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F-REGULARITY RELATIVE TO MODULES

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Abstract. In this paper we will generalize some of known results on the tight closure of an ideal to the tight closure of an ideal relative to a module .

1. introduction

Throughout of this paper, R will denote a commutative Noetherian ring with identity and with a positive prime characteristic p. Further $\mathbf{N}$ will denote the set of nonnegative integers and R° will denote the subset of R consisting of all elements which are not contained in any minimal prime ideal of R.

The main idea of tight closure of an ideal in a commutative Noetherian ring (with a positive prime characteristic) was introduce by Hochster and Huneke in [4].

Let I be an ideal of R. We recall that an element x of R is said to be in tight closure, I^*, of I, if there exists an element $c \in R^\circ$ such that for all sufficiently large e, $cx^{p^e} \in (i^{p^e} : i \in I)$. The ideal $(i^{p^e} : i \in I)$ is denoted by $I^{[p^e]}$ and is called the eth Frobenius power of I. In particular if $I = (a_1, a_2, ..., a_n)$, then we have $I^{[p^e]} = (a_1^{p^e}, a_2^{p^e}, ..., a_n^{p^e})$. The reader is referred to [6] for the tight closure of an ideal.

In the remainder of this paper, to simplify notation, we will write q to stand for a power p^e of p. For any ideals I and J, $I^{[q]} + J^{[q]} = (I + J)^{[q]}$, $I^{[q]} J^{[q]} = (IJ)^{[q]}$.

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In [2], the dual notion of tight closure of ideals relative to modules was introduced and some properties of this concept which reflect results of tight closure in the classical situation were obtained. It is appropriate for us to begin by briefly summarizing some of main aspects.

Let M be an $R-$module and let I and J be ideals of R. Then I is an $F-$reduction of J relative to M if $I \subseteq J$ and there exists $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cJ^{[q]}) \ for\ all\ q \gg 0.$$

It is straightforward to see that the set of ideals of R which have I as an $F-$reduction relative to M has a unique maximal member, denoted by $I^{*[M]}$, and called the tight closure of I relative to M. This is in fact the largest ideal which has I as $F-$reduction relative to M (see [2]).

An element x of R is said to be tightly dependent on I relative to M if there exists an element $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q) \ for\ all\ q \gg 0.$$

Moreover in [2], it was shown that

$$I^{*[M]} = \{x \in R : x\ is\ tight\ dependent\ on\ I\ relative\ to\ M\}.$$

In this paper we will prove some new properties for tight closure of an ideal relative to a module which reflect some results of tight closure in the classical situation.[1, 3.17]

2. Tightly closed relative to a module

In this section, we study some related results for these ideals which are tightly closed relative to a module.

Definition 2.1. (See [2]) Let I be an ideal of R and let M be an $R-$module. I is said to be tightly closed relative to M if $I^{*[M]} = I$.

Remark 2.2. Let M be an $R-$module and let S be a multiplicatively closed subset of R. Let $\frac{u}{1} \in (S^{-1}I)^{*[S^{-1}M]}$. Then there exists $\frac{c}{1} \in (S^{-1}R)^\circ$ such that

$$(0 :_{S^{-1}M} (S^{-1}I)^{[q]}) \subseteq (0 :_{S^{-1}M} \frac{c}{1}(\frac{u}{1})^q) \ for\ all\ q \gg 0.$$

By using the similar method which is used in by [5, 4.14], without losing the generality, we can assume $c \in R^\circ$.

Let $h : R \to T$ be a homomorphism of a ring R into a ring T. For every ideal I of R, I^e will denote the extension of I to T. Also for every ideal J of T, J^c will denote the contraction of J to R.

Proposition 2.3. *Let T be another commutative Noetherian ring of positive prime characteristic p and let $h : R \to T$ be a ring homomorphism such that $h(R^\circ) \subseteq T^\circ$. Further assume that M is a $T-$module.*

(a) *If I is an ideal of R, then $(I^{*[M]})^e \subseteq (I^e)^{*[M]}$.*
(b) *If the ideal J of T is tightly closed relative to M, then J^c is tightly closed relative to M.*

Proof. (a) Let $y \in h(I^{*[M]})$. Then there exists $x \in I^{*[M]}$ such that $y = h(x)$. Since $x \in I^{*[M]}$, there exists $c \in R^\circ$ such that

$$(0 :_M I^{[q]}) \subseteq (0 :_M cx^q) \text{ for all } q \gg 0.$$

It follows that

$$(0 :_M (I^e)^{[q]}) \subseteq (0 :_M h(c)y^q) \text{ for all } q \gg 0,$$

where $h(c) \in h(R^\circ) \subseteq T^\circ$. This shows that $(I^{*[M]})^e \subseteq (I^e)^{*[M]}$.
(b) This follows from (a). □

Corollary 2.4. *Let I be an ideal of R. For every $P \in Spec(R)$, we have*

$$I^{*[E(R/P)]}R_P = (IR_P)^{*[E(R/P)]}.$$

Proof. This follows from Proposition 2.3(a), and Remark 2.2, □

Theorem 2.5. *Let (R, m) be a local Noethrian ring of characteristic p. Let $E = E(R/\mathrm{m})$ and let $x_1, x_2, ..., x_d \in \mathrm{m}$ be a regular sequence.*

(a) *If $J = (x_1, x_2, ..., x_{d-1})$, then $(J^{*[E]} :_R x_d) = J^{*[E]}$;*
(b) *If the ideal $(x_1, x_2, ..., x_d)$ is tightly closed relative to E, then the ideal $(x_1, x_2, ..., x_i)$ is tightly closed relative to E for every $1 \le i \le d$.*

Proof. (a) Let $I = (x_1, x_2, ..., x_d)$. Since $I \cap (R - \mathrm{m}) = \emptyset$, $\frac{x_1}{1}, \frac{x_2}{1}, ..., \frac{x_d}{1}$ is a regular sequence in R_{m}. Now let $u \in (J^{*[E]} :_R x_d)$. Then there exists $c \in R^\circ$ such that

$$(0 :_E J^{[q]}) \subseteq (0 :_E c(x_d u)^q) \text{ for all } q \gg 0.$$

By [3, Corollary 1.6], we have

$$\frac{c}{1}\frac{(x_d u)^q}{1} \in J^{[q]}R_{\mathrm{m}}.$$

Since $\frac{x_1{}^q}{1}, \frac{x_2{}^q}{1}, ..., \frac{x_d{}^q}{1}$ is a regular sequence in R_{m}, we have $\frac{c}{1}\frac{u^q}{1} \in J^{[q]}R_{\mathrm{m}}$. Now by [3, Corollary 1.6], we can conclude that

$$(0 :_E J^{[q]}) \subseteq (0 :_E cu^q) \ for\ all\ q \gg 0.$$

Hence $u \in J^{*[E]}$.

(b) Without loss of generality, we may assume that $i = d-1$. Let $J = (x_1, x_2, ..., x_{d-1})$ and $u \in J^{*[E]}$. Then $u \in (J + x_dR)^{*[E]} = J + x_dR$. So there exist $j \in J$ and $r \in R$ such that $u = j + x_dr$. By using part (a), we see that $r \in J^{*[E]}$. Thus $J^{*[E]} = J + x_dJ^{*[E]}$. Now Nakayama's Lemma shows that $J^{*[E]} = J$. □

Theorem 2.6. *Let $(R, \mathbf{m})$ be a local ring and let $x_1, x_2, ..., x_t \in \mathbf{m}$ be an $R-$sequence. Let E be an injective $R-$module such that $|Ass_R(E)| < \infty$. Then the ideal $(x_1, x_2, ..., x_t)$ of R is tightly closed relative to E if and only if the ideal $(x_1{}^n, x_2{}^n, ..., x_t{}^n)$ of R is tightly closed relative to E for all $n \in \mathbf{N}$.*

Proof. ($\Leftarrow$) This is clear.

For the converse, assume that the ideal $(x_1, x_2, ..., x_t)$ of R is tightly closed relative to M. For any positive integer n, we set

$$J_n = (x_1{}^n, x_2{}^n, ..., x_t{}^n).$$

We will show $J_n{}^{*[E]} = J_n$ for all $n \geq 2$. Let $z \in (J_n)^{*[E]} \setminus J_n$ and suppose that there exists $1 \leq i_1 \leq t$, such that $z_1 = zx_{i_1} \notin J_n$. We can choose $x_{i_1}, x_{i_2}, ..., x_{i_k} \in \{x_1, x_2, ..., x_t\}$ (not necessary distinct) such that $z_k = zx_{i_1}x_{i_2}...x_{i_k} \notin J_n$ but $z_kx_i \in J_n$ for all $1 \leq i \leq n$. Then $z_k(x_1, x_2, ..., x_t) \subseteq J_n$. Since $x_1, x_2, ..., x_t$ is an $R-$sequence,

$$z_k \in (J_n : (x_1, ..., x_t)) = (J_n, y^{n-1}),$$

where $y = x_1x_2...x_t$. So

$$z_k = \sum_{i=1}^{t} r_ix_i{}^n + uy^{n-1},$$

where $u, r_1, r_2, ..., r_t \in R$. Now let $E = \bigoplus\limits_{i\in I} E(R/P_i)$. Since $|Ass_R(E)| < \infty$, $J_n{}^{*[E]} = \bigcap\limits_{i\in I} J_n{}^{*[E(R/P_i)]}$. This shows that

$$uy^{n-1} \in (J^n)^{*[E(R/P_i)]} \quad \forall i \in I.$$

For every $i \in I$, if $\{x_1, x_2, ..., x_t\} \cap P_i \neq \emptyset$, then $u \in J_1{}^{*E(R/P_i)}$. Otherwise, since $uy^{n-1} \in (J^n)^{*[E(R/P_i)]}$, there exists $c \in R^\circ$ such that

$$(0 :_{E(R/P_i)} J_n{}^{[q]}) \subseteq (0 :_{E(R/P_i)} c(uy^{n-1})^q) \quad for\ all\ q \gg 0.$$

By [3, Corollary 1.6], we have

$$\frac{c(uy^{n-1})^q}{1} \in {J_n}^{[q]} R_{P_i}.$$

Thus

$$\frac{cu^q}{1} \in (\frac{{x_1}^{nq}}{1}, ..., \frac{{x_t}^{nq}}{1} : \frac{y^{qn-1)}}{1}) = {J_1}^{[q]} R_{P_i}.$$

Hence by [3, 1.6],

$$(0 :_{E(R/P_i)} {J_1}^{[q]}) \subseteq (0 :_{E(R/P_i)} cu^q) \quad for\ all\ q \gg 0.$$

This implies that

$$u \in {J_1}^{*[E(R/P_i)]} \quad \forall i \in I.$$

Hence $u \in {J_1}^{*[E]} = J_1$. So $uy^{n-1} \in J_n$ and this shows that $z_k \in J_n$. This contradiction shows $(J_n)^{*[E]} = J_n$. □

Corollary 2.7. *Let $(R, \mathbf{m})$ be a Cohen-Macaulay ring and let $x_1, x_2, ..., x_t \in \mathbf{m}$ be a system of parameters of R. Let E be an injective $R-$module such that $|Ass_R(E)| < \infty$. Then the ideal $(x_1, x_2, ..., x_t)$ of R is tightly closed relative to E if and only if the ideal $({x_1}^n, {x_2}^n, ..., {x_t}^n)$ of R is tightly closed relative to E for all $n \in \mathbf{N}$.*

Proof. By assumption $x_1, x_2, ..., x_t \in \mathbf{m}$ is an $R-$sequence. Now the proof follows from Theorem 2.6. □

3. F-regularity and weakly $F-$regularity relative to a module

Definition 3.1. Let M be an $R-$module. Then R is said to be weakly $F-$regular relative to M if every ideal of R is tightly closed relative to M. Furthermore, if for every multiplicative closed subset S of R, $S^{-1}R$ is weakly $F-$regular relative to $S^{-1}M$, then R is said to be $F-$regular relative to M.

Lemma 3.2. *Let M be an $R-$module. If R is $F-$regular relative to M, then R is weakly $F-$regular relative to M.*

Proof. Let I be an ideal of R and let $x \in I^{*[M]} \setminus I$. Then there exists a prime ideal P of R, such that $(I : x) \subseteq P$. Since R is $F-$regular relative to M, R_P is weakly $F-$regular relative to M. Thus

$$(IR_P)^{*[M_P]} = IR_P.$$

By Proposition 2.3, we have $\frac{x}{1} \in (I^{*[M]})R_P \subseteq (IR_P)^{*[M_P]} = IR_P$. Then there exists $a \in I$ and $t \in R \setminus P$ such that $\frac{x}{1} = \frac{a}{t}$. It follows that there exists $w \in R \setminus P$ with $wt \in (I : x) \subseteq P$. This is a contradiction. □

Lemma 3.3. *R is $F-$regular relative to M if and only if for every prime ideal P of R, R_P is weakly $F-$regular relative to M_P.*

Proof. ($\Rightarrow$) This is clear.
($\Leftarrow$) Let S be a multiplicatively closed subset of R. we will show that $S^{-1}R$ is weakly $F-$regular relative to $S^{-1}M$. Let $\frac{x}{1} \in (S^{-1}I)^{*[S^{-1}M]} \setminus S^{-1}I$. Then there exists $\frac{c}{1} \in (S^{-1}R)^{\circ}$ such that

$$(0:_{S^{-1}M}(S^{-1}I)^{[q]}) \subseteq (0:_{S^{-1}M}\frac{c}{1}(\frac{x}{1})^q) \text{ for all } q \gg 0.$$

By Remark 2.2, we can assume that $c \in R^{\circ}$. Since $\frac{x}{1} \notin S^{-1}I$, $(I:x) \cap S = \emptyset$. Let P be a prime ideal such that $(I:x) \subseteq P$ and $P \cap S = \emptyset$. Now we can see that $\frac{c}{1} \in (R_P)^{\circ}$ and

$$(0:_{M_P}(IR_P)^{[q]}) \subseteq (0:_{M_P}\frac{c}{1}(\frac{x}{1})^q) \text{ for all } q \gg 0.$$

Then $\frac{x}{1} \in (IR_P)^{*[M_P]}$. Since $(I:x) \subseteq P$, $\frac{x}{1} \in (IR_P)^{*[M_P]} \setminus IR_P$. But This is a contradiction by assumption. This contradiction shows that $(S^{-1}I)^{*[S^{-1}M]} = S^{-1}I$. Hence $S^{-1}R$ is weakly $F-$regular relative to $S^{-1}M$. $\square$

Lemma 3.4. *Let I be an ideal of R and let M be an $R-$module. Let S be a multiplicatively closed subset of R. Then we have the following*

(a) $S^{-1}(I^{*[M]}) \subseteq (S^{-1}I)^{*[S^{-1}M]}$.
(b) *If $S \cap Zd(M) = \emptyset$, then*

$$S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}.$$

Proof. The proof is straightforward. $\square$

Theorem 3.5. *Let I be an ideal of R and let M be a Noetherian $R-$module. Let S be a multiplicatively closed subset of R. Then*

$$S^{-1}(I^{*[M]}) = (S^{-1}I)^{*[S^{-1}M]}.$$

Proof. Since M is a Noetherian $R-$module, there exists q' such that

$$(0:_M I^{[q]}) = (0:_M I^{[q']})$$

for every $q \geq q'$. Let $\frac{x}{1} \in (S^{-1}I)^{*[S^{-1}M]}$. Then there exists $\frac{c}{1} \in (S^{-1}R)^{\circ}$ such that

$$(0:_{S^{-1}M}(S^{-1}I)^{[q]}) \subseteq (0:_{S^{-1}M}\frac{c}{1}(\frac{x}{1})^q) \text{ for all } q \gg 0.$$

By Remark 2.2, we can choose $c \in R^{\circ}$ and $t \in S$ such that

$$(0:_M I^{[q]}) \subseteq (0:_M c(tx)^q) \text{ for all } q \gg q'.$$

Then $tx \in I^{*[M]}$ and

$$\frac{x}{1} = \frac{tx}{t} \in S^{-1}(I^{*[M]}).$$

This shows that $(S^{-1}I)^{*[S^{-1}M]} \subseteq S^{-1}(I^{*[M]})$. Now the proof is completed by Lemma 3.4. □

Corollary 3.6. *Let M be a Noetherian $R-$module. If R is weakly $F-$regular relative to M then R is $F-$regular relative to M.*

Proof. This is clear from Theorem 3.5. □

Theorem 3.7. *Let R be a regular ring and let I be an ideal of R.*

(a) *For every $P \in Spec(R)$, we have $I^{*[E(\frac{R}{P})]} = I$.*
(b) *For every injective $R-$module E, we have $I^{*[E]} = I$.*

Proof. (a) Let $x \in I^{*[E(\frac{R}{P})]} \setminus I$. Then there exists an element $c \in R^{\circ}$ such that

$$(0 :_{E(\frac{R}{P})} I^{[q]}) \subseteq (0 :_{E(\frac{R}{P})} cx^q) \ for \ all \ q \gg 0.$$

By [3, 1.6], we have

$$\frac{cx^q}{1} \in I^{[q]}R_P \ for \ all \ q \gg 0.$$

This follows that $\frac{c}{1} \in \bigcap_q (I^{[q]}R_P : \frac{x^q}{1})$. By assumption, R is a regular ring. Then [4, 4.3] implies that

$$\frac{c}{1} \in \bigcap_q (IR_P : \frac{x}{1})^{[q]} \subseteq \bigcap_q (PR_P)^q = 0,$$

which is a contradiction. So we conclude that $I^{*[E(\frac{R}{P})]} = I$.

(b) We have $E \cong \bigoplus_{P \in Ass_R(E)} E(R/P)$. Then $I^{*[E]} \subseteq \bigcap_{P \in Ass_R(E)} I^{*[E(R/P)]}$. But by part (a), for every $P \in Ass_R(E)$, we have $I^{*[E(R/P)]} = I$. This shows that $I^{*[E]} \subseteq I$. Now the assertion follows from [2, 2.7(a)]. □

Corollary 3.8. *Let R be a regular ring. Then we have the following.*

(a) *R is weakly $F-$regular relative to every injective $R-$module E.*
(b) *R is $F-$regular relative to every injective $R-$module E.*

Proof. (a) This follows from Theorem 3.7(b).

(b) Let S be a multiplicative subset of R and let E be an injective R-module. Then $S^{-1}R$ is a regular ring and $S^{-1}E$ is an injective $S^{-1}R-$module. Now the claim follows from part (a). □

4. Integral closure of an ideal relative to some module and the Briançon-Skoda theorem

Definition 4.1. (See [1]). Let I and J be ideals of R and let M be an $R-$module. Then I is said to be a reduction of J relative to M if $I \subseteq J$ and there exists a positive integer n such that

$$(0 :_M IJ^n) = (0 :_M J^{n+1}).$$

Since R is Noetherian ring, the set of ideals of R which have I as a reduction relative to M has a unique maximal member which is denoted by $I^{*(M)}$ and is called the integral closure of I relative to M.

Definition 4.2. (See [1]). Let I be an ideal of R and let M be an $R-$module. An element x of R is said to be integrally dependent on I relative to M if there exists a positive integer n such that

$$(0 :_M \sum_{i=1}^{n} x^{n-i} I^i) \subseteq (0 :_M x^n).$$

Definition 4.3. (See [1]). A subset T of $Ass(R)$ has reduced property if for every $P \in T$, there exists an element $x \in R$ such that $P = Ann(x)$ and $x^2 \neq 0$.

Remark 4.4. Let I be an ideal of R and M be an $R-$module. Then

$$I^{*(M)} = \{x \in R : x \textit{ is integrally dependent on } I \textit{ relative to } M\}$$

in each of the following cases:

(a) M is an Artinian $R-$module (see [5]);
(b) M is an injective $R-$module (see [3, 2.7];
(c) $Ass_R(M) \subseteq Ass(R)$ and $Ass_R(M)$ has reduced property (see [1, 3.10]).

The following theorem is proved by a similar technique used in [4, 5.4].

Theorem 4.5. *(generalized Briançon-Skoda theorem). Let I be an ideal of R generated by n elements and let $ht(I) > 0$. Let M be an $R-$module. Then for every $m \in \mathbf{N}$, we have*

$$(I^{n+m})^{*(M)} \subseteq (I^{m+1})^{*[M]}$$

and in particular $(I^n)^{*(M)} \subseteq (I^{*[M]}$ in each of the following cases:

(a) *M is an Artinian $R-$module ;*
(b) *M is an injective $R-$module ;*
(c) *$Ass_R(M) \subseteq Ass(R)$ and $Ass_R(M)$ has reduced property.*

Proof. Suppose I is generated by $u_1, u_2, ..., u_n$. If

$$(I^{n+m})^{*(M)} \subseteq (I^{m+1})^{*[M]} \cup \bigcup_{P \in Min(R)} P,$$

then since $ht(I) > 0$, we have $(I^{n+m})^{*(M)} \subseteq (I^{m+1})^{*[M]}$. So we assume to the contrary that $(I^{n+m})^{*(M)}$ is not contained in the union $(I^{m+1})^{*[M]} \cup \bigcup_{P \in Min(R)} P$. Then there exists $y \in (I^{n+m})^{*(M)} - (I^{m+1})^{*[M]}$ such that $y \in R^{\circ}$. By Remark 4.4, y is integrally dependent on I^{n+m} relative to M. Let $J = I^{n+m}$. Then J is a reduction of $J + Ry$ relative to M by [1, 2.11]. This implies that there is a positive integer k such that

$$(0 :_M (J + yR)^{k+h}) = (0 :_M J^{h+1}(J + yR)^{k-1})$$

for every $h \in \mathbf{N}$. But

$$(0 :_M J^h) \subseteq (0 :_M J^{h+1}(J+yR)^{k-1}) = (0 :_M (J+yR)^{k+h}) \subseteq (0 :_M y^h y^k)$$

for all $h \in \mathbf{N}$. Just as proved in [4, 5.4], $J^h = I^{hn+hm}$ is generated by monomials of degree $hn + hm$ in the u_i. So $J^h = I^{hn+hm} \subseteq ({u_1}^h, {u_2}^h, ..., {u_n}^h)^{m+1}$. This shows that

$$(0 :_M ({u_1}^h, {u_2}^h, ..., {u_n}^h)^{m+1}) \subseteq (0 :_M y^h y^k),$$

for every $h \in \mathbf{N}$. When h has the form $q = p^e$, we have

$$(0 :_M (I^{m+1})^{[q]}) = (0 :_M ({u_1}^q, {u_2}^q, ..., {u_n}^q)^{m+1}) \subseteq (0 :_M y^q y^k)$$

where $y^k \in R^{\circ}$. Hence $y \in (I^{m+1})^{*[M]}$. □

Corollary 4.6. *Let I is an ideal of R generated by n elements and $ht(I) > 0$. Further assume that M is an $R-$module such that*

(a) *M is an Artinian $R-$module or*
(b) *M is an injective $R-$module or*
(c) *$Ass_R(M) \subseteq Ass(R)$ and $Ass_R(M)$ has reduced property.*

If R is weakly $F-$regular relative to M, then

$$(I^n)^{*(M)} \subseteq I.$$

Proof. This follows from Theorem 4.5.

□

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