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A numerical algorithm for solving a class of matrix equations

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Abstract. In this paper, we present a numerical algorithm for solving matrix equations $(A \otimes B)X = F$ by extending the well-known Gaussian elimination for $Ax = b$. The proposed algorithm has a high computational efficiency. Two numerical examples are provided to show the effectiveness of the proposed algorithm.

Keywords: Gaussian elimination, Kronecker product, matrix equation.
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1 Introduction

Numerical solutions or iterative algorithms for different matrix equations have received much attention [34, 22, 23, 11]. For example, Charnsethikul presented a numerical algorithm for solving $n \times n$ linear equations $\boldsymbol{AX} = \boldsymbol{b}$ with parameters covariances [2]. The iterative algorithms can solve linear matrix equations [10, 9, 25, 29, 17] but the Gaussian elimination method is direct and important for solving linear equations [20, 15]. In order to avoid

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the error accumulations and to improve the numerical stability, several pivoting strategies have been adopted [15, 14], e.g., the partial pivoting strategy, the complete pivoting strategy and the rook pivoting strategy. Studies on Gaussian elimination include the pivoting strategies [28], stabilities [27] and coefficient matrices [15].

The matrix equations play an important role in system theory [32, 12, 3, 5], control theory [26, 31, 30, 18], stability analysis [21, 24, 13, 4]. A conventional method for solving equations $\boldsymbol{AXB} = \boldsymbol{F}$ is to use the Kronecker product [15]. However, high dimensions of the associated matrices result in heavy computational burden [15]. There exist many methods which transform the matrix into forms for which solutions may be readily computed, such as the Jordan canonical form [19], the companion form [1] and the Hessenberg-Schur form [16]. However, these methods require computing additional matrix transformations or decompositions. Besides these methods, the iterative algorithms [32, 33] and the hierarchical identification principle [6, 7, 8] have also been used to solve the linear equations. Recently, the solution of matrix equation $\boldsymbol{AXB} = \boldsymbol{F}$ has been discussed under different conditions [6]. In this paper, we consider the matrix equation $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$ and present a new and efficient algorithm based on the Gaussian elimination.

This paper is organized as follows. Section 2 introduces the Gaussian elimination for equations $\boldsymbol{AX} = \boldsymbol{F}$. Section 3 discusses numerical algorithms for matrix equations $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$. Section 4 gives two numerical examples to illustrate the effectiveness of the proposed algorithm. Finally, we provide some concluding remarks in Section 5.

2 Gaussian elimination for $\boldsymbol{AX}$=$\boldsymbol{F}$

Consider the following matrix equation

$$\boldsymbol{AX} = \boldsymbol{F}, \tag{1}$$

where $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ and $\boldsymbol{F} \in \mathbb{R}^{n\times m}$ are given constant matrices, $\boldsymbol{X} \in \mathbb{R}^{n\times m}$ is the unknown matrix to be solved. Let

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_n \end{bmatrix} \in \mathbb{R}^{n\times m},\ \ \boldsymbol{f}_i \in \mathbb{R}^{1\times m},\ \ i = 1, 2, \ldots, n.$$

Assume that $\boldsymbol{A}$ is invertible and let $[\boldsymbol{A}|\boldsymbol{F}]^{(1)} := [\boldsymbol{A}|\boldsymbol{F}]$ be the augmented matrix of system (1), and denoted as

$$[\boldsymbol{A}|\boldsymbol{F}]^{(1)} = \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} & \boldsymbol{f}_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} & \boldsymbol{f}_n^{(1)} \end{array}\right],$$

where

$$\begin{aligned} a_{ij}^{(1)} &= a_{ij},\ i,j = 1,2,\ldots,n, \\ \boldsymbol{f}_i^{(1)} &= \boldsymbol{f}_i \in \mathbb{R}^{1\times m},\ i = 1,2,\ldots,n. \end{aligned}$$

With these symbols, we give the Gaussian elimination for solving matrix equations $\boldsymbol{AX} = \boldsymbol{F}$.

Algorithm 1.

1. For $i = 1$, let

$$|a_{j1}^{(1)}| := \max\{|a_{11}^{(1)}|, |a_{21}^{(1)}|, \ldots, |a_{n1}^{(1)}|\},$$

interchange the 1st row and jth row. If $\boldsymbol{A}$ is invertible, then $a_{11}^{(1)} \neq 0$ can be used to eliminate $a_{21}^{(1)}, a_{31}^{(1)}, \ldots, a_{n1}^{(1)}$. Let $m_{k1} := a_{k1}^{(1)}/a_{11}^{(1)}$, $k = 2,3,\ldots,n$, we have

$$[\boldsymbol{A}|\boldsymbol{F}]^{(2)} := \left[\begin{array}{cccc|c} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} & \boldsymbol{f}_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & a_{n2}^{(2)} & \cdots & a_{nn}^{(2)} & \boldsymbol{f}_n^{(2)} \end{array}\right],$$

where

$$\begin{aligned} a_{kj}^{(2)} &= a_{kj}^{(1)} - m_{k1}a_{1j}^{(1)},\ k = 2,3,\ldots,n,\ j = 2,3,\ldots,n, \\ \boldsymbol{f}_k^{(2)} &= \boldsymbol{f}_k^{(1)} - m_{k1}\boldsymbol{f}_1^{(1)},\ k = 2,3,\ldots,n. \end{aligned}$$

2. For $i = 2$, let

$$|a_{j2}^{(2)}| := \max\{|a_{22}^{(2)}|, |a_{32}^{(2)}|, \ldots, |a_{n2}^{(2)}|\},$$

interchange the 2nd row and jth row. If $\boldsymbol{A}$ is invertible, then $a_{22}^{(2)} \neq 0$ can be used to eliminate $a_{32}^{(2)}, a_{42}^{(2)}, \ldots, a_{n2}^{(2)}$. Set

$$m_{k2} := \frac{a_{k2}^{(2)}}{a_{22}^{(2)}}, \ k = 3, 4, \ldots, n,$$

and subtract m_{k2} times the second row of $[\boldsymbol{A}|\boldsymbol{F}]^{(2)}$ from the kth row gives

$$[\boldsymbol{A}|\boldsymbol{F}]^{(3)} := \left[\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & \boldsymbol{f}_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & \boldsymbol{f}_3^{(3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & a_{n3}^{(3)} & \cdots & a_{nn}^{(3)} & \boldsymbol{f}_n^{(3)} \end{array}\right],$$

where

$$\begin{aligned} a_{kj}^{(3)} &= a_{kj}^{(2)} - m_{k2} a_{2j}^{(2)}, \ k = 3, 4, \ldots, n, \ j = 3, 4, \ldots, n, \\ \boldsymbol{f}_k^{(3)} &= \boldsymbol{f}_k^{(2)} - m_{k2} \boldsymbol{f}_2^{(2)}, \ k = 3, 4, \ldots, n. \end{aligned}$$

3. For $i = 3, 4, \ldots, n$, continuing in this way, let

$$|a_{ji}^{(i)}| = \max\{|a_{ii}^{(i)}|, |a_{i+1,i}^{(i)}|, \ldots, |a_{ni}^{(i)}|\},$$

interchange the ith row and jth row. If $\boldsymbol{A}$ is invertible then $a_{ii}^{(i)} \neq 0, \ i = 3, 4, \ldots, n$. Set

$$m_{ki} := \frac{a_{ki}^{(i)}}{a_{ii}^{(i)}}, \ i = 3, 4, \ldots, n, \ k = i+1, i+2, \ldots, n$$

and subtract m_{ki} times the ith row of $[\boldsymbol{A}|\boldsymbol{F}]^{(i)}$ from the kth row. After $n-3$ steps we end up with

$$[\boldsymbol{A}|\boldsymbol{F}]^{(n)} := \left[\begin{array}{ccccc|c} a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & \cdots & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & \cdots & a_{2n}^{(2)} & \boldsymbol{f}_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & \cdots & a_{3n}^{(3)} & \boldsymbol{f}_3^{(3)} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & a_{nn}^{(n)} & \boldsymbol{f}_n^{(n)} \end{array}\right]. \tag{2}$$

Here $a_{kj}^{(i+1)}$ and $\boldsymbol{f}_k^{(i+1)}$ satisfy

$$\begin{aligned} a_{kj}^{(i+1)} &= a_{kj}^{(i)} - m_{ki}a_{ij}^{(i)},\ i = 3, 4, \ldots, n, \\ &\quad k = i+1, i+2, \cdots, n,\ j = i+1, i+2, \ldots, n, \\ \boldsymbol{f}_k^{(i+1)} &= \boldsymbol{f}_k^{(i)} - m_{ki}\boldsymbol{f}_i^{(i)},\ i = 3, 4, \ldots, n,\ k = i+1, i+2, \ldots, n. \end{aligned}$$

4. Referring to (2), we can get the linear system,

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1^{(1)} \\ \boldsymbol{f}_2^{(2)} \\ \vdots \\ \boldsymbol{f}_n^{(n)} \end{bmatrix}, \tag{3}$$

where

$$\boldsymbol{X} := \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix} \in \mathbb{R}^{n\times m},\ \boldsymbol{X}_i \in \mathbb{R}^{1\times m},\ i = 1, 2, \ldots, n.$$

From (3), we have

$$\boldsymbol{X}_n = \frac{\boldsymbol{f}_n^{(n)}}{a_{nn}^{(n)}} := \boldsymbol{P}_n. \tag{4}$$

The current augmented matrix corresponding to (3) is denoted as

$$[\boldsymbol{A}|\boldsymbol{F}]_{(1)}^{(n)} = \left[\begin{array}{ccccc|c} a_{11}^{(1)} & \cdots & a_{1,n-2}^{(1)} & a_{1,n-1}^{(1)} & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & a_{n-2,n-2}^{(n-2)} & a_{n-2,n-1}^{(n-2)} & a_{n-2,n}^{(n-2)} & \boldsymbol{f}_{n-2}^{(n-2)} \\ 0 & \ddots & 0 & a_{n-1,n-1}^{(n-1)} & a_{n-1,n}^{(n-1)} & \boldsymbol{f}_{n-1}^{(n-1)} \\ 0 & \cdots & 0 & 0 & 1 & \boldsymbol{P}_n \end{array}\right].$$

5. According to (3) and (4), we get

$$\boldsymbol{X}_{n-1} = \frac{1}{a_{n-1,n-1}^{(n-1)}} \left[\boldsymbol{f}_{n-1}^{(n-1)} - a_{n-1,n}^{(n-1)}\boldsymbol{X}_n\right] := \boldsymbol{P}_{n-1}. \tag{5}$$

The current augmented matrix corresponding to (3) is denoted as

$$[\boldsymbol{A}|\boldsymbol{F}]_{(2)}^{(n)} = \left[\begin{array}{ccccc|c} a_{11}^{(1)} & \cdots & a_{1,n-2}^{(1)} & a_{1,n-1}^{(1)} & a_{1n}^{(1)} & \boldsymbol{f}_1^{(1)} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{n-2,n-2}^{(n-2)} & a_{n-2,n-1}^{(n-2)} & a_{n-2,n}^{(n-2)} & \boldsymbol{f}_{n-2}^{(n-2)} \\ 0 & \cdots & 0 & 1 & 0 & \boldsymbol{P}_{n-1} \\ 0 & \cdots & 0 & 0 & 1 & \boldsymbol{P}_n \end{array}\right].$$

6. According to (3), (4) and (5), we have

$$\boldsymbol{X}_i = \frac{1}{a_{ii}^{(i)}}\left[\boldsymbol{f}_i^{(i)} - \sum_{j=i+1}^{n} a_{ij}^{(i)}\boldsymbol{X}_j\right] := \boldsymbol{P}_i,\ \ i = n-2, n-3, \ldots, 1. \quad (6)$$

It follows from (4), (5) and (6) that

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \\ \vdots \\ \boldsymbol{P}_n \end{bmatrix},$$

and its augmented matrix is denoted as

$$[\boldsymbol{A}|\boldsymbol{F}]_{(n)}^{(n)} = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \boldsymbol{P}_1 \\ 0 & 1 & \cdots & 0 & \boldsymbol{P}_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \boldsymbol{P}_n \end{array}\right].$$

From the above discussion, we get a solution to the equation $\boldsymbol{AX} = \boldsymbol{F}$ by Algorithm 1. In the following section we will tackle matrix equation $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$ by using the result in Section 2.

3 The matrix equation $(\mathbf{A} \otimes \mathbf{B})\mathbf{X} = \mathbf{F}$

Consider the matrix equation

$$(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}, \quad (7)$$

where $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{F} \in \mathbb{R}^{(nm)\times l}$ are given constant matrices, $\boldsymbol{X} \in \mathbb{R}^{(nm)\times l}$ is the unknown matrix to be solved.

Let $\boldsymbol{I}_n$ denote an $n \times n$ identity matrix. For an $m \times l$ matrix

$$\boldsymbol{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_l] \in \mathbb{R}^{m \times l},\ \boldsymbol{y}_i \in \mathbb{R}^m,$$

Let col$[\boldsymbol{Y}]$ represent an ml-dimensional vector formed by the columns of $\boldsymbol{Y}$, i.e.,

$$\mathrm{col}[\boldsymbol{Y}] := \begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \boldsymbol{y}_l \end{bmatrix} \in \mathbb{R}^{ml}.$$

Using the relationship $\boldsymbol{A} \otimes \boldsymbol{B} = (\boldsymbol{A} \otimes \boldsymbol{I}_m)(\boldsymbol{I}_n \otimes \boldsymbol{B})$ in [35] and from Eq. (7), we have

$$(\boldsymbol{A} \otimes \boldsymbol{I}_m)(\boldsymbol{I}_n \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}.$$

It follows that

$$\begin{bmatrix} a_{11}\boldsymbol{I}_m & a_{12}\boldsymbol{I}_m & \cdots & a_{1n}\boldsymbol{I}_m \\ a_{21}\boldsymbol{I}_m & a_{22}\boldsymbol{I}_m & \cdots & a_{2n}\boldsymbol{I}_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{I}_m & a_{n2}\boldsymbol{I}_m & \cdots & a_{nn}\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{B} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \vdots \\ \boldsymbol{F}_n \end{bmatrix}, \tag{8}$$

where

$$\boldsymbol{X} := \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix}, \quad \boldsymbol{F} := \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \vdots \\ \boldsymbol{F}_n \end{bmatrix},\ \boldsymbol{X}_i \in \mathbb{R}^{m \times l},\ \boldsymbol{F}_i \in \mathbb{R}^{m \times l}.$$

Eq. (8) can be written as

$$\begin{bmatrix} a_{11}\boldsymbol{I}_m & a_{12}\boldsymbol{I}_m & \cdots & a_{1n}\boldsymbol{I}_m \\ a_{21}\boldsymbol{I}_m & a_{22}\boldsymbol{I}_m & \cdots & a_{2n}\boldsymbol{I}_m \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{I}_m & a_{n2}\boldsymbol{I}_m & \cdots & a_{nn}\boldsymbol{I}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{B}\boldsymbol{X}_1 \\ \boldsymbol{B}\boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{B}\boldsymbol{X}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \vdots \\ \boldsymbol{F}_n \end{bmatrix},$$

or in a compact form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \{\mathrm{col}[(\boldsymbol{B}\boldsymbol{X}_1)^{\mathrm{T}}]\}^{\mathrm{T}} \\ \{\mathrm{col}[(\boldsymbol{B}\boldsymbol{X}_2)^{\mathrm{T}}]\}^{\mathrm{T}} \\ \vdots \\ \{\mathrm{col}[(\boldsymbol{B}\boldsymbol{X}_n)^{\mathrm{T}}]\}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \{\mathrm{col}[\boldsymbol{F}_1^{\mathrm{T}}]\}^{\mathrm{T}} \\ \{\mathrm{col}[\boldsymbol{F}_2^{\mathrm{T}}]\}^{\mathrm{T}} \\ \vdots \\ \{\mathrm{col}[\boldsymbol{F}_n^{\mathrm{T}}]\}^{\mathrm{T}} \end{bmatrix}. \tag{9}$$

Let

$$\boldsymbol{G} := \begin{bmatrix} \{\mathrm{col}[\boldsymbol{F}_1^{\mathrm{T}}]\}^{\mathrm{T}} \\ \{\mathrm{col}[\boldsymbol{F}_2^{\mathrm{T}}]\}^{\mathrm{T}} \\ \vdots \\ \{\mathrm{col}[\boldsymbol{F}_n^{\mathrm{T}}]\}^{\mathrm{T}} \end{bmatrix} \in \mathbb{R}^{n\times(ml)}, \tag{10}$$

and $[\boldsymbol{A}|\boldsymbol{G}]$ be the augmented matrix of Eq. (9). According to Algorithm 1, simplifying $[\boldsymbol{A}|\boldsymbol{G}]$ gives

$$[\boldsymbol{A}|\boldsymbol{G}]_{(n)}^{(n)} = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \boldsymbol{P}_1 \\ 0 & 1 & \ddots & \vdots & \boldsymbol{P}_2 \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & \boldsymbol{P}_n \end{array}\right]. \tag{11}$$

Thus, we obtain an important intermediate result

$$\{\mathrm{col}[(\boldsymbol{B}\boldsymbol{X}_i)^{\mathrm{T}}]\}^{\mathrm{T}} = \boldsymbol{P}_i \in \mathbb{R}^{1\times(ml)},\ i = 1, 2, \ldots, n.$$

Let

$$\begin{cases} \boldsymbol{P}_i & = & [\boldsymbol{P}_{i1}, \boldsymbol{P}_{i2}, \ldots, \boldsymbol{P}_{im}],\ \boldsymbol{P}_{ij} \in \mathbb{R}^{1\times l},\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m, \\ \boldsymbol{H}_i & = & \begin{bmatrix} \boldsymbol{P}_{i1} \\ \boldsymbol{P}_{i2} \\ \vdots \\ \boldsymbol{P}_{im} \end{bmatrix} \in \mathbb{R}^{m\times l},\ i = 1, 2, \ldots, n. \end{cases} \tag{12}$$

According to the definition of $\mathrm{col}[\boldsymbol{X}]$, we have $\boldsymbol{B}\boldsymbol{X}_i = \boldsymbol{H}_i$, $i = 1, 2, \ldots, n$. This means that

$$\boldsymbol{B}[\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n] = [\boldsymbol{H}_1, \boldsymbol{H}_2, \ldots, \boldsymbol{H}_n]. \tag{13}$$

Then the solution of Eq. (7) can be obtained by Algorithm 1. The above procedures can be summarized as Algorithm 2.

Algorithm 2.

1. Form $\boldsymbol{G}$ by (10).
2. According to Algorithm 1, simplify the augmented matrix $[\boldsymbol{A}|\boldsymbol{G}]$ by (11).
3. Form $\boldsymbol{H}_i$ by (12).
4. Obtain the solution of Eq. (7) by solving (13).

4 Numerical examples

Example 1. Suppose that $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 13 & 7 \\ 5 & 6 \\ -7 & 2 \end{bmatrix}.$$

According Algorithm 2, we construct matrix $\boldsymbol{G}$. Letting $[\boldsymbol{A}|\boldsymbol{G}]^{(1)} := [\boldsymbol{A}|\boldsymbol{G}]$ gives

$$[\boldsymbol{A}|\boldsymbol{G}]^{(1)} = \left[\begin{array}{cc|cccc} 1 & 1 & 7 & 15 & 13 & 7 \\ 2 & -1 & 5 & 6 & -7 & 2 \end{array}\right].$$

Consider the entries of the first column, due to $2 > 1$, interchange these two rows, we have

$$\left[\begin{array}{cc|cccc} 2 & -1 & 5 & 6 & -7 & 2 \\ 1 & 1 & 7 & 15 & 13 & 7 \end{array}\right].$$

Adding $-1/2$ times the first row to the second row gives

$$[\boldsymbol{A}|\boldsymbol{G}]^{(2)} = \left[\begin{array}{cc|cccc} 2 & -1 & 5 & 6 & -7 & 2 \\ 0 & 1.5 & 4.5 & 12 & 16.5 & 6 \end{array}\right].$$

Dividing the second row of $[\boldsymbol{A}|\boldsymbol{G}]^{(2)}$ by $a_{22}^{(2)} = 1.5$ gives

$$[\boldsymbol{A}|\boldsymbol{G}]_{(1)}^{(2)} = \left[\begin{array}{cc|cccc} 2 & -1 & 5 & 6 & -7 & 2 \\ 0 & 1 & 3 & 8 & 11 & 4 \end{array}\right].$$

Adding the second row to the first row of the matrix $[\boldsymbol{A}|\boldsymbol{G}]_{(1)}^{(2)}$, we have

$$\left[\begin{array}{cc|cccc} 2 & 0 & 8 & 14 & 4 & 6 \\ 0 & 1 & 3 & 8 & 11 & 4 \end{array}\right].$$

Dividing the first row by $a_{11}^{(1)} = 2$ gives

$$[\boldsymbol{A}|\boldsymbol{G}]_{(2)}^{(2)} = \left[\begin{array}{cc|cccc} 1 & 0 & 4 & 7 & 2 & 3 \\ 0 & 1 & 3 & 8 & 11 & 4 \end{array}\right].$$

Then we have

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_1 \\ \boldsymbol{P}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} \end{bmatrix} = \begin{bmatrix} 4 & 7 & 2 & 3 \\ 3 & 8 & 11 & 4 \end{bmatrix},$$

$$\boldsymbol{H}_1 = \begin{bmatrix} \boldsymbol{P}_{11} \\ \boldsymbol{P}_{12} \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix}, \quad \boldsymbol{H}_2 = \begin{bmatrix} \boldsymbol{P}_{21} \\ \boldsymbol{P}_{22} \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 11 & 4 \end{bmatrix},$$

$$[\boldsymbol{B}|\boldsymbol{H}_1,\boldsymbol{H}_2] = \left[\begin{array}{rr|rrrr} 1 & 1 & 4 & 7 & 3 & 8 \\ -1 & 1 & 2 & 3 & 11 & 4 \end{array}\right].$$

According to Algorithm 1, we have

$$[\boldsymbol{B}|\boldsymbol{H}_1,\boldsymbol{H}_2]^{(2)}_{(2)} = \left[\begin{array}{rr|rrrr} 1 & 0 & 1 & 2 & -4 & 2 \\ 0 & 1 & 3 & 5 & 7 & 6 \end{array}\right].$$

Finally, we obtain the solution for the equation $(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{X}=\boldsymbol{F}$ with

$$\boldsymbol{X} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ -4 & 2 \\ 7 & 6 \end{bmatrix}.$$

Example 2. Consider matrix equation $(\boldsymbol{A}\otimes\boldsymbol{B})\boldsymbol{X}=\boldsymbol{F}$, where

$$\boldsymbol{A} = \begin{bmatrix} 2 & -3 \\ -1 & -2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 3 & -2 & 1 \\ 4 & 0 & 2 \\ -1 & -3 & -4 \end{bmatrix},$$

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \end{bmatrix} = \begin{bmatrix} -60 & -77 \\ -58 & -84 \\ 31 & 44 \\ -19 & -28 \\ -6 & -42 \\ -12 & 13 \end{bmatrix}.$$

According to Algorithm 2, $\boldsymbol{G}$ can be obtained by

$$\boldsymbol{G} = \begin{bmatrix} \{\mathrm{col}[\boldsymbol{F}_1^{\mathrm{T}}]\}^{\mathrm{T}} \\ \{\mathrm{col}[\boldsymbol{F}_2^{\mathrm{T}}]\}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} -60 & -77 & -58 & -84 & 31 & 44 \\ -19 & -28 & -6 & -42 & -12 & 13 \end{bmatrix},$$

and the augmented matrix $[\boldsymbol{A}|\boldsymbol{G}]$ can be written as

$$[\boldsymbol{A}|\boldsymbol{G}]^{(1)} = \left[\begin{array}{rr|rrrrrr} 2 & -3 & -60 & -77 & -58 & -84 & 31 & 44 \\ -1 & -2 & -19 & -28 & -6 & -42 & -12 & 13 \end{array}\right].$$

Simplifying the augmented matrix $[\boldsymbol{A}|\boldsymbol{G}]^{(1)}$ gives

$$[\boldsymbol{A}|\boldsymbol{G}]^{(2)}_{(2)} = \left[\begin{array}{rr|rrrrrr} 1 & 0 & -9 & -10 & -14 & -6 & 14 & 7 \\ 0 & 1 & 14 & 19 & 10 & 24 & -1 & -10 \end{array}\right],$$

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{12} & \boldsymbol{P}_{13} \\ \boldsymbol{P}_{21} & \boldsymbol{P}_{22} & \boldsymbol{P}_{23} \end{bmatrix} = \begin{bmatrix} -9 & -10 & -14 & -6 & 14 & 7 \\ 14 & 19 & 10 & 24 & -1 & -10 \end{bmatrix}.$$

Constructing the matrix

$$[\boldsymbol{H}_1, \boldsymbol{H}_2] = \begin{bmatrix} \boldsymbol{P}_{11} & \boldsymbol{P}_{21} \\ \boldsymbol{P}_{12} & \boldsymbol{P}_{22} \\ \boldsymbol{P}_{13} & \boldsymbol{P}_{23} \end{bmatrix} = \begin{bmatrix} -9 & -10 & 14 & 19 \\ -14 & -6 & 10 & 24 \\ 14 & 7 & -1 & -10 \end{bmatrix},$$

we write the augmented $[\boldsymbol{B}|\boldsymbol{H}_1, \boldsymbol{H}_2]$,

$$[\boldsymbol{B}|\boldsymbol{H}_1, \boldsymbol{H}_2] = \left[\begin{array}{rrr|rrrr} 3 & -2 & 1 & -9 & -10 & 14 & 19 \\ 4 & 0 & 2 & -14 & -6 & 10 & 24 \\ -1 & -3 & -4 & 14 & 7 & -1 & -10 \end{array}\right],$$

which can be transformed into

$$[\boldsymbol{B}|\boldsymbol{H}_1, \boldsymbol{H}_2]^{(3)}_{(3)} = \left[\begin{array}{rrr|rrrr} 1 & 0 & 0 & -2 & 1 & 1 & 5 \\ 0 & 1 & 0 & 0 & 4 & -4 & -1 \\ 0 & 0 & 1 & -3 & -5 & 3 & 2 \end{array}\right].$$

Finally, we obtain the solution for equation $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$,

$$\boldsymbol{X} = \begin{bmatrix} -2 & 1 \\ 0 & 4 \\ -3 & -5 \\ 1 & 5 \\ -4 & -1 \\ 3 & 2 \end{bmatrix}.$$

5 Conclusions

A new and efficient algorithm for solving linear matrix equation $(\boldsymbol{A} \otimes \boldsymbol{B})\boldsymbol{X} = \boldsymbol{F}$ has been presented by using the Gaussian elimination. Two examples have illustrated the effectiveness of the proposed algorithm.

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