We choose the following values of parameters for $\bar{\bar{E}}_{23}$:

$$D_1 = 0.001, A_1 = 1.35, A_2 = 0.501, B_2 = 12.0.$$

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{\tilde{E}}_{21}$, it is found that the interior equilibrium $\bar{\bar{E}}_{23} = (9.5375, 2.2163, 1.3688)$ is locally asymptotically stable (see Figs. 3 and 4).

Now, we study the Hopf-bifurcation of the Model 2, taking A_1 as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_1 = \bar{\bar{A}}_1 = 0.6377$. It is clear that the interior equilibrium point $\bar{\bar{E}}_{23}$ of Model 2 is stable when $A_1 > \bar{\bar{A}}_1$ and unstable when $A_1 \leq \bar{\bar{A}}_1$ for which Hopf-bifurcation occurs (see Figs. 5 and 6).

6.2 Numerical Example for *Model* 1:

We choose the following values of parameters for $\tilde{E}_{11}$:

$$r_1 = 0.081, \quad a_1 = 1.61, \quad d_1 = 0.01, \quad \beta = 0.8, \quad \beta_{10} = 0.6, \quad \beta_{20} = 0.1,$$
$$\theta = 0.31, \quad l_1 = 1.2, \quad l_2 = 0.599, \quad g_0 = 0.2, \quad k_1 = 0.1, \quad k_2 = 0.11,$$
$$r_0 = 0.41, \quad B = 153.3, \quad c = 0.1, \quad a = 0.5, \quad A_1 = 1.5, \quad B_1 = 43.1,$$
$$D_1 = 0.9, \quad A_2 = 0.5, \quad B_2 = 8.9, \quad D_2 = 0.01.$$

It is found that under the above set of parameters, the equilibrium point $\tilde{E}_{11} = (11.9736, 0.0000, 0.0000, 0.2401, 0.2679)$ is locally asymptotically stable (see Fig. 7).

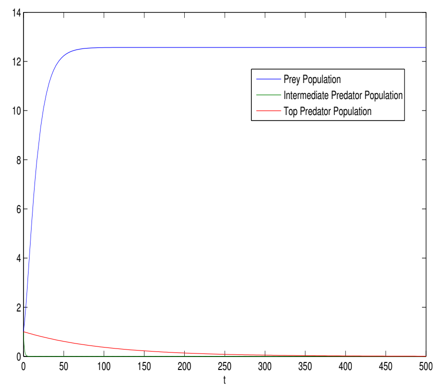

Figure 1: Time graph for the Model 2, around the equilibrium point $\tilde{\tilde{E}}_{21}$, showing the stability behavior.

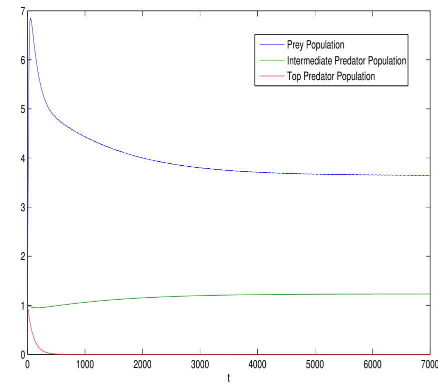

Figure 2: Time graph for the Model 2, around the equilibrium point $\hat{\hat{E}}_{22}$, showing the stability behavior.

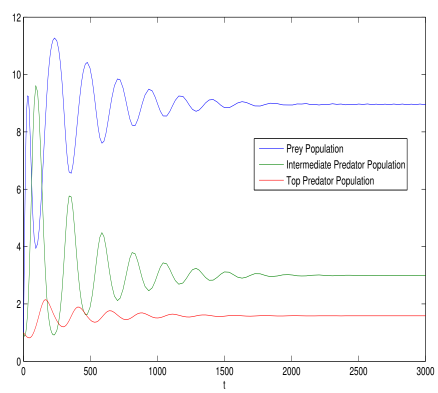

Figure 3: Time graph for the Model 2, around the equilibrium point $\bar{\bar{E}}_{23}$, showing the stability behavior.

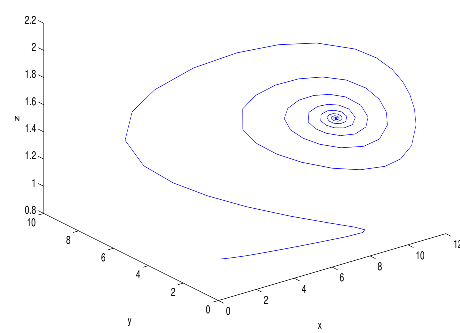

Figure 4: Phase graph for the Model 2, around the equilibrium point $\bar{\bar{E}}_{23}$, showing the stability behavior.

Now, we choose the following values of parameters for $\hat{E}_{12}$:

$$r_1 = 0.02, \quad \beta_{10} = 0.003, \quad \beta_{20} = 0.02 \quad a_1 = 1.6, \quad B = 140.3, \quad A_1 = 1.3,$$
$$B_1 = 5.1, \quad D_1 = 0.001, \quad A_2 = 0.008, \quad B_2 = 6.5.$$

With the above values of parameters and taking the remaining param-

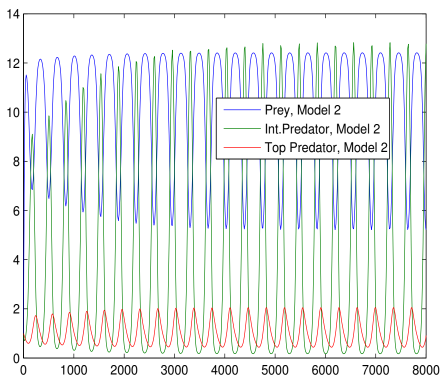

Figure 5: Time graph for the *Model* 2, around the equilibrium point $\bar{\bar{E}}_{23}$, showing the bifurcation behavior.

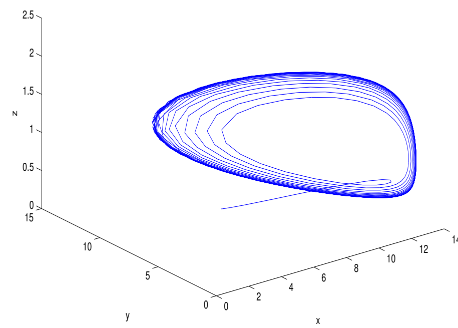

Figure 6: Phase graph for the *Model* 2 around the equilibrium point $\bar{\bar{E}}_{23}$, showing the bifurcation behavior.

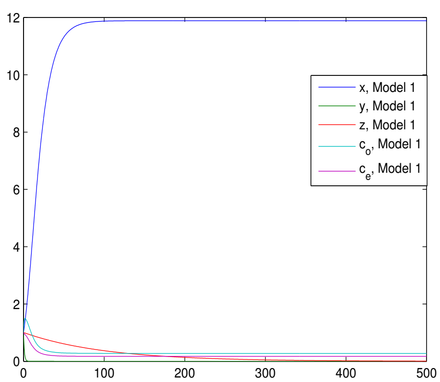

Figure 7: Time graph for the *Model* 1, around the equilibrium point $\tilde{E}_{11}$, showing the stability behavior.

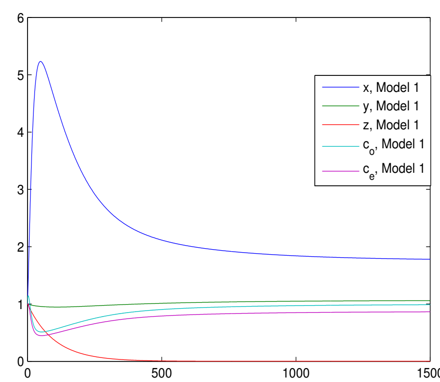

Figure 8: Time graph for the *Model* 1, around the equilibrium point $\hat{E}_{12}$, showing the stability behavior.

eters to be the same as considered for $\tilde{E}_{11}$ of *Model* 1, it is found that the equilibrium $\hat{E}_{12} = (1.7587, 1.0694, 0.0000, 0.8796, 0.9603)$ is locally asymptotically stable (see 8).

Now, we choose the following values of parameters for $\bar{E}_{13}$:

$$r_1 = 0.2, \quad a_1 = 1.00, \quad A_1 = 1.1, \quad D_1 = 0.001, \quad B_2 = 9.0, \quad A_2 = 0.6.$$

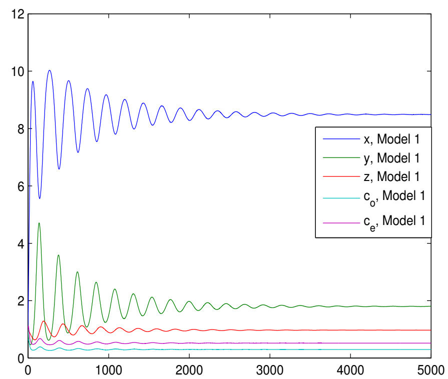

Figure 9: Time graph for the *Model* 1, around the equilibrium point $\bar{E_{13}}$, showing the stability behavior.

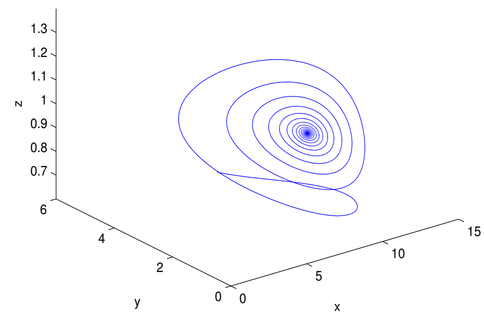

Figure 10: Phase graph for the *Model* 1, around the equilibrium point $\bar{E_{13}}$, showing the stability behavior.

With the above values of parameters and taking the remaining parameters to be the same as considered for $\tilde{E_{11}}$ of *Model* 1, it is found that the interior equilibrium $\bar{E_{13}} = (8.4842, 1.8039, 0.9731, 0.2989, 0.5211)$ is locally asymptotically stable (see Figs. 9 and 10).

Now, we study the Hopf-bifurcation of the *Model* 1, taking A_1 as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_1 = \bar{A_1} = 0.7015$. It is clear that the interior equilibrium point $\bar{E_{13}}$ of *Model* 1 is stable when $A_1 > \bar{A_1}$ and unstable when $A_1 \leq \bar{A_1}$ for which Hopf-bifurcation occurs (see Figs. 11 and 12).

6.3 Effect of Toxicant on *Model* 1 and Comparison with *Model* 2:

Now, we compare the equilibrium levels of the population for both the *Model*s. From the Tables 1, 2 and Fig. 13, we can see that the populations are decreasing under the stress of toxicant.

7 Conclusion

In this paper we have proposed and analyzed a nonlinear mathematical model to study the effect of toxicant on a three species food chain system. The local stability analysis of all the equilibrium points of the *Model* 1 and 2 has been carried out. The global stability analysis of only the non-trivial

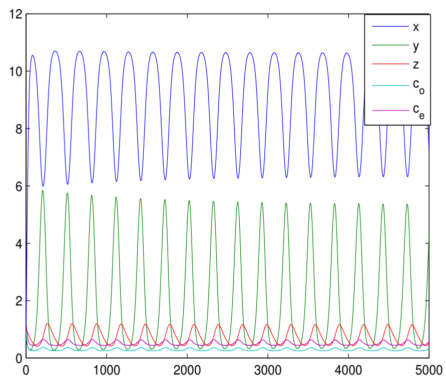

Figure 11: Time graph for the $Model$ 1, around the equilibrium point $\bar{E_{13}}$, showing the bifurcation behavior.

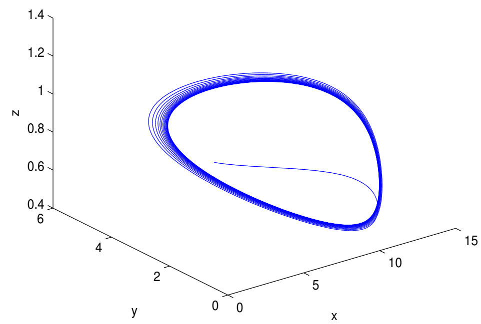

Figure 12: Phase graph for the $Model$ 1, around the equilibrium point $\bar{E_{13}}$, showing the bifurcation behavior.

Table 1: Numerical values of equilibrium points of $Model$ 2.

Equilibrium Points	Numerical values of Model 2
$\tilde{\tilde{E_{21}}}(\tilde{\tilde{x}}, 0, 0)$	(12.5706, 0, 0)
$\hat{\hat{E_{22}}}(\hat{\hat{x}}, \hat{\hat{y}}, 0)$	(3.6744, 1.2250, 0)
$\bar{\bar{E_{23}}}(\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}})$	(9.5375, 2.2163, 1.3688)

positive equilibrium points of both the $Model$s has been conducted. From the stability of $\tilde{\tilde{E_{21}}}$ of $Model$ 2, it is concluded that only the prey population will survive and both the predator populations would tend to extinction. From the stability of $\tilde{E_{11}}$ of $Model$ 1 we derive the same dynamical behavior of prey and predator populations as observed for $\tilde{\tilde{E_{21}}}$ of $Model$ 2 with the only difference that equilibrium level of prey population reduces due to the presence of toxicant (see Figs. 1 and 7). From the stability of $\hat{\hat{E_{22}}}$ of $Model$ 2, it is concluded that only prey and intermediate predator populations would survive and the top predator population may die out. Similar dynamical behavior has been observed for prey and predator populations from the stability analysis of $\hat{E_{12}}$ as being observed from the stability analysis of $\hat{\hat{E_{22}}}$. However, in this case also the equilibrium level of prey and predator populations decrease due to the presence of toxicant (see Figs. 2 and 8). The interior equilibrium points of both the $Model$s are locally sta-

Table 2: Numerical values of equilibrium points of *Model* 1.

Equilibrium Points	Numerical values of Model 1
$\tilde{E_{11}}(\tilde{x}, 0, 0, \tilde{C_0}, \tilde{C_E})$	(11.9736, 0, 0, 0.2401, 0.2679)
$\hat{E_{12}}(\hat{x}, \hat{y}, 0, \hat{C_0}, \hat{C_E})$	(1.7587, 1.0694, 0, 0.8796, 0.9603)
$\bar{E_{13}}(\bar{x}, \bar{y}, \bar{z}, \bar{C_0}, \bar{C_E})$	(8.4842, 1.8039, 0.9731, 0.2989, 0.5211)

ble showing the same dynamical behavior and co-existence of all the three populations of prey and predator species. However, from the equilibrium values it is seen that the equilibrium density of top predator reduces due to the presence of toxicant in prey and intermediate predator (see Figs. 3, 4, 9, 10 and 13). It may be also noted from the equilibrium of the intermediate predator population that the level of intermediate predator population may increase due to the presence of toxicant in the top predator.

The interior equilibrium points of both the *Model*s are globally asymptotically stable in the region Ω. Looking at Ω, it may be concluded that the region of global stability shrinks when the toxicant is introduced in the underlying system of prey and predator species. It is noted from the stability conditions of the equilibrium of the *Model*s that the system with toxicant seems to be more stable than that of the system with no toxicant effects. It is further concluded that the system with toxicant moves faster towards equilibrium after given perturbation than that of the system without toxicant for same parametric values. Finally, we have demonstrated the dynamical behavior of a three species food chain system with toxicant and without toxicant with the help of numerical simulation to support analytical results.

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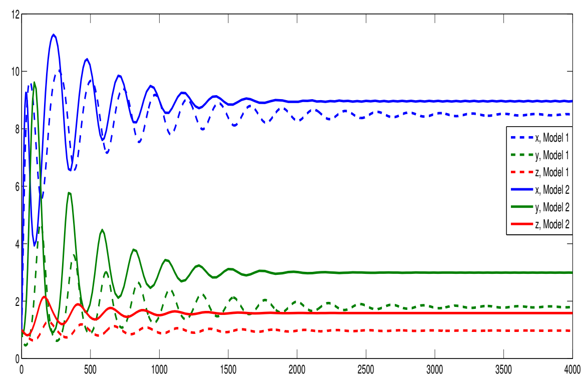

Figure 13: Time graph for Model 2 compared with Model 1 around the equilibrium points $\bar{\bar{E}}_{23}$ and $\bar{E}_{13}$ respectively, showing the stability behavior.

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