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STRONGLY COTOP MODULES

H. ANSARI-TOROGHY *, S. S. POURMORTAZAVI, AND S. KEYVANI

Abstract. In this paper, we introduce the dual notion of strongly top modules and study some of the basic properties of this class of modules.

1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary. Also the notation $\mathbb{Z}$ (resp. $\mathbb{Q}$) will denote the ring of integers (resp. the field of fractions of $\mathbb{Z}$). If N is a subset of an R-module M, then $N \leq M$ denotes N is an R-submodule of M. For any ideal I of R containing $Ann_R(M)$, $\bar{R}$ and $\bar{I}$ denote $R/Ann_R(M)$ and $I/Ann_R(M)$, respectively. The *colon ideal of* M *into* N is defined to be $(N : M) = \{r \in R : rM \subseteq N\} = Ann_R(M/N)$.

Let M be an R-module. A non-zero submodule S of M is said to be *second* if for each $a \in R$ the homomorphism $S \xrightarrow{a} S$ is either surjective or zero. This implies that $Ann_R(S) = p$ is a prime ideal of R and S is said to be *p-second* (see [16]).

The second spectrum of M is defined as the set of all second submodules of M and denoted by $Spec^s(M)$ or X^s. We call the map $\psi : X^s \to Spec(\overline{R})$ given by $S \mapsto \overline{Ann_R(S)}$ as the *natural map* of X^s.

Let N be a submodule of M. Define $V^s(N) := \{S \in Spec^s(M) : Ann_R(N) \subseteq Ann_R(S)\}$ and set $\zeta^s(M) := \{V^s(N) : N \leq M\}$. Then there exists a topology, τ^s say, on $Spec^s(M)$ having ζ^s as the family

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∗Corresponding author.

of all its closed sets. This topology is called the *Zariski topology on* $Spec^s(M)$ (see [2]).

For any submodule N of M, define $V^{s*}(N) = \{S \in Spec^s(M) : S \subseteq N\}$. Set $\zeta^{s*}(M) = \{V^{s*}(N) : N \subseteq M\}$. Then $\zeta^{s*}(M)$ contains the empty set and $Spec^s(M)$, and it is closed under arbitrary intersections. In general $\zeta^{s*}(M)$ is not closed under finite unions. A module M is called a cotop module if $\zeta^{s*}(M)$ is closed under finite unions. In this case, $\zeta^{s*}(M)$ is called the quasi Zariski topology (see [2]).

For a submodule N of M, the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and denoted by $sec(N)$ (or $soc(N)$). In case N does not contain any second submodule, the socle of N is defined to be (0) . Also, $N \neq (0)$ is said to be a *socle submodule* of M if $sec(N) = N$ (see [4, 10]).

M is said to be X^s*-injective* (resp. *secondful*) if the natural map of X^s is injective (resp. surjective). Equivalently, M is X^s-injective if and only if $Ann_R(S_1) = Ann_R(S_2)$, $S_1, S_2 \in X^s$, implies that $S_1 = S_2$ if and only if for every $p \in Spec(R)$, $|Spec^s_p(M)| \leq 1$ (see [7, 11]).

A second submodule S of M is said to be *extraordinary* if whenever N and K are socle submodules of M with $S \subseteq N + K$ then $S \subseteq N$ or $S \subseteq K$ (see [2]).

A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M, implies that $N = N_i$ for some $i \in I$ (see [12]).

Let p be a prime ideal of R and let N be a submodule of M. Then $N^{ec} = \{m \in M : cm \in N \text{ for some } c \in R \backslash p\}$ and it is called the p-closure of N and denoted by $cl_P(N)$ (see [14, p. 92]). The dual of this notion, i.e., p-interior of N relative to M is defined as the set $I^M_p(N) := \bigcap \{L | \, L$ *is a completely irreducible submodule of* M *and* $rN \subseteq L$ *for some* $r \in R \backslash p\}$ (see [4]).

Let R be an integral domain. A submodule N of M is said to be *cotorsion-free* (resp. *cotorsion*) if $I^M_0(N) = N$ (resp. $I^M_0(N) = (0)$) (see [3]).

M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ (see [5]).

The concept of strongly top modules was introduced and investigated in [1] and [6].

In this paper, we introduce the dual notion of strongly top R-modules and obtain some related results.

In section 2, among other results, we obtain some useful characterization for strongly cotop modules (see Theorem 2.5). In Theorem 2.9, we consider some conditions under which an R-module is an strongly cotop module. Furthermore, in Proposition 2.17 and Corollary 2.20, we study the behavior of an strongly cotop module under colocalization. More information about colocalization of certain modules can be found in Theorem 2.19. Finally, in Theorem 2.22 we investigate the interplay between strongly cotop modules and spectral spaces.

2. Main results

Definition 2.1. Let M be a cotop R-module. We say that M is a *strongly cotop* module (or simply s-cotop module) if $\tau^s_M = \tau^{s*}_M$.

Remark 2.2. ([2, Theorem 2.5]). Let M be an R-module. Then the following statements are equivalent.

(a) M is a cotop module.
(b) Every second submodule of M is extraordinary.
(c) $V^{s*}(N) \cup V^{s*}(K) = V^{s*}(N + K)$ for any socle submodules N and K of M.

Example 2.3. (a) Every comultiplication R-module is an s-cotop module.
(b) Not every s-cotop module is a comultiplication module. For example, let $M = \mathbb{Z}$. Then $Spec^s_{\mathbb{Z}}(M) = \emptyset$. Hence M is an s-cotop $\mathbb{Z}$-module, But it is not a comultiplication $\mathbb{Z}$-module.
(c) Every s-cotop R-module is a cotop module.
(d) Not every cotop module is an s-cotop module. For example, let $M = \mathbb{Q} \oplus \mathbb{Z}_p$ (as $\mathbb{Z}$-module) for some prime integer p. Then $Spec^s_{\mathbb{Z}}(M) = \{\mathbb{Q} \oplus 0, 0 \oplus \mathbb{Z}_p\}$. Hence by Remark 2.2, M is a cotop $\mathbb{Z}$-module but it is not an s-cotop $\mathbb{Z}$-module.

Proposition 2.4. (a) *Every s-cotop R-module is an X^s-injective R-module.*
(b) *Every submodule of an s-cotop R-module is s-cotop.*
(c) *Every homomorphic image of an s-cotop module is not necessarily s-cotop.*
(d) *If $M = \oplus_{i\in I} M_i$ is an s-cotop module, then each M_i is s-cotop for $i \in I$.*

Proof. (a) Let $S_1, S_2 \in Spec^s(M)$ and $(0 :_R S_1) = (0 :_R S_2)$. Thus $(0 :_M (0 :_R S_1)) = (0 :_M (0 :_R S_2))$. Since M is an s-cotop module, we have $V^{s*}(S_1) = V^{s*}(S_2)$. This implies that $S_1 = S_2$.
(b) It is trivial.
(c) Set $M = \mathbb{Z} \oplus \mathbb{Z}_p$. Then M is an s-cotop $\mathbb{Z}$-module, but its homomorphic image $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is not. In fact we have

$$\mathbb{Z}_p \oplus \mathbb{Z}_p \subseteq (\mathbb{Z}_p \oplus 0) + (0 \oplus \mathbb{Z}_p).$$

This implies that $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is not s-cotop module by Remark 2.2 and Example 2.3 (c).

(d) By part(b).

□

For an R-module M the set $\Omega(M)$ is defined as

$$\Omega(M) = \{p \in V(Ann_R(M)) \mid I_p^M((0 :_M p)) \neq (0)\}.$$

Theorem 2.5. *Consider the following statements for an R-module M.*

(a) *M is an s-cotop R-module;*
(b) *For every submodule N of M, there exists an ideal I of R such that $V_M^{s*}(N) = V_M^{s*}((0 :_M I))$;*
(c) *$V_M^{s*}(N) = V_M^s(sec(N))$ for every submodule N of M, where $sec(N) = \sum_{S \in Spec_R^s(N)} S$;*
(d) *M is an X^s-injective cotop R-module and $\psi : (X^s, \tau^{s*}) \to Im(\psi)$ is a closed map, where ψ is a natural map of X^s.*
(e) *For any $p \in \Omega(M)$ and for every family $\{p_i\}$, where $p_i \in \Omega(M)$, we have $\bigcap_{i \in I} p_i \subseteq p \Rightarrow I_p^M((0 :_M p)) \subseteq \sum_{i \in I} I_{p_i}^M((0 :_M p_i))$.*

Then $(a) - (d)$ are equivalent. Moreover, if M is an X^s-injective Artinian R-module, then $(c) \Leftrightarrow (e)$.

Proof. (a) $\Leftrightarrow$ (b). Let M be an s-cotop R-module and let N be a submodule of M. Since $\tau_M^s = \tau_M^{s*}$, $V_M^{s*}(N)$ is a closed subset of (X^s, τ_M^s). This implies that there exists a submodule K of M such that $V_M^{s*}(N) = V_M^s(K)$. It is not difficult to see that

$$V_M^{s*}((0 :_M (0 :_R K))) = V_M^s(K)$$

as desired. The reverse implication follows from the following fact. Let M be an R-module and let I be an ideal of R. Then

(i) $\tau_M^s \subseteq \tau_M^{s*}$.
(ii) $V_M^{s*}((0 :_M I)) = V_M^s((0 :_M I))$.

(a) $\Leftrightarrow$ (c). Let the situation be as in part (a). Let N be a submodule of M. Then $Y = V_M^{s*}(N)$ is a closed subset of (X^s, τ_M^s). This implies that $Y = cl(Y)$, where $cl(Y)$ is the topological closure of Y in (X^s, τ_M^s). It is easy to check that $cl(Y) = V_M^s(\Sigma_{S \in Y} S)$. On the other hand, $\Sigma_{S \in Y} S = sec(N)$. By the above arguments, we have $V_M^{s*}(N) = V_M^s(sec(N))$. The reverse implication follows from the fact that $\tau_M^s \subseteq \tau_M^{s*}$.

(a) $\Leftrightarrow$ (d). Let M be an s-cotop R-module and let N be a submodule of M. It is enough to prove that

$$\psi^s(V^{s*}_M(N)) = V_{\bar{R}}(\overline{(0:_R N)}) \cap Im(\psi^s).$$

To see this, let $L \in V_{\bar{R}}(\overline{(0:_R N)}) \cap Im(\psi^s)$. Then there exists $S \in X^s$ such that $L = \psi^s(S) = \overline{(0:_R S)}$ and $\overline{(0:_R N)} \subseteq \overline{(0:_R S)}$. This implies that

$$S \subseteq (0:_M (0:_R S)) \subseteq (0:_M (0:_R N)).$$

Therefore $S \in V^s_M((0:_M (0:_R N)))$. As in the proof (a) $\Rightarrow$ (b) we have

$$V^{s*}_M(N) = V^{s*}_M((0:_M (0:_R N))) = V^s_M((0:_M (0:_R N))).$$

Hence $L \in \psi^s(V^{s*}_M(N))$ so that

$$V_{\bar{R}}(\overline{(0:_R N)}) \cap Im(\psi^s) \subseteq \psi^s(V^{s*}_M(N)).$$

The reverse implication is clear.

(c) $\Leftrightarrow$ (e). Assume that $V^{s*}_M(N) = V^s_M(sec(N))$ for every submodule N of M. Also let $p \in \Omega(M)$ and $\{p_i\}_{i\in I}$ be a family of elements of $\Omega(M)$ with $\bigcap_{i\in I} p_i \subseteq p$. As M is Artinian, $I^M_p((0:_M p))$ is a p-second submodule of M by [4, Corollary 2.10]. Now our assumption implies that

$$I^M_p((0:_M p)) \in V^s(\sum_{i\in I} I^M_{p_i}((0:_M p_i))).$$

Since

$$V^s(sec(\sum_{i\in I} I^M_{p_i}((0:_M p_i)))) = V^s(\sum_{i\in I} I^M_{p_i}((0:_M p_i))),$$

we have

$$I^M_p((0:_M p)) \in V^s(sec(\sum_{i\in I} I^M_{p_i}((0:_M p_i)))).$$

Assumption (c) implies that

$$I^M_p((0:_M p)) \subseteq \sum_{i\in I} I^M_{p_i}((0:_M p_i))).$$

For the reverse implication, let N be a submodule of M and set

$$\Gamma = \{p \in \Omega(M) \mid I^M_p((0:_M p)) \subseteq N\}.$$

Then $sec(N) = \sum_{p\in\Gamma} I^M_p((0:_M p))$. It turns out that $V^{s*}_M(N) = V^s_M(sec(N))$, as required.

$\square$

An R-module M is said to be a *weak comultiplication module* if it does not have any second submodule or for every second submodule S of M, $S = (0 :_M I)$ for some ideal I of R (see [3]).

Corollary 2.6. *Let M be an R-module. Then M is an s-cotop module in the following cases.*

(a) $|Spec^s_R(M)| \leq 1$.

(b) *M is weak comultiplication and $|Spec(R)| < \infty$. However, not every weak comultiplication module is an s-cotop module.*

(c) *For every submodule N of M, $sec(N) = (0 :_M \sqrt{Ann_R(N)})$.*

Proof. (a) The proof is clear, if $Spec^s_R(M) = \emptyset$. Now we assume that $Spec^s_R(M) = \{K\}$, where K is a second submodule of M. Let N be a submodule of M. By Theorem 2.5, it is enough to prove that $V^{s*}_M(N) = V^s_M(sec(N))$. If $K \subseteq N$, we are done. So assume that $K \not\subseteq N$. Then we have $sec(N) = (0)$. This implies that $V^{s*}_M(N) = V^s_M(sec(N)) = \emptyset$.

(b) Let N be a submodule of M. By Theorem 2.5, it is enough to prove that $V^{s*}_M(N) = V^s_M(sec(N))$. It is clear that $V^{s*}_M(N) \subseteq V^s_M(sec(N))$. Conversely, assume that $S \in V^s_M(sec(N))$. Then

$$(0 :_R S) \supseteq (0 :_R sec(N)) = \bigcap_{W \in Spec^s_R(N)} (0 :_R W).$$

Since $|Spec(R)| < \infty$, there exists $T \in Spec^s_R(N)$ such that $(0 :_R S) \supseteq (0 :_R T)$. Hence

$$S = (0 :_M (0 :_R S)) \subseteq (0 :_M (0 :_R T)) = T.$$

Therefore $S \in V^{s*}_M(N)$ so that $V^s_M(sec(N)) \subseteq V^{s*}_M(N)$. To see the second assertion, set $M = \bigoplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z}$, where $\{p_i\}_{i \in I}$ is the set of all prime integers. Then $Spec^s_{\mathbb{Z}}(M) = \{(0 :_M p_i) \mid i \in I\}$ so that M is a weak comultiplication module. Now let $N = \bigoplus_{j \neq i \in I} \mathbb{Z}/p_i\mathbb{Z}$. Then $V^{s*}(N) = \{(0 :_M p_i) \mid j \neq i \in I\}$ and $V^s(sec(N)) = V^s(N) = X^s$ so that M is not an s-cotop module by Theorem 2.5.

(c) let $N \leq M$. Then by [2, Lemma 3.3 (c)], we have

$$\begin{aligned} V^{s*}(N) = V^{s*}(sec(N)) &= V^{s*}((0 :_M \sqrt{Ann_R(N)})) \\ &= V^s((0 :_M \sqrt{Ann_R(N)})) = V^s(sec(N)). \end{aligned}$$

Hence M is a strongly cotop module by Theorem 2.5.

□

Remark 2.7. Let S be a commutative ring with identity. S is said to be a *perfect ring* if it satisfies DCC on principal ideals. Clearly, every Artinian ring is perfect. Note that if S is a perfect ring and $p \in Spec(S)$, then by [9, Lemma 2.2], S/p is a perfect domain so that it is a field. Hence $dim(S) = 0$. Furthermore, every perfect ring is a semilocal ring by [9, Theorem P or p. 475, Examples (6)].

Lemma 2.8. *Let R be a one dimensional integral domain and let M be a non-zero X^s-injective R-module.*

(i) $Spec^s_R(M) = Min(M) \cup Spec^s_{(0)}(M)$, *where*

$$Min(M) = \{(0 :_M p) \mid p \in V(Ann_R(M)) \cap Max(R),\ (0 :_M p) \neq (0)\}.$$

(ii) *If M is secondful, then $Spec^s_R(M) = Min(M) \cup Spec^s_{(0)}(M)$, where $Min(M) = \{(0 :_M p) \mid p \in V(Ann_R(M)) \cap Max(R)\}$.*

Proof. Use the technique of [7, Theorem 3.16]. □

Theorem 2.9. *Let M be an X^s-injective R-module. Then M is an s-cotop module in the following cases.*

(a) $|Spec(R)| = |Max(R)| < \infty$. *In particular R is an Artinian or a perfect ring.*
(b) *R is a PID or one dimensional Noetherian domain and M is a non faithful R-module.*
(c) *R is a PID and M is a non faithful Artinian R-module.*
(d) *R is a one dimensional integral domain and the summation of every infinite number of minimal submodules of M is equal M and $sec(M) = \sum_{S \in Spec^s_{(0)}(M)} S$.*

Proof. (a) Let N be a submodule of M. By Theorem 2.5, it is enough to prove that $V^{s*}_M(N) = V^s_M(sec(N))$. To see this, let $L \in V^s_M(sec(N))$. Then

$$\bigcap_{W \in V^{s*}_M(N)} (0 :_R W) = (0 :_R sec(N)) \subseteq (0 :_R L).$$

By assumption, there exists $S \in V^{s*}_M(N)$ such that $(0 :_R W) = (0 :_R L)$. Since M is X^s-injective, $S = L$. Therefore $L \in V^{s*}_M(N)$ so that $V^s_M(sec(N)) \subseteq V^{s*}_M(N)$. The reverse inclusion is clear.

(b) We have similar arguments as in part (a).

(c) Let $\{p_i\}_{i \in I}$ be a family of elements of $\Omega(M)$ such that $\bigcap_{i \in I} p_i \subseteq p$, where $p \in \Omega(M)$. Since $Ann_R(M) \neq (0)$, $|Spec(\overline{R})| < \infty$. Thus I is a finite set, so $p_k = p$ for some $k \in I$. This implies that $I^M_{p_k}((0 :_M p_k)) = I^M_p((0 :_M p))$. Hence $I^M_p((0 :_M p)) \subseteq \sum_{i\in I} I^M_{p_i}((0 :_M p_i))$. Therefore M is an s-cotop R-module by Theorem 2.5 (e).

(d) Suppose that $M \neq (0)$. By Lemma 2.8, $Spec^s_R(M) = Spec^s_0(M) \cup Min(M)$, where $Min(M) = \{(0 :_M p) \mid p \in Max(R), (0 :_M p) \neq (0)\}$. Let $N \leq M$. As the summation of every infinite number of minimal submodules of M is M, assume that $sec(N) = \sum_{i=1}^n (0 :_M p_i)$. It is easy to see that

$$V^{s*}(\sum_{i=1}^n (0 :_M p_i)) = V^{s*}((0 :_M \bigcap_{i=1}^n p_i)).$$

Therefore, by [2, Lemma 3.3 (c)], we have

$$V^{s*}(N) = V^{s*}(sec(N)) = V^{s*}(\sum_{i=1}^n (0 :_M p_i)) = V^{s*}((0 :_M \bigcap_{i=1}^n p_i))$$
$$= V^s((0 :_M \bigcap_{i=1}^n p_i)) = V^s(\sum_{i=1}^n (0 :_M p_i)) = V^s(sec(N)).$$

Hence M is a strongly cotop module.

□

Let p be a prime ideal of R. For an R-module M, $Spec^s_p(M)$ denotes the set of all p-second submodules of M.

Remark 2.10. (a) ([7, Proposition 3.12]). Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. If M is an X^s-injective module, then

$$Spec^s_R(M) = \left\{ S \oplus \left(\bigoplus_{j\neq i\in I} (0) \right) \mid j \in I,\, S \in Spec^s_R(M_j) \right\}.$$

(b) A family $(M_i)_{i\in I}$ of R-modules is said to be *second-compatible* if for all $i \neq j$ in I, there doesn't exist a prime ideal p in R with $Spec^s_p(M_i)$ and $Spec^s_p(M_j)$ both nonempty.

(c) ([7, Theorem 3.14]). Let $(M_i)_{i\in I}$ be a family of R-modules and let $M = \bigoplus_{i\in I} M_i$. Then M is an X^s-injective R-module if and only if $(M_i)_{i\in I}$ is a family of second-compatible X^s-injective R-modules.

Proposition 2.11. *Let R be a domain and $(M_i)_{i\in I}$ be a family of R-modules such that $M_t \in Spec^s_{(0)}(M_t)$ for some $t \in I$. Consider the following statements.*

(a) $M = \bigoplus_{i\in I} M_i$ *is an s-cotop module.*

(b) *$Spec^s_R(M_j) = \emptyset$ for every $t \neq j \in I$.*

Then (a) $\Rightarrow$ (b). Moreover, if $Spec^s_R(M_t) = \{M_t\}$, then (b) $\Rightarrow$ (a).

Proof. $(a) \Rightarrow (b)$. Let M be an s-cotop module and let $j \in I$ with $j \neq t$. We show that $Spec^s_R(M_j) = \emptyset$. To see this, let $S_j \in Spec^s_R(M_j)$. Then by Remark 2.10 (a), $K = S_j \oplus (\bigoplus_{j\neq i\in I} 0) \in Spec^s_R(M)$. Also we have $L = M_t \oplus (\bigoplus_{t\neq i\in I} 0) \in Spec^s_{(0)}(M)$. Hence $K \in V^s(L)$ so that $K \in V^{s*}(L)$ by Theorem 2.5. Thus $S_j = (0)$, a contradiction.

$(b) \Rightarrow (a)$. Let $Spec^s_R(M_t) = \{M_t\}$. Then $Spec^s_R(M) = \{M_t \oplus (\bigoplus_{t\neq i\in I} 0)\}$ by Remark 2.10 (a) and Remark 2.10 (c). Now by Corollary 2.6 (a), M is an s-cotop module. Hence the proof is compeleted. □

Corollary 2.12. *Let R be a domain with the field of fractions Q and M' be an R-module, then the R-module $M = Q \oplus M'$ is an s-cotop module if and only if $Spec^s_R(M') = \emptyset$.*

Proof. This follows from Proposition 2.11 and the fact that for every domain R with the field of fractions Q, we have $Spec^s_R(Q) = \{Q\}$. □

Definition 2.13. We say that an R-module M is a semi-comultiplication module if it does not have any socle submodule or each socle submodule of M is of the form $(0 :_M I)$ for some ideal I of R.

It is easy to see that if S is a socle submodule of a semi-comultiplication R-module M, then $S = (0 :_M Ann_R(S))$. For example, every comultiplication module is a semi-comultiplication module. On the other hand, the $\mathbb{Z}$-module $\mathbb{Q}$ is semi-comultiplication which is not comultiplication.

Remark 2.14. ([7, Lemma 3.6]). Let p and q be prime ideals of R with $q \subseteq p$. Let M be an R-module with $cl_p(0) = (0)$ and let $\phi : Hom_R(R_p, M) \to M$ be the natural homomorphism given by $f \mapsto f(1/1)$. Then we have the following.

(i) If S is an R_p submodule of $Hom_R(R_p, M)$, then we have $S^{ec} = S = Hom_R(R_p, L)$, where $L = S^e$. (Here T^e, where $T \subseteq Hom_R(R_p, M)$, and N^c, where $N \subseteq M$, denote $\phi(T)$ and $\phi^{-1}(N)$, respectively.)

(ii) If M is an Artinian R-module and K is a q-second submodule of M, then $Hom_R(R_p, K)$ is a qR_p-second submodule of $Hom_R(R_p, M)$ and $K^c = Hom_R(R_p, K)$. Further we have $(Hom_R(R_p, K))^e = \phi(Hom_R(R_p, K)) = I_p(K) = K$ and $K^{ce} = K$.

Theorem 2.15. *Let M be a semi-comultiplication R-module. Then*

(a) *M is an s-cotop R-module.*
(b) *M is a weak comultiplication R-module; but the converse is not true in general.*
(c) *Every submodule of M is a semi-comultiplication R-module. But every homomorphic image of M is not necessary semi-comultiplication R-module.*
(d) *If R is an integral domain and M is an Artinian R-module, then M is either cotorsion or cotorsion-free.*
(e) *Let $p \in Spec(R)$ and $cl_p^M(0) = (0)$, then $Hom_R(R_p, M)$ is a semi-comultiplication R_p-module.*
(f) *If R is a Noetherian ring and M is an Artinian R-module with finite length, then M is a comultiplication R-module.*

Proof. (a) Let N be a submodule of M. If $sec(N) = (0)$, then $V^{s*}(N) = V^{s*}((0)) = V^{s*}((0 :_M R))$, so we are done. If $sec(N) \neq (0)$, then $sec(N)$ is a socle submodule of M. Hence,

$$V^{s*}(N) = V^{s*}(sec(N)) = V^{s*}((0 :_M Ann_R(sec(N)))).$$

Therefore, M is an s-cotop R-module.

(b) This is clear. To see the last assertion, set $M = \bigoplus_{i \in I} \mathbb{Z}/p_i\mathbb{Z}$, where $\{p_i\}_{i \in I}$ is the set of all prime integers. Then by part (a) and Corollary 2.6 (b), M is not a semi-comultiplication module.

(c) The assertion one is clear. For assertion two, set $M = \mathbb{Q}$. $\mathbb{Q}$ is a semi-comultiplication $\mathbb{Z}$-module, but its homomorphic image $\mathbb{Q}/\mathbb{Z}$ is not a semi-comultiplication $\mathbb{Z}$-module. Because for every prime integer p,

$$\mathbb{Z}_{p^\infty} \neq (0 :_{\mathbb{Q}/\mathbb{Z}} Ann_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})) = (0 :_{\mathbb{Q}/\mathbb{Z}} (0)) = \mathbb{Q}/\mathbb{Z}.$$

(d) If $I^M_{(0)}(M) = (0)$, then M is cotorsion. We assume that $I^M_{(0)}(M) \neq (0)$. Hence $I^M_{(0)}(M)$ is a (0)-second submodule by [4, Corollary 2.10]. Since M is semi-comultiplication,

$$I^M_{(0)}(M) = (0 :_M Ann_R(I^M_{(0)}(M))) = M.$$

Thus M is cotorsion-free.

(e) Let S be a socle R_p-submodule of $Hom_R(R_p, M)$. It is easy to see that $\phi(S)$ is a socle submodule of M, where $\phi : Hom_R(R_p, M) \to M$ given by $f \mapsto f(1/1)$. Hence, $\phi(S) = (0 :_M I)$ for some ideal I of R. Now by Remark 2.14, we have $S = Hom_R(R_p, \phi(S))$ and so $S = Hom_R(R_p, (0 :_M I))$. One can see that

$$Hom_R(R_p, (0 :_M I)) = (0 :_{Hom_R(R_p,M)} IR_p).$$

Therefore $Hom_R(R_p, M)$ is a semi-comultiplication R_p-module.

(f) By part (a) and Proposition 2.4 (a), M is an X^s-injective R-module. Now the proof follows from [2, Theorem 2.11].

□

We are going to give an example of a module which is not semi-comultiplication. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$, where p is a prime integer. Then $S = (1/p + \mathbb{Z}) \oplus \mathbb{Z}_{p^\infty}$ is a socle submodule of M, but $(1/p + \mathbb{Z}) \oplus \mathbb{Z}_{p^\infty} \neq (0 :_M Ann_{\mathbb{Z}}(S)) = M$. Moreover, M is a cotorsion-free module. This example shows that the converse of part (d) of Theorem 2.15 is not true in general.

The next proposition shows the behavior of s-cotop modules over colocalizations. First we need the following lemma.

Lemma 2.16. *Let p and q be prime ideals of R with $q \subseteq p$. Let M be an Artinian R-module such that $cl_p^M(0) = (0)$. Then we have the following.*

(a) *If S is a q-second submodule of M and $r \in R$, then*

$$rHom_R(R_p, S) = Hom_R(R_p, rS).$$

(b) *If S is a q-second submodule of M, then*

$$Ann_R(S) = Ann_R(Hom_R(R_p, S)).$$

Proof. (a) If $r \in Ann_R(S)$, then $rHom_R(R_p, S) = Hom_R(R_p, (0)) = (0)$. Otherwise, the sequence $0 \to S \xrightarrow{r} S \to 0$ is exact because $cl_p^M(0) = (0)$. Therefore, by [15, Proposition 2.4], the sequence

$$0 \to Hom_R(R_p, S) \xrightarrow{r} Hom_R(R_p, S) \to 0$$

is exact. This implies that

$$rHom_R(R_p, S) = Hom_R(R_p, rS) = Hom_R(R_p, S).$$

(b) This follows from part(a).

□

Proposition 2.17. *Let p be a prime ideal of R and let M be an Artinian s-cotop R-module with $cl_p^M(0) = (0)$. Then $Hom_R(R_p, M)$ is an s-cotop R_p-module.*

Proof. Let N be a submodule of the R_p-module $Hom_R(R_p, M)$. By Theorem 2.5, it is enough to prove that $V^{s*}(N) = V^s(sec(N))$. It is clear that $V^{s*}(N) \subseteq V^s(sec(N))$. Conversely, assume that $W \in V^s(sec(N))$. Since $Ann_{R_p}(sec(N)) \subseteq Ann_{R_p}(W)$, we have $Ann_R(sec(N)) \subseteq Ann_R(W) \subseteq p$. By Remark 2.14, $W = Hom_R(R_p, S)$, $sec(N) =$

$\sum_{i\in I} Hom_R(R_p, S_i)$, and $N = Hom_R(R_p, L)$, where $L \leq M$, and S, $S_i \in Spec^s_R(M)$. It is easy to see that $\sum_{i\in I} S_i \subseteq sec(L)$ and so

$$sec(N) = \sum_{i\in I} Hom_R(R_p, S_i) \subseteq Hom_R(R_p, \sum_{i\in I} S_i) \subseteq Hom_R(R_p, sec(L)).$$

By the above arguments and Lemma 2.16, we have

$$\begin{aligned} Ann_R(sec(L)) &\subseteq Ann_R(Hom_R(R_p, sec(L)) \subseteq Ann_R(sec(N)) \\ &\subseteq Ann_R(W) = Ann_R(S). \end{aligned}$$

Therefore, $S \in V^s(sec(L))$ so that $S \in V^{s*}(L)$ by Theorem 2.5 (c). This implies that $W \in V^{s*}(N)$. Hence the proof is completed.

□

Remark 2.18. (a) Let M be an Artinian R-module and let $p \in Spec(R)$ such that $cl^M_p(0) = (0)$. Then by Remark 2.14, the second submodules of the R_p-module $Hom_R(R_p, M)$ are in a one-to-one correspondence with those second submodules S of M which satisfy $Ann_R(S) \subseteq p$.

(b) Let (R, m) be a quasi local ring and let $\{p_i\}_{i\in I}$ be a collection of prime ideals of R. Then $(\bigcap_{i\in I} p_i)_m = \bigcap_{i\in I}(p_i)_m$.

(c) Let X, Y be two sets and let $g : X \to Y$ be a map from X into Y. Suppose τ is an arbitrary topology on X. Set $U = \{A \subseteq Y | \, g^{-1}(A) \in \tau\}$. Then U is a topology in Y, called the induced topology by g in Y. We denote this topology by $g(\tau)$. In fact U is the coarser topology in Y that $g : (X, \tau) \to (Y, U)$ is continuous. Moreover, if g is bijective, then $g(\tau) = \{g(w) | \, w \in \tau\}$.

We use f_p to denote the natural map $f_p : Spec^s_{R_p}(Hom_R(R_p, M)) \to Spec^s_R(M)$ defined by $W \mapsto \phi(W)$, where $\phi : Hom_R(R_p, M) \to M$ given by $f \mapsto f(1/1)$.

Theorem 2.19. *Let M be an Artinian R-module and $p \in Spec(R)$ such that $cl^M_p(0) = (0)$. Let $f : \overline{R} \to \overline{R_p}$ be the canonical homomorphism and let $f^* : Spec(\overline{R_p}) \to Spec(\overline{R})$ be the associated mapping. Consider the following diagram.*

$$\begin{array}{ccc} (Spec^s_{R_p}(Hom_R(R_p, M)), \tau^{s*}_{Hom_R(R_p,M)}) & \xrightarrow{f_p} & (Spec^{s*}_R(M), \tau^{s*}_M) \\ \Big\downarrow{\psi_p} & & \Big\downarrow{\psi} \\ Spec(\overline{R_p}) & \xrightarrow{f^*} & Spec(\overline{R}) \end{array}$$

with natural maps. Then we have the following.

(a) *The above diagram is commutative.*
(b) *If (R, p) is a quasi local ring, then*
  (i) *f_p is bijective.*
  (ii) *If $Hom_R(R_p, M)$ is a secondful (X^s-injective) R_p-module, then ψ is surjective (injective) so that M is a secondful (X^s-injective) R-module.*
  (iii) *If $Hom_R(R_p, M)$ is a cotop R_p-module, then we have*

$$f_p(\tau^s_{Hom_R(R_p,M)}) \subseteq \tau^s_M \subseteq f_p(\tau^{s*}_{Hom_R(R_p,M)}) = \tau^{s*}_M.$$

Consequently, M is cotop R-module and all maps in the above diagram are continuous.

Proof. (a) Use Lemma 2.16 (b) and Remark 2.18 (a).

(b) (i) By Remark 2.18 (a).

(b) (ii) Consider the map $f^* : Spec(\overline{R_p}) \to Spec(\overline{R})$ given by $f^*(q) = f^{-1}(q)$, where $f : \overline{R} \to \overline{R_p}$ is the canonical homomorphism and $q \in Spec(\overline{R_p})$. Then by [8, p. 46, Exercise 21], f^* is a homeomorphism of $Spec(\overline{R_p})$ onto its image in $Spec(\overline{R})$. Since (R, p) is a quasi-local ring, f^* is a surjective map so that it is a homeomorphism. Now the claim follows from part (a) and part (b)(i).

(b) (iii) First we show that $f_p(\tau^s_{Hom_R(R_p,M)}) \subseteq \tau^s_M$. It is enough to prove

$$f_p(\{V^s(W) \mid W \leq Hom_R(R_p, M)\}) \subseteq \{V^s(N) \mid N \leq M\}.$$

To see this, let

$$A \in f_p(\{V^s(W) \mid W \leq Hom_R(R_p, M)\})$$

so that

$$A = f_p(V^s(Hom_R(R_p, L)))$$

for some submodule L of M by Remark 2.14. We show that A is closed in $(Spec^s_R(M), \tau^s_M)$ or equivalently, $V^s(\sum_{S\in A} S) = A$ by [2, Proposition 5.1]. Clearly, $A \subseteq V^s(\sum_{S\in A} S)$. Now let $S' \in V^s(\sum_{S\in A} S)$. It follows that

$$\bigcap_{S\in A} Ann_R(S) = Ann_R(\sum_{S\in A} S) \subseteq Ann_R(S').$$

So that $\bigcap_{S\in A}(Ann_R(S))_p \subseteq (Ann_R(S'))_p$ by Remark 2.18 (b). On the other hand we have

$$(Ann_R(S'))_p = (Ann_R(Hom_R(R_p, S')))_p = Ann_{R_p}(Hom_R(R_p, S')).$$

Thus
$$\bigcap_{S\in A} Ann_{R_p}(Hom_R(R_p,S) \subseteq Ann_{R_p}(Hom_R(R_p,S')).$$
But $S \in A$ implies that $S = f_p(Hom_R(R_p,S))$ and so
$$Hom_R(R_p,S) \in V^s(Hom_R(R_p,L))$$
or
$$Ann_{R_p}(Hom_R(R_p,L)) \subseteq Ann_{R_p}(Hom_R(R_p,S')).$$
It follows that $S' \in f_p(V^s(Hom_R(R_P,L)) = A$. To complete the first assertion, since $\tau^s_M \subseteq \tau^{s*}_M$, it is enough to show that $f_p(\tau^{s*}_{Hom_R(R_p,M)}) = \tau^{s*}_M$. As f_p is bijective, it suffices to show that
$$D := f_p(\{V^{s*}(W) \mid W \leq Hom_R(R_p,M)\}) = \{V^{s*}(N) \mid N \leq M\}.$$
If $K \in D$, there exists a submodule L of M such that
$$K = f_P(V^{s*}(Hom_R(R_p,L))$$
by Remark 2.14. It is easy to check that $K = V^{s*}(L)$. This implies that $K \in \{V^{s*}(N) \mid N \leq M\}$. We have similar arguments for the reverse inclusion. Therefore, $f_p(\tau^{s*}_{Hom_R(R_p,M)}) = \tau^{s*}_M$, so M is a cotop R-module. To see the last assertion, we note that
$$f_p : (Spec^s_{R_p}(Hom_R(R_p,M)), \tau^{s*}_{Hom_R(R_p,M)}) \to (Spec^s_R(M), \tau^{s*}_M)$$
is a continuous map by the above arguments. Also f^* is continuous map by [8, p. 46, Exercize 21 (i)]. Moreover, if L is a cotop R-module, then the natural map $\psi : (Spec^s_R(L), \tau^{s*}_L) \to Spec(\overline{R})$ is always a continuous map (for, if $\overline{I}$ is an ideal of $\overline{R}$, $\psi^{-1}(V(\overline{I})) = V^s((0 :_M I)) = V^{s*}((0 :_M I))$ by [2, Proposition 3.6] and [2, Lemma 3.3 (c)]). As $Hom_R(R_p,M)$ and M are cotop modules, it follows that ψ and ψ_p are continuous as desired. This completes the proof.

□

Corollary 2.20. *Let (R,p) be a quasi-local ring and M an Artinian R-module with $cl^M_p(0) = (0)$. Then M is an s-cotop R-module if and only if $Hom_R(R_p,M)$ is an s-cotop R_p-module.*

Proof. ($\Rightarrow$) This follows from Proposition 2.17. To see the reverse implication, since $Hom_R(R_p,M)$ is an s-cotop R_p-module, we have $\tau^s_{Hom_R(R_p,M)} = \tau^{s*}_{Hom_R(R_p,M)}$. Thus $f_p(\tau^s_{Hom_R(R_p,M)}) = f_p(\tau^{s*}_{Hom_R(R_p,M)})$. Now by Theorem 2.19 (b), we have $\tau^s_M = \tau^{s*}_M$. This completes the proof.

□

A spectral space is a topological space homeomorphic to the prime spectrum of a commutative ring equipped with the Zariski topology.

Spectral spaces have been characterized by M. Hochster as quasi-compact T_0-space having a quasi-compact open base closed under finite intersections and each irreducible closed subset has a generic point [13].

Lemma 2.21. *Let M be a strongly cotop module and ψ be the natural map of X^s. Then $(X^s, \tau^s) = (X^s, \tau^{s*}) \cong Im(\psi)$.*

Proof. $\psi|_{Im(\psi)}$ is bijective. Also we have

$$\begin{aligned}\psi(V^s(N)) &= \{\overline{Ann_R(S)} \mid S \in X^s,\ Ann_R(N) \subseteq Ann_R(S)\}\\ &= V(Ann_R(N)) \cap Im(\psi).\end{aligned}$$

This implies that ψ is continuous and a closed map by [2, Proposition 3.6]. Consequently

$$(X^s, \tau^s) = (X^s, \tau^{s*}) \cong Im(\psi).$$

□

Theorem 2.22. *Let M be an R-module. Then (X^s, τ^s_M) and (X^s, τ^{s*}) are spectral spaces in each of the following cases.*

(a) *M is an s-cotop R-module and $Im(\psi)$ is a closed subspace of $Spec(\overline{R})$, where ψ is a natural map of X^s (in particular, when M is a secondful R-module).*
(b) *$Spec(\overline{R})$ is a Noetherian space (in particular, when R is a Noetherian ring).*
(c) *(R, p) is a quasi-local ring, M is an Artinian R-module with $cl^M_p(0) = (0)$, and $Hom_R(R_p, M)$ is a secondful s-cotop R_p-module.*
(d) *R is a PID and M is an s-cotop R-module .*
(e) *R is PID and M has the property listed in (b), (c), and (d) of Theorem 2.9.*

Proof. (a) This follows from [2, Theorem 6.5].

(b) As $Spec(\overline{R})$ is a Noetherian space, (X^s, τ^{s*}) is also Noetherian by Lemma 2.21. Now the claim follows from [7, Corollary 3.3].
(c) By Theorem 2.19 (b) and Corollary 2.20, M is a secondful s-cotop R-module. Hence by part(a), (X^s, τ^s_M) and (X^s, τ^{s*}_M) are both spectral spaces.
(d) Let $V^s(N)$ be a closed subset of X^s for some submodule N of M. If $V^s(N)$ is infinite, then $Ann_R(N)$ is contained in infinite number of prime ideals of R because M is X^s-injective by Proposition 2.4 (a). Since R is PID, this implies that $Ann_R(N) = (0)$, so $V^{s*}(N) = X^s$. It follows that $\tau^s = \tau^{s*} \subseteq \tau^{fc}$. Thus

$(X^s, \tau^{s*} = \tau^s)$ is a Noetherian topological space and hence by [7, Corollary 3.3], $(X^s, \tau^{s*} = \tau^s)$ is a spectral space.

(e) This is an immediate consequence of part (d) and Theorem 2.9. $\square$

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H. Ansari-Toroghy

Department of pure Mathematics, Faculty of mathematical Sciences, University of

Guilan, P. O. Box 41335-19141, Rasht, Iran.
Email: ansari@guilan.ac.ir

S. S. Pourmortazavi
Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
Email: mortazavi@phd.guilan.ac.ir

S. Keyvani
Department of Mathematics, Bandar Anzali Branch, Islamic Azad University, Bandar Anzali, Iran.
Email: keivani@bandaranzaliiau.ac.ir