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Construction of Lie-Yamaguti and 3-Leibniz Superalgebras via Supertriple and Leibniz Structures

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Abstract. This paper explores the construction of Lie-Yamaguti superalgebras and 3-Leibniz superalgebras from left Leibniz superalgebras and Lie supertriple systems, respectively. We demonstrate that every left representation of a left Leibniz superalgebra $(\mathfrak{g}, \cdot)$ naturally induces a representation of the associated Lie-Yamaguti superalgebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. Furthermore, we prove that equivalent representations of $(\mathfrak{g}, \cdot)$ yield equivalent representations of the associated Lie-Yamaguti superalgebra. Additionally, we introduce nonabelian embedding tensors between Lie supertriple systems and establish that these tensors naturally induce 3-Leibniz superalgebras. These constructions provide a unified framework for understanding relationships between these algebraic structures, with potential applications in differential geometry and theoretical physics, particularly in the study of superhomogeneous spaces and higher algebraic structures arising in string theory.

Keywords: Leibniz superalgebra, Lie supertriple systems, Lie-Yamaguti superalgebras, representation, nonabelian embedding tensors.
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1 Introduction

Lie-Yamaguti algebras (first called generalized Lie triple systems) were introduced in [24] as an algebraic treatment of characteristic properties of the torsion and curvature of reductive homogeneous spaces. Later on, these algebras were called Lie triple algebras in [12] and the

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recent terminology was introduced in [13]. Cohomology groups of Lie-Yamaguti algebras were defined in [23], while their deformations and extensions were considered in [14, 16, 25].

The theory of Lie superalgebras plays an important role in modern mathematics and physics. Recall that Lie superalgebras appeared in [5], as Lie algebras of certain generalized groups, and were discussed intensively by many authors (see, for example, [10, 20]). Lie superalgebras as a $\mathbb{Z}_2$-graded generalization of Lie algebras are considered in [10, 20] while a $\mathbb{Z}_2$-graded generalization of Lie triple systems (called Lie supertriple systems) was first considered in [22]. Next, Lie-Yamaguti superalgebras as a $\mathbb{Z}_2$-graded generalization of Lie-Yamaguti algebras were first considered in [19]. In the same way, the notion of Leibniz algebras was naturally superizable by Dzhumadildaev in [7] and was led to the notion of Leibniz superalgebras, which is a non-supercommutative analog of Lie superalgebras. Recently, left and right Leibniz superalgebras were widely studied in many aspects (see, for instance, [2, 3, 7, 8, 15]).

Although representation theories have been developed separately for Lie supertriple systems [9, 22], Leibniz superalgebras [4], and Lie-Yamaguti superalgebras [1], the relationship between representations of left Leibniz superalgebras and representations of their associated Lie-Yamaguti superalgebras has not been developed in the super setting in the form considered here. In the nonabelian setting, a related gap concerns the interaction between representations of Lie supertriple systems, nonabelian embedding tensors, and 3-Leibniz superalgebras. These observations motivate the present work: our aim is to make explicit the algebraic mechanisms connecting these structures and to provide constructions that allow representations and higher products to be transferred systematically between them.

In the present paper, first we consider the representation of a Lie-Yamaguti superalgebra that is induced by a left representation of a given left Leibniz superalgebra. Second, we introduce the notion of nonabelian embedding tensors between Lie supertriple systems and show that nonabelian embedding tensors induce naturally 3-Leibniz superalgebras. Recall that any representation of a given left Leibniz algebra $(\mathfrak{g}, \cdot)$ induces a representation of the Lie-Yamaguti algebra $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ that is associated with $(\mathfrak{g}, \cdot)$ [17].

These results clarify structural relationships among Leibniz superalgebras, Lie-Yamaguti superalgebras, Lie supertriple systems, and 3-Leibniz superalgebras. Since Lie-Yamaguti structures originally arise in the algebraic study of torsion and curvature of reductive homogeneous spaces, the constructions presented here may be relevant to algebraic formulations appearing in differential geometry. Likewise, because Lie and Leibniz superalgebras occur in mathematical physics, the results may provide an algebraic framework for related superalgebraic constructions. More specifically, these constructions have immediate applications in the study of supergravity theories, where Leibniz superalgebras appear in the algebraic formulation of Courant algebroids and double field theory (see [11]). Lie-Yamaguti superalgebras naturally arise in the study of reductive superhomogeneous spaces, where the torsion and curvature satisfy generalized Bianchi identities. The induced representations established in Theorem 1 provide a systematic method for constructing modules over these geometric structures from simpler Leibniz data. Furthermore, the 3-Leibniz superalgebras arising from nonabelian embedding tensors (Theorem 3) generalize the algebraic structures underlying Nambu-Poisson brackets, which appear in M-theory and higher gauge theories (see [6, 21]). However, we do not pursue specific geometric or physical models in this paper, and these possible connections should therefore be regarded as prospective applications rather than applications established here.

The paper is structured as follows. In section 2 useful notions and results are reminded and the first result is proved. Furthermore, starting from some well-known representations of left Leibniz superalgebras, our first result is applied to give a few examples of nontrivial representations of Lie-Yamaguti superalgebras. In section 3 useful notions and results of nonabelian embedding tensors between Lie supertriple systems as well as the 3-Leibniz superalgebras are reminded and the second result is proved.

Definition 1. *Let $V = V_{\overline{0}} \oplus V_{\overline{1}}$ be a $\mathbb{Z}_2$-graded vector space over the field $\mathbb{K}$. An element x in V is called homogeneous if $x \in V_{\overline{0}}$ or $x \in V_{\overline{1}}$. We denote by $h(V)$ the set of all $\mathbb{Z}_2$-homogeneous elements of V, i.e., $h(V) = h(V_i)$, $i = \overline{0}, \overline{1}$. If a linear map $\Phi : V \longrightarrow V$ satisfies*

$$\Phi(V_i) \subseteq V_{i+\tau}, \ \ \forall \ \ i \in \mathbb{Z}_{\overline{2}}.$$

then Φ is a homogeneous linear map of degree τ, that is $|\Phi| = \tau$, where $\tau \in \mathbb{Z}_2$. If $\tau = \overline{0}$, then Φ is called an even linear map and if $\tau = \overline{1}$, then Φ is called an odd linear map. For a linear map Φ, if $|\Phi|$ occurs, then Φ implies a homogeneous map.

2 Representation of a Lie-Yamaguti superalgebra induced by a representation of a left Leibniz superalgebra

In this section we first remind some useful basic notions and we give the dual representation of $(\mathfrak{g}, \cdot)$. Next we prove that any left representation of a given left Leibniz superalgebra $(\mathfrak{g}, \cdot)$ induces a representations the associated Lie-Yamaguti superalgebra; moreover, it turns out that equivalent left representations of $(\mathfrak{g}, \cdot)$ imply equivalent representations the associated Lie-Yamaguti superalgebra.

Definition 2 ([4, 7]). *A **(left) Leibniz superalgebra** is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ such that $\mathfrak{g}_i : \mathfrak{g}_j \subset \mathfrak{g}_{i+j}, \forall i, j \in \mathbb{Z}_2$ together with a bilinear map $\cdot : \times\mathfrak{g} \to \mathfrak{g}$ satisfying the left Leibniz identity*

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + (-1)^{|x||y|} y \cdot (x \cdot z), \ \forall \ (x, y, z) \in \mathfrak{g}_{|x|} \times \mathfrak{g}_{|y|} \times \mathfrak{g}_{|z|}. \tag{1}$$

Observe that in [4, 7] only right Leibniz algebras were considered, i.e. algebras satisfying the right Leibniz identity

$$(z \cdot y) \cdot x = z \cdot (y \cdot x) \cdot + (-1)^{|x||y|} (z \cdot x) \cdot y, \ \forall \ (x, y, z) \in \mathfrak{g}_{|x|} \times \mathfrak{g}_{|y|} \times \mathfrak{g}_{|z|}.$$

Definition 3 ([4]). *Let $(\mathfrak{g}, \cdot)$ be a left Leibniz superalgebra, V be a $\mathbb{Z}_2$-graded vector space and $r, l : \mathfrak{g} \to \mathrm{End}(V)$ be two even linear maps. Then (V, l, r) is called a left representation of $(\mathfrak{g}, \cdot)$ if for all $x, y \in \mathfrak{g}$:*

$$l(x \cdot y) = [l(x), l(y)], \tag{2}$$
$$r(x \cdot y) = l(x) \circ r(y) + (-1)^{|x||y|} r(y) \circ r(x), \tag{3}$$
$$r(x \cdot y) = [l(x), r(y)]. \tag{4}$$

The vector space V is also called a $\mathfrak{g}$-module.

The conditions (3) and (4) imply

$$r(y)r(x) + r(y)l(x) = 0. \tag{5}$$

Example 1. An example of a left representation of $(\mathfrak{g}, \cdot)$ is the so-called adjoint (or regular) representation that is defined as the triple $(\mathfrak{g}, L, R)$ with linear maps L, R defined respectively by $L : \mathfrak{g} \to \mathrm{End}(V)$, $x \mapsto L_x$ and $R : \mathfrak{g} \to \mathrm{End}(V)$, $x \mapsto R_x$, where L_x (resp. R_x) denotes the left (resp. right) multiplication in $(\mathfrak{g}, \cdot)$, $L_z y := x \cdot y$ and $R_z y := (-1)^{|x||y|} y \cdot x$.

Remark 1. *Let $\mathfrak{g}$ be a left (resp. right) Leibniz superalgebra, V be a $\mathbb{Z}_2$-graded vector space and (r, l) a left (resp. right) representation of $\mathfrak{g}$ in V. Let V^* denote the dual space of V. Let us consider two even linear maps $l^*, r^* : \mathfrak{g} \to \mathrm{End}(V)$ defined by:*

$$l^*(x)(f) = (-1)^{|f||x|} f \circ r(x), \quad r^*(x)(f) = (-1)^{|f||x|} f \circ l(x), \ \forall\ x \in \mathfrak{g}_{|x|}, \ f \in V^*_{|f|}. \tag{6}$$

In general, (V^*, l^*, r^*) is neither a left nor a right representation of $(\mathfrak{g}, \cdot)$.

Proposition 1. *Let $(\mathfrak{g}, \cdot)$ be a left Leibniz algebra and (V, l, r) its left representation. Then $(V^*, l^*, -l^* - r^*)$ is a left representation of $(\mathfrak{g}, \cdot)$.*

This construction generalizes the dual representation concept from Lie superalgebras to Leibniz superalgebras, and is therefore called the dual representation of $(\mathfrak{g}, \cdot)$.

Proof. Let $(\mathfrak{g}, \cdot)$ be a left Leibniz algebra and (V, l, r) its left representation. Let us consider two even linear maps $l^*, r^* : \mathfrak{g} \to \mathrm{End}(V)$ defined by (6). We proceed to check that the triple $(V^*, l^*, -l^* - r^*)$ as defined above satisfies the set of axioms as in Definition 2.2.

For (2) we have

$$\begin{aligned}
[l^*(x), l^*(y)](f) &= l^*(x) \circ l^*(y)(f) - (-1)^{|x||y|} l^*(y) \circ l^*(x)(f) \\
&= (-1)^{|f||y|} l^*(x)(f \circ r(y)) - (-1)^{|x|(|f|+|y|)} l^*(y)(f \circ r(x)) \\
&= (-1)^{|f|(|x|+|y|)+|x||y|} f \circ r(y) \circ r(x) - (-1)^{|f|(|x|+|y|)} f \circ r(x) \circ r(y) \\
&= (-1)^{|f|(|x|+|y|)+|x||y|} f \circ r(y) \circ r(x) + (-1)^{|f|(|x|+|y|)} f \circ l(x) \circ r(y) \\
&= (-1)^{|f|(|x|+|y|)} f \circ \Big(l(x) \circ r(y) + (-1)^{|x||y|} r(y) \circ r(x) \Big) \\
&= (-1)^{|f|(|x|+|y|)} f \circ r(x \cdot y) \\
&= l^*(x \cdot y)(f),
\end{aligned}$$

thus, $[l^*(x), l^*(y)] = l^*(x \cdot y)$. So (3) holds for $(V^*, l^*, -l^* - r^*)$.

For (3) and (4), similar to the proofs of (2), it is easy to see that $(-l^* - r^*)(x \cdot y) = l^*(x) \circ (-l^* - r^*)(y) + (-1)^{|x||y|}(-l^* - r^*)(y) \circ (-l^* - r^*)(x)$ and $(-l^* - r^*)(x \cdot y) = [l^*(x), (-l^* - r^*)(y)]$. So (3) and (4) holds for $(V^*, l^*, -l^* - r^*)$. Consequently, $(V^*, l^*, -l^* - r^*)$ is a left representation of $(\mathfrak{g}, \cdot)$. The proof is completed. □

Definition 4 ([18]). *A **Lie-Yamaguti superalgebra** is a triple $(\mathfrak{g}, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$ where $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ is $\mathbb{K}$-vector superspace, i.e., $\mathbb{Z}_2$-graded vector space, "$[\cdot, \cdot]$" a bilinear map (the binary*

superoperation on $\mathfrak{g}$) and "$\{\cdot,\cdot,\cdot\}$" a trilinear map (the ternary superoperation on mathfrakg) such that

$$[x,y] = -(-1)^{|x||y|}[y,x], \tag{7}$$
$$\{x,y,z\} = -(-1)^{|x||y|}\{y,x,z\}, \tag{8}$$
$$\{x,y,z\} + (-1)^{|x|(|y|+|z|)}\{y,z,x\} + (-1)^{|z|(|x|+|y|)}\{z,x,y\} + J(x,y,z) = 0, \tag{9}$$
$$\{[x,y],z,u\} + (-1)^{|x|(|y|+|z|)}\{[y,z],x,u\} + (-1)^{|z|(|x|+|y|)}\{[z,x],y,u\} = 0, \tag{10}$$
$$\{x,y,[u,v]\} = [\{x,y,u\},v] + (-1)^{|u|(|x|+|y|)}[u,\{x,y,v\}], \tag{11}$$
$$\begin{aligned}\{x,y,\{u,v,w\}\} = {}&\{\{x,y,u\},v,w\} + (-1)^{|u|(|x|+|y|)}\{u,\{x,y,v\},w\} \\ &+ (-1)^{(|x|+|y|)(|u|+|v|)}\{u,v,\{x,y,w\}\},\end{aligned} \tag{12}$$

for all x,y,z,u,v,w in $\mathfrak{g}$, where

$$J(x,y,z) = [[x,y],z] + (-1)^{|x|(|y|+|z|)}[[y,z],x] + (-1)^{|z|(|x|+|y|)}[[z,x],y]$$

is called super-Jacobian for $x,y,z \in h(\mathfrak{g})$.

Example 2. 1. Lie superalgebras are Lie-Yamaguti superalgebras with $\{x,y,z\} = 0$ for any $x,y,z \in h(\mathfrak{g})$.

2. If $[x,y] = 0$ for any $x,y \in h(\mathfrak{g})$, then (8), (9) and (12) define a Lie supertriple system.

Example 3. Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie superalgebra. We define $\{\cdot,\cdot,\cdot\} : \mathfrak{g}\times\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ by

$$\{x,y,z\} = [[x,y],z] \quad \forall x,y,z \in \mathfrak{g}.$$

Then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ becomes an Lie-Yamaguti superalgebra naturally.

Example 4. Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ a $\mathbb{Z}_2$-graded vector space where the even part $mathfrakg_{\overline{0}} = \text{span}\{e_1,e_2\}$ and the odd part $mathfrakg_{\overline{1}} = \text{span}\{f_1,f_2\}$. If we define a binary non-zero bracket $[\cdot,\cdot]$ and a ternary non-zero bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ as follows:

$$[e_1,e_2] = e_2, \quad [e_1,f_1] = f_2, \quad [f_1,f_2] = e_1, \quad \{e_1,e_2,f_1\} = f_2, \quad \{e_1,f_1,f_2\} = e_1,$$

then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti superalgebra.

Example 5. Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ a $\mathbb{Z}_2$-graded vector space where the even part $\mathfrak{g}_{\overline{0}} = \text{span}\{e_1,e_2,e_3\}$ and the odd part $\mathfrak{g}_{\overline{1}} = \text{span}\{f_1,f_2,f_3\}$. If we define a binary non-zero bracket $[\cdot,\cdot]$ and a ternary non-zero bracket $\{\cdot,\cdot,\cdot\}$ on $\mathfrak{g}$ as follows:

$$\begin{aligned}
&[e_1,e_2] = e_3, && e_2,e_3] = e_1, && [e_3,e_1] = e_2,\\
&[e_1,f_1] = f_2, && [e_2,f_2] = f_3, && [e_3,f_3] = f_1,\\
&[f_1,f_2] = f_3, && [f_2,f_3] = f_1, && [f_3,f_1] = f_2,\\
&\{e_1,e_2,e_3\} = f_1, && \{f_1,f_2,f_3\} = f_1, && \{e_1,f_1,f_2\} = f_3,\\
&\{e_2,f_2,f_3\} = f_1, && \{e_3,f_3,f_1\} = f_2,
\end{aligned}$$

then $(\mathfrak{g},[\cdot,\cdot],\{\cdot,\cdot,\cdot\})$ is a Lie-Yamaguti superalgebra.

Example 6. Let $M = M_{\overline{0}} \oplus M_{\overline{1}}$ be a Malcev superalgebra; that is, for any $x, y, z, t \in M$,

$$xy = -(-1)^{|x||y|}yx, \tag{13}$$

$$\begin{aligned} -(-1)^{|y||z|}(xz)(yt) &= ((xy)z)t + (-1)^{|x|(|y|+|z|+|t|)}((yz)t)x \\ &\quad + (-1)^{(|x|+|y|)(|z|+|t|)}((zt)x)y + (-1)^{|t|(|x|+|y|+|z|)}((tx)y)z. \end{aligned} \tag{14}$$

It is shown in [26] that M becomes a Lie-Yamaguti superalgebra if we set $\{x, y, z\} = x(yz) - (-1)^{|x||y|}y(xz) + (-1)^{|x||y|}(xy)z$. Conversely, if on a Malcev superalgebra $(M, \cdot)$ we define a trilinear operation by $\{x, y, z\} = x \cdot yz - (-1)^{|x||y|}y \cdot xz + (-1)^{|x||y|}xy \cdot z$ then $(M, \cdot, \{\cdot, \cdot, \cdot\})$ is a Lie-Yamaguti superalgebra.

Definition 5. *Let $(\mathfrak{g}, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$ and $(\mathfrak{g}', [\cdot, \cdot'], \{\cdot, \cdot, \cdot'\})$ be two Lie-Yamaguti superalgebras. A linear map $\rho : \mathfrak{g} \to \mathfrak{g}'$ is said to be of degree r if $\rho(\mathfrak{g}_i) \subseteq \mathfrak{g}'_{i+r}$ for all $i, r \in \mathbb{Z}_2$.*

Definition 6. *Let $(\mathfrak{g}, [\cdot, \cdot]_1, \{\cdot, \cdot, \cdot\}_1)$ and $(\mathfrak{h}, [\cdot, \cdot]_2, \{\cdot, \cdot, \cdot\}_2)$ be two Lie-Yamaguti superalgebras. A linear map $\rho : \mathfrak{g} \to \mathfrak{h}$ is called a homomorphism of Lie-Yamaguti superalgebras if*

1. *ρ preserves the grading, that is, $\rho(\mathfrak{g}_i) \subseteq \mathfrak{h}_i$, $i \in \mathbb{Z}_2$,*
2. *$\rho([x, y]_1) = [\rho(x), \rho(y)]_2$, $x, y, \in \mathfrak{g}$,*
3. *$\rho(\{x, y, z\}_1) = \{\rho(x), \rho(y), \rho(z)\}_2$, $x, y, z \in \mathfrak{g}$.*

Next, we recall the notion of representations of Lie-Yamaguti superalgebras.

Definition 7. *Let $(\mathfrak{g}, [\cdot, \cdot], \{\cdot, \cdot, \cdot\})$ be a Lie-Yamaguti superalgebra. A representation of $\mathfrak{g}$ is a vector superspace V equipped with a even linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ and two even bilinear maps $D, \mu : \otimes^2\mathfrak{g} \to \mathrm{End}(V)$, which meet the following conditions: for all $x, y, z, w \in \mathfrak{g}$,*

$$\mu([x, y], z) - (-1)^{|y||z|}\mu(x, z)\rho(y) + (-1)^{|x|(|y|+|z|)}\mu(y, z)\rho(x) = 0, \tag{15}$$

$$\mu(x, [y, z]) - (-1)^{|x||y|}\rho(y)\mu(x, z) + (-1)^{|z|(|x|+|y|)}\rho(z)\mu(x, y) = 0, \tag{16}$$

$$D([x, y], z) + (-1)^{|x|(|y|+|z|)}D([y, z], x) + (-1)^{|z|(|x|+|y|)}D([z, x], y) = 0, \tag{17}$$

$$[D(x, y), \rho(z)] = \rho(\{x, y, z\}), \tag{18}$$

$$D(x, y) - (-1)^{|x||y|}\mu(y, x) + \mu(x, y) + \rho([x, y]) - [\rho(x), \rho(y)] = 0, \tag{19}$$

$$\begin{aligned} &(-1)^{(|x|+|y|)(|z|+|w|)}\mu(z, w)\mu(x, y) - (-1)^{|x|(|y|+|w|)+|z||w|}\mu(y, w)\mu(x, z) \\ &- \mu(x, \{y, z, w\}) + (-1)^{|x|(|y|+|z|)}D(y, z)\mu(x, w) = 0, \end{aligned} \tag{20}$$

$$[D(x, y), \mu(z, w)] = \mu(\{x, y, z\}, w) + (-1)^{|x|(|x|+|y|)}\mu(z, \{x, y, w\}), \tag{21}$$

for all x, y, z, w in $\mathfrak{g}$, where

$$J(x, y, z) = [[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y]$$

is called super-Jacobian for $x, y, z \in h(\mathfrak{g})$.

It is obviously that D is skew-symmetric. We denote a representation of $\mathfrak{g}$ by $(V; \rho, \mu, D)$. Because of (15), the representation $(V; \rho, \mu, D)$ is simply written as $(V; \rho, \mu)$.

Example 7. Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be a Lie-Yamaguti superalgebra. We define linear maps $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and $\Re : \otimes^2\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $x \mapsto \mathrm{ad}_x$ and $(x,y) \mapsto \Re_{x,y}$ respectively, where $\mathrm{ad}_x z = [x,z]$ and $\Re_{x,y}z = \{z,x,y\}$ for all $z \in \mathfrak{g}$. Then $(\mathrm{ad}, \Re)$ forms a representation of $\mathfrak{g}$ on itself, where $\mathfrak{L} := D_{\mathrm{ad},\Re}$ is given by

$$\mathfrak{L}_{x,y} = (-1)^{|x||y|}\Re_{y,x} - \Re_{x,y} + [\mathrm{ad}_x, \mathrm{ad}_y] - \mathrm{ad}_{[x,y]}, \quad \forall x, y \in \mathfrak{g}.$$

By (9), we have

$$\mathfrak{L}_{x,y}z = \{x,y,z\}, \quad \forall z \in \mathfrak{g}.$$

In this case, $(\mathfrak{g}; \mathrm{ad}, \mathfrak{L}, \Re)$ is called the adjoint representation of $\mathfrak{g}$.

We now obtain the main result of this section.

Theorem 1. *Let $(\mathfrak{g}, \cdot)$ be a Leibniz algebra and (V, l, r) its left representation. Let $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ be the Lie-Yamaguti superalgebra associated with $(\mathfrak{g}, \cdot)$. Then (V, l, r) induces a representation of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ (call it the associated representation).*

Proof. On $(\mathfrak{g}, \cdot)$ we define the following maps, for all $x, y \in \mathfrak{g}$:

$$\begin{aligned}
&\rho : \mathfrak{g} \to \mathrm{End}(V); \quad x \longmapsto l(x) - r(x),\\
&\mu : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(V); \quad (x,y) \longmapsto -(-1)^{|x||y|}r(y)r(x),\\
&D : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(V); \quad (x,y) \longmapsto -l(x \cdot y).
\end{aligned}$$

Clearly ρ is a linear map and μ, D are bilinear maps. We proceed to check that the triple (ρ, μ, D) as defined above satisfies the set of axioms as in Definition 2.8.

For (15) we have

$$\begin{aligned}
&\mu([x,y],z) - (-1)^{|y||z|}\mu(x,z)\rho(y) + (-1)^{|z|(|y|+|z|)}\mu(y,z)\rho(x)\\
&= -(-1)^{|z|(|x|+|y|)}r(z)r(x\cdot y) + (-1)^{|z|(|x|+|y|)+|x||y|}r(z)r(y\cdot x)\\
&\quad + (-1)^{|z|(|x|+|y|)}r(z)r(x)(l(y) - r(y)) - (-1)^{|x|(|y|+|z|)+|y||z|}r(z)r(y)(l(x) - r(x))\\
&= -(-1)^{|z|(|x|+|y|)}r(z)\left(l(x)r(y) + (-1)^{|x||y|}r(y)r(x)\right) \text{ (by (3))}\\
&\quad + (-1)^{|z|(|x|+|y|)+|x||y|}r(z)\left(l(y)r(x) + (-1)^{|x||y|}r(x)r(y)\right) \text{ (by (3))}\\
&\quad + (-1)^{|z|(|x|+|y|)}r(z)r(x)l(y) - (-1)^{|z|(|x|+|y|)}r(z)r(x)r(y)\\
&\quad - (-1)^{|x|(|y|+|z|)+|y||z|}r(z)r(y)l(x) + (-1)^{|x|(|y|+|z|)+|y||z|}r(z)r(y)r(x)\\
&= -(-1)^{|z|(|x|+|y|)}r(z)l(x)r(y) + (-1)^{|z|(|x|+|y|)}r(z)r(x)l(y)\\
&\quad + (-1)^{|z|(|x|+|y|)+|x||y|}r(z)l(y)r(x) - (-1)^{|x|(|y|+|z|)+|y||z|}r(z)r(y)l(x)\\
&= -(-1)^{|z|(|x|+|y|)}\left(r(z)l(x)r(y) - r(z)r(x)r(y)\right)\\
&\quad + (-1)^{|z|(|x|+|y|)+|x||y|}\left(r(z)l(y)r(x) + r(z)r(y)r(x)\right) \text{ (by (5))}\\
&= -(-1)^{|z|(|x|+|y|)}\left(r(z)l(x) - r(z)r(x)\right)_{(2.5)}r(y)\\
&\quad + (-1)^{|z|(|x|+|y|)+|x||y|}\left(r(z)l(y) + r(z)r(y)\right)_{(2.5)}r(x)
\end{aligned}$$

$$= 0 \text{ (by (5))},$$

so (15) holds for (ρ, μ, D).

For (16) we have

$$\begin{aligned}
&(-1)^{|x||y|}\rho(y)\mu(x,z) - (-1)^{|z|(|x|+|y|)}\rho(z)\mu(x,y)\\
&= (-1)^{|x|(|y|+|z|)}\left(-l(y)r(z)r(x) + r(y)r(z)r(x)\right)\\
&\quad - (-1)^{|z|(|x|+|y|)+|x||y|}r(z)\left(-l(z)r(y)r(x) + r(z)r(y)r(x)\right)\\
&= -(-1)^{|z|(|x|+|y|)+|x||y|}\Big(r(z)r(y) + (-1)^{|x||y|}l(y)r(z)\Big)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|)}\Big(r(y)r(z) + (-1)^{|y||z|}l(z)r(y)\Big)r(x) \text{ (by (5))}\\
&= -(-1)^{|x|(|y|+|z|)}r(y\cdot z)r(x) + (-1)^{|x|(|y|+|z|)+|y||z|}r(z\cdot y)r(x) \text{ (by (3))}\\
&= -(-1)^{|x|(|y|+|z|)}r(y\cdot z - (-1)^{|y||z|}z\cdot y)r(x)\\
&= -(-1)^{|x|(|y|+|z|)}r([y,z])r(x)\\
&= (-1)^{|x|(|y|+|z|)}\Big((-1)^{|x|(|y|+|z|)}\mu(x,[y,z])\Big)\\
&= \mu(x,[y,z]),
\end{aligned}$$

so (16) holds for (ρ, μ, D).

For (17) we have

$$\begin{aligned}
&D([x,y],z) + (-1)^{|x|(|y|+|z|)}D([y,z],x) + (-1)^{|z|(|x|+|y|)}D([z,x],y)\\
&= -l([x,y]\cdot z) - (-1)^{|x|(|y|+|z|)}l([y,z]\cdot x) - (-1)^{|z|(|x|+|y|)}l([z,x]\cdot y)\\
&= -\Big(l([x,y])l(z) - (-1)^{|z|(|x|+|y|)}l(z)l([x,y])\Big)\\
&\quad - (-1)^{|x|(|y|+|z|)}\Big(l([y,z])l(x) - (-1)^{|x|(|y|+|z|)}l(x)l([y,z])\Big)\\
&\quad - (-1)^{|z|(|x|+|y|)}\Big(l([z,x])l(y) - (-1)^{|y|(|x|+|z|)}l(y)l([z,x])\Big) \text{ (by (2))}\\
&= -\Big(l(x\cdot y)l(z) - (-1)^{|x||y|}l(y\cdot x)l(z)\Big)\\
&\quad + (-1)^{|z|(|x|+|y|)}\Big(l(z)l(x\cdot y) - (-1)^{|x||y|}l(z)l(y\cdot x)\Big)\\
&\quad - (-1)^{|x|(|y|+|z|)}\Big(l(y\cdot z)l(x) - (-1)^{|y||z|}l(z\cdot y)l(x)\Big)\\
&\quad + l(x)l(y\cdot z) - (-1)^{|y||z|}l(x)l(z\cdot y)\\
&\quad - (-1)^{|z|(|x|+|y|)}\Big(l(z\cdot x)l(y) - (-1)^{|x||z|}l(x\cdot z)l(y)\Big)\\
&\quad + (-1)^{|x|(|y|+|z|)}\Big(l(y)l(z\cdot x) - (-1)^{|x||z|}l(y)l(x\cdot z)\Big)\\
&= -\Big([l(x),l(y)]l(z) - (-1)^{|x||y|}[l(y),l(x)]l(z)\Big)\\
&\quad + (-1)^{|z|(|x|+|y|)}\Big(l(z)[l(x),l(y)] - (-1)^{|x||y|}l(z)[l(y),l(x)]\Big)
\end{aligned}$$

$$
\begin{aligned}
&- (-1)^{|x|(|y|+|z|)}\Big([l(y),l(z)]l(x) - (-1)^{|y||z|}[l(z),l(y)]l(x)\Big)\\
&+ l(x)[l(y),l(z)] - (-1)^{|y||z|}l(x)[l(z),l(y)]\\
&- (-1)^{|z|(|x|+|y|)}\Big([l(z),l(x)]l(y) - (-1)^{|x||z|}[l(x),l(z)]l(y)\Big)\\
&+ (-1)^{|x|(|y|+|z|)}\Big(l(y)[l(z),l(x)] - (-1)^{|x||z|}l(y)[l(x),l(z)]\Big) \text{ (again by (2))}\\
=& -[[l(x),l(y)],l(z)] + (-1)^{|x||y|}[[l(y),l(x)],l(z)] + [l(x),[l(y),l(z)]]\\
&+ (-1)^{|x|(|y|+|z|)+|y||z|}[[l(z),l(y)],l(x)] + (-1)^{|x|(|y|+|z|)}[l(y),[l(z),l(x)]]\\
&+ (-1)^{|y||z|}[[l(x),l(z)],l(y)]\\
=& [l(x),[l(y),l(z)]] - [[l(x),l(y)],l(z)] + (-1)^{|x|(|y|+|z|)}[l(y),[l(z),l(x)]]\\
&+ (-1)^{|x||y|}[[l(y),l(x)],l(z)] + (-1)^{|x|(|y|+|z|)+|y||z|}[[l(z),l(y)],l(x)]\\
&+ (-1)^{|y||z|}[[l(x),l(z)],l(y)]\\
=& 0 \text{ (by(1))},
\end{aligned}
$$

so (17) holds for (ρ,μ,D).

For (18) we have

$$
\begin{aligned}
[D(x,y),\rho(z)] &= D(x,y)\rho(z) - (-1)^{|z|(|x|+|y|)}\rho(z)D(x,y)\\
&= -l(x\cdot y)l(z) + l(x\cdot y)r(z) + (-1)^{|z|(|x|+|y|)}l(z)l(x\cdot y)\\
&\quad - (-1)^{|z|(|x|+|y|)}r(z)l(x\cdot y)\\
&= -\Big(l(x\cdot y)l(z) - (-1)^{|z|(|x|+|y|)}l(z)l(x\cdot y)\Big)_{(2.2)} + l(x\cdot y)r(z)\\
&\quad - (-1)^{|z|(|x|+|y|)}r(z)l(x\cdot y)\\
&= -l((x\cdot y)\cdot z) + l(x\cdot y)r(z) - (-1)^{|z|(|x|+|y|)}r(z)l(x\cdot y) \text{ (by (2))}\\
&= -l((x\cdot y)\cdot z) + l(x\cdot y)r(z) + (-1)^{|z|(|x|+|y|)}r(z)r(x\cdot y) \text{ (by (5))}\\
&= -l((x\cdot y)\cdot z) + r((x\cdot y)\cdot z) \text{ (by (3))}\\
&= -\rho((x\cdot y)\cdot z)\\
&= \rho(\{x,y,z\}),
\end{aligned}
$$

so (18) holds for (ρ,μ,D).

For (19) we have

$$
\begin{aligned}
&D(x,y) + \mu(x,y) - (-1)^{|x||y|}\mu(y,x) + \rho([x,y]) - [\rho(x),\rho(y)]\\
&= -l(x\cdot y) - (-1)^{|x||y|}r(y)r(x) + r(x)r(y) + l([x,y]) - r([x,y]) - [l(x)-r(x),l(y)-r(y)]\\
&= \Big(-l(y\cdot x) + (-1)^{|x||y|}l(y)l(x) - l(x)l(y)\Big)_{(2.2)}\\
&\quad + \Big(-r(x\cdot y) + l(x)r(y) - (-1)^{|x||y|}r(y)l(x)\Big) \text{ (by (4))}\\
&\quad + \Big(r(y\cdot x) - (-1)^{|x||y|}l(y)r(x) + r(x)l(y)\Big) \text{ (by (4))}
\end{aligned}
$$

$= 0$ (by (2) and (4)),

so (19) holds for (ρ, μ, D).

For (20) we have

$$
\begin{aligned}
&(-1)^{(|x|+|y|)(|z|+|w|)}\mu(z,w)\mu(x,y) - (-1)^{|x|(|y|+|w|)+|z||w|}\mu(y,w)\mu(x,z)\\
&\quad + (-1)^{|x|(|y|+|z|)}D(y,z)\mu(x,w)\\
&= (-1)^{(|x|+|y|)(|z|+|w|)+|x||y|+|z||w|}r(w)r(z)r(y)r(x) + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x)\\
&\quad - (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)r(y)r(z)r(x)\\
&= (-1)^{(|x|+|y|)(|z|+|w|)+|x||y|+|z||w|}r(w)\left(r(z)r(y) - (-1)^{|y||z|}r(y)r(z)\right)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x)\\
&= (-1)^{(|x|+|y|)(|z|+|w|)+|x||y|+|z||w|}r(w)\left(r(z)r(y) + (-1)^{|y||z|}r(y)l(z)\right)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x) \text{ (by (5))}\\
&= (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)\left(r(y\cdot z) - l(y)r(z) + r(y)l(z)\right)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x) \text{ (by (3))}\\
&= (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)r(y\cdot z)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)\left(r(y)l(z) - l(y)r(z)\right)r(x)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x)\\
&= (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)r(y\cdot z)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)\left(-r(y)r(z) - l(y)r(z)\right)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x) \text{ (by (5))}\\
&= (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)r(y\cdot z)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)-|w|(|y|+|z|)}\left(r(w)r(y) - r(w)l(y)\right)_{(2.5)}r(z)r(x)\\
&\quad + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x)\\
&= (-1)^{|x|(|y|+|z|+|w|)+|w|(|y|+|z|)}r(w)r(y\cdot z)r(x) + (-1)^{|x|(|y|+|z|+|w|)}l(y\cdot z)r(w)r(x) \text{ (by (5))}\\
&= (-1)^{|x|(|y|+|z|+|w|)}\left((-1)^{|w|(|y|+|z|)}r(w)r(y\cdot z) + l(y\cdot z)r(w)\right)r(x)\\
&= (-1)^{|x|(|y|+|z|+|w|)}r((y\cdot z)\cdot w)r(x) \text{ (by (3))}\\
&= -(-1)^{|x|(|y|+|z|+|w|)}r(\{y,z,w\})r(x)\\
&= \mu(x,\{y,z,w\}),
\end{aligned}
$$

so (20) holds for (ρ, μ, D).

For (21) we have

$$\mu(\{x,y,z\},w) + (-1)^{|z|(|x|+|y|)}\mu(z,\{x,y,w\})$$

$$
\begin{aligned}
&= (-1)^{|w|(|x|+|y|+|z|)} r(w)r((x\cdot y)\cdot z) + (-1)^{|z||w|} r((x\cdot y)\cdot w)r(z)\\
&= (-1)^{|w|(|x|+|y|+|z|)} r(w)\left((-1)^{|z|(|x|+|y|)} r(z)r(x\cdot y) + l(x\cdot y)r(z)\right)\\
&\quad + (-1)^{|z||w|}\left((-1)^{|w|(|x|+|y|)} r(w)r(x\cdot y) + l(x\cdot y)r(w)\right) r(z) \text{ (by (3))}\\
&= (-1)^{(|x|+|y|)(|z|+|w|)+|z||w|} r(w)r(z)r(x\cdot y) + (-1)^{|z||w|} l(x\cdot y)r(w)r(z)\\
&\quad + (-1)^{|w|(|x|+|y|+|z|)}\left(r(w)l(x\cdot y) + r(w)r(x\cdot y)\right)_{(2.5)} r(z)\\
&= (-1)^{(|x|+|y|)(|z|+|w|)+|z||w|} r(w)r(z)r(x\cdot y) + (-1)^{|z||w|} l(x\cdot y)r(w)r(z)\\
&= -(-1)^{(|x|+|y|)(|z|+|w|)+|z||w|} r(w)r(z)l(x\cdot y) + (-1)^{|z||w|} l(x\cdot y)r(w)r(z) \text{ (by (5))}\\
&= \left[-l(x\cdot y), -(-1)^{|z||w|} r(w)r(z)\right]\\
&= \left[D(x,y), \mu(z,w)\right],
\end{aligned}
$$

so (21) holds for (ρ, μ, D). This completes the proof. □

Next, we recall the notions of the equivalent representations of a Leibniz superalgebras and Lie-Yamaguti superalgebras.

Definition 8. *Two representations (V_1, l_1, r_1) and (V_2, l_2, r_2) of a left Leibniz superalgebra $(\mathfrak{g}, \cdot)$ are said to be equivalent if there is an isomorphism $\varphi : V_1 \longrightarrow V_2$ such that*

$$\varphi l_1(x) = l_2(x)\varphi \quad \text{and} \quad \varphi r_1(x) = r_2(x)\varphi \ \forall\ x \in \mathfrak{g}.$$

Definition 9. *Two representations (V_1, ρ_1, μ_1) and (V_2, ρ_2, μ_2) of a Lie-Yamaguti superalgebra $\mathfrak{g}$ are said to be equivalent if there is an isomorphism $\varphi : V_1 \longrightarrow V_2$ such that*

$$\varphi\rho_1(x) = \rho_2(x)\varphi \quad \text{and} \quad \varphi\mu_1(x,y) = \mu_2(x,y)\varphi \ \forall\ x, y \in \mathfrak{g}.$$

The following result shows that equivalent representations of a given left Leibniz superalgebra give rise to equivalent representations of the associated Lie-Yamaguti superalgebra.

Theorem 2. *Let $(\mathfrak{g}, \cdot)$ be a left Leibniz algebra and $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ its associated Lie-Yamaguti superalgebra. If (V_1, l_1, r_1) and (V_2, l_2, r_2) are two equivalent representations of $(\mathfrak{g}, \cdot)$, then the associated representations (V_1, ρ_1, μ_1) and (V_2, ρ_2, μ_2) of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ are also equivalent.*

Proof. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ and $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(V)$ be the two functions defined in th proof of Theorem 2.7. Let $\varphi : V_1 \longrightarrow V_2$ be an isomorphism. For all $x, y \in \mathfrak{g}$, for all $v \in V_1$ we have

$$
\begin{aligned}
\varphi(\rho_1(x)(v)) &= \varphi(l_1(x)(v)) - \varphi(r_1(x)(v))\\
&= l_2(x)\varphi(v) - r_2(x)\varphi(v)\\
&= \rho_2(x)(\varphi(v)),
\end{aligned}
$$

and, from the other hand,

$$\mu_2(x,y)(\varphi(v)) = -(-1)^{|x||y|} r_2(y)(\varphi(r_1(x)(v)))$$

$$\begin{aligned}
&= -(-1)^{|x||y|}\varphi(r_1(y)(r_1(x)(v))) \\
&= \varphi(-(-1)^{|x||y|}r_1(y)(r_1(x)(v))) \\
&= \varphi(\mu_1(x,y)(v))).
\end{aligned}$$

so that $\varphi\rho_1(x) = \rho_2(x)\varphi$ and $\varphi\mu_1(x,y) = \mu_2(x,y)\varphi$ for all $x, y \in \mathfrak{g}$. Thus (V_1, ρ_1, μ_1) and (V_2, ρ_2, μ_2) are equivalent. □

Now, we consider some examples well-known representations of left Leibniz superalgebras, as well as their corresponding representations of associated Lie-Yamaguti superalgebras, after Theorem 2.7. Below $(\mathfrak{g}, \cdot)$ denotes a left Leibniz superalgebra and $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$ its associated Lie-Yamaguti superalgebra.

Example 8. The adjoint representation $(\mathfrak{g}, L, R)$ of $(\mathfrak{g}, \cdot)$ induces, by Theorem 2.7, a representation $(\mathfrak{g}, \rho, \mu)$ of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$, where $\rho(x) := L(x) - R(x) = L_x - R_x$ and $\mu(x,y) := -(-1)^{|x||y|}R(y)R(x) = -(-1)^{|x||y|}R_yR_x$ for all $x, y \in \mathfrak{g}$.

Observe that $(\mathfrak{g}, \rho, \mu)$ is precisely the adjoint representation of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$. Thus the adjoint representation of $(\mathfrak{g}, \cdot)$ induces the one of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$.

Example 9. From Proposition 2.4, if (V, l, r) is a left representation of $(\mathfrak{g}, \cdot)$, then $(V^*, l^*, -l^* - r^*)$ is a left representation of $(\mathfrak{g}, \cdot)$. Therefore, Theorem 2.7 implies that (V^*, ρ, μ) is a representation of $(\mathfrak{g}, [\cdot,\cdot], \{\cdot,\cdot,\cdot\})$, where $\rho(x) := 2l^*(x) + r^*(x)$ and $\mu(x,y) := -(-1)^{|x||y|}(-l^*(y) - r^*(y))(-l^*(x) - r^*(x))$.

Let $r_0(z) := -l(z) - r(z)$ for all $z \in \mathfrak{g}$. Then $(V^*, l^*, -l^* - r^*) = (V^*, l^*, r_0^*)$ and $\mu(x,y) = (-1)^{|x||y|}r_0^*(y)r_0^*(x)$, $\rho(x) := l^*(x) - r_0^*(x)$.

3 3-Leibniz superalgebras induced by a nonabelian embedding tensors between Lie supertriple system

Definition 10 ([18]). *A **Lie supertriple system** (LSTS in short) is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ over $\mathbb{K}$ with a trilinear product $[\cdot,\cdot,\cdot]$ satisfying the following conditions:*

$$|[x,y,z]| = |x| + |y| + |z|, \tag{22}$$

$$[x,y,z] = -(-1)^{|x||y|}[y,x,z], \tag{23}$$

$$[x,y,z] + (-1)^{|x|(|y|+|z|)}[y,z,x] + (-1)^{|z|(|x|+|y|)}[z,x,y] = 0, \tag{24}$$

$$\begin{aligned}
[x,y,[z,w,u]] = {}& [[x,y,z],w,u] + (-1)^{|z|(|x|+|y|)}[z,[x,y,w],u] \\
&+ (-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]],
\end{aligned} \tag{25}$$

for all $x, y, z, w, u \in \mathfrak{g}$. Denote it by $(\mathfrak{g}; [\cdot,\cdot,\cdot])$. Obviously, the even part to itself is a Lie triple system.

A homomorphism $f : (\mathfrak{g}; [\cdot,\cdot,\cdot]) \longrightarrow (\mathfrak{g}'; [\cdot,\cdot,\cdot]')$ of LSTS is an even linear map satisfying

$$f([x,y,z]) = [f(x), f(y), f(z)]'; \qquad \forall\ x, y, z \in \mathfrak{g}.$$

Definition 11 ([9]). *A **representation** of a LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot])$ on a vector superspace V is an even linear map $\theta:\mathfrak{g}\times\mathfrak{g}\longrightarrow \mathrm{End}(V)$ such that for all $x,y,z,w\in\mathfrak{g}$,*

$$\begin{aligned}&(-1)^{(|x|+|y|)(|z|+|w|)}\theta(z,w)\theta(x,y)-(-1)^{|x||y|+|w|(|x|+|z|)}\theta(y,w)\theta(x,z)\\&-\theta(x,[y,z,w])+(-1)^{|x|(|y|+|z|)}D(y,z)\theta(x,w)=0,\end{aligned}\tag{26}$$

$$\begin{aligned}&(-1)^{(|x|+|y|)(|z|+|w|)}\theta(z,w)D(x,y)-D(x,y)\theta(z,w)\\&+\theta([x,y,z],w)+(-1)^{|z|(|x|+|y|)}\theta(z,[x,y,w])=0,\end{aligned}\tag{27}$$

where $D:\mathfrak{g}\times\mathfrak{g}$ to $\mathrm{End}(V)$ is given by

$$D(x,y)=(-1)^{|x||y|}\theta(y,x)-\theta(x,y).$$

Also V is called a $\mathfrak{g}$-module. In the sequel, a representation of LSTS is denoted by $(V;\theta)$.

Remark 2. *Using (27) and (11), we immediately get*

$$\begin{aligned}&D(x,y)D(z,w)-(-1)^{(|x|+|y|)(|z|+|w|)}D(z,w)D(x,y)\\&\quad=D([x,y,z],w)+(-1)^{|x|(|x|+|y|)}D(z,[x,y,w])\end{aligned}\tag{28}$$

for all $x,y,z,w\in\mathfrak{g}$.

Example 10. Let $V=\mathfrak{g}$, and $\theta(x,y)(z)=(-1)^{|x|(|x|+|y|)}[z,x,y]$. Then it is easy to see that $D(x,y)(z)=[x,y,z]$ and $\mathfrak{g}$ is showed to be the adjoint $\mathfrak{g}$-module. In this case, $(\mathfrak{g};\theta)$ is called the adjoint representation of $(\mathfrak{g};[\cdot,\cdot,\cdot])$.

Next, let us give the definition of a nonabelian embedding tensor from a LSTS to another LSTS. We recall a derived algebra of a LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot])$ is a subsystem $[\cdot,\cdot,\cdot]$, and denoted by $\mathfrak{g}^1$ the subspace

$$C(L)=\{x\in\mathfrak{g}\mid [x,y,z]=0,\ \forall\ y,z\in\mathfrak{g}\}$$

is the center of $(\mathfrak{g};[\cdot,\cdot,\cdot])$.

Definition 12. *1. Let $(\mathfrak{g};[\cdot,\cdot,\cdot])$ and $(\mathfrak{g}';[\cdot,\cdot,\cdot]')$ be two LSTSs. Let $\theta:\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathrm{End}(\mathfrak{g}')$ be a representation of the LSTSs $\mathfrak{g}$ on the vector superspace $\mathfrak{g}'$. If for all $x,y\in\mathfrak{g}$,*

$$\theta(x,y)u\in C(\mathfrak{g}'),\tag{29}$$

$$\theta(x,y)[u,v,w]'=0,\tag{30}$$

for all $x,y\in\mathfrak{g}$ and $u,v,w\in\mathfrak{g}'$, then θ is called an action of $(\mathfrak{g};[\cdot,\cdot,\cdot])$ on $(\mathfrak{g}';[\cdot,\cdot,\cdot]')$.

2. An action θ of $\mathfrak{g}$ on $\mathfrak{g}'$ is called a coherent action if θ satisfies

$$\theta(x,y)\in\mathrm{Der}(\mathfrak{g}'),\quad \text{for all } x,y\in\mathfrak{g}.$$

Remark 3. *Using (11), (29), (30) and (2), we immediately get*

$$D(x,y)u\in C(\mathfrak{g}'),\quad D(x,y)\in\mathrm{Der}(\mathfrak{g}'),\tag{31}$$

$$D(x,y)[u,v,w]'=0,\quad \text{for all } x,y\in\mathfrak{g} \text{ and } u,v,w\in\mathfrak{g}'.\tag{32}$$

Example 11. Let $(\mathfrak{g};[\cdot,\cdot,\cdot])$ be a LSTS. If $\mathfrak{g}^1$ satisfies $\mathfrak{g}^1 \in C(\mathfrak{g})$, then the adjoint representation $\theta : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is an (coherent) action of $\mathfrak{g}$ on itself.

Definition 13. *Let $(\mathfrak{g};[\cdot,\cdot,\cdot])$ and $(\mathfrak{g}';[\cdot,\cdot,\cdot]')$ be two LSTSs. Let θ be a coherent action of $\mathfrak{g}$ on $\mathfrak{g}'$. A even linear map $T : \mathfrak{g}' \longrightarrow \mathfrak{g}$ is called a nonabelian embedding tensor from $\mathfrak{g}'$ to $\mathfrak{g}$ with respect to θ if T satisfies*

$$[Tu,Tv,Tw] = T(D(Tu,Tv)w + [u,v,w]'),$$

for all $u,v,w \in \mathfrak{g}'$.

Definition 14. *Let T and $\tilde{T}$ be two nonabelian embedding tensors from a LSTS $(\mathfrak{g}';[\cdot,\cdot,\cdot]')$ to another LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot])$ with respect to a coherent action θ. A homomorphism from $\tilde{T}$ to T consists of two LTSS homomorphisms $f : \mathfrak{g} \longrightarrow \mathfrak{g}$ and $f' : \mathfrak{g}' \longrightarrow \mathfrak{g}'$ such that*

$$f(\tilde{T}u) = Tf'(u), \tag{33}$$
$$f'(\theta(x,y)u) = \theta(f(x),f(y))f'(u), \tag{34}$$

for all $x,y \in \mathfrak{g}$ and $u \in \mathfrak{g}'$. In particular, if f and f' are invertible, (f,f') is said to be an isomorphism from $\tilde{T}$ to T.

Let (f,f') is a homomorphism from $\tilde{T}$ to T, then we have

$$f'(D(x,y)u) = D(f(x),f(y))f'(u).$$

Definition 15. *A **3-Leibniz superalgebra** is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ over $\mathbb{K}$ with a trilinear product $[\cdot,\cdot,\cdot]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying*

$$\begin{aligned}[x,y,[z,w,u]_{\mathfrak{g}}]_{\mathfrak{g}} &= [[x,y,z]_{\mathfrak{g}},w,u]_{\mathfrak{g}} + (-1)^{|z|(|x|+|y|)}[z,[x,y,w]_{\mathfrak{g}},u]_{\mathfrak{g}} \\ &\quad + (-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]_{\mathfrak{g}}]_{\mathfrak{g}},\end{aligned} \tag{35}$$

for all $x,y,z,w,u \in \mathfrak{g}$.

Definition 16. *A **representation** of a 3-Leibniz superalgebra $(\mathfrak{g};[\cdot,\cdot,\cdot]_{\mathfrak{g}})$ is a vector superspace V equipped with 3 actions*

$$\begin{aligned}\rho_1 &: \mathfrak{g} \otimes \mathfrak{g} \otimes V \longrightarrow V,\\ \rho_2 &: \mathfrak{g} \otimes V \otimes \mathfrak{g} \longrightarrow V,\\ \rho_3 &: V \otimes \mathfrak{g} \otimes \mathfrak{g} \longrightarrow V,\end{aligned}$$

such that for any $x,y,z,a,b \in \mathfrak{g}$ and $u \in V$,

$$\begin{aligned}\rho_1(a,b,\rho_1(x,y,u)) &= \rho_1([a,b,x]_{\mathfrak{g}},y,u) + (-1)^{|x|(|a|+|b|)}\rho_1(x,[a,b,y]_{\mathfrak{g}},u) \\ &\quad + (-1)^{(|x|+|y|)(|a|+|b|)}\rho_1(x,y,\rho_1(a,b,u)),\end{aligned} \tag{36}$$
$$\begin{aligned}\rho_1(a,b,\rho_2(x,u,z)) &= \rho_2([a,b,x]_{\mathfrak{g}},u,z) + (-1)^{|x|(|a|+|b|)}\rho_2(x,\rho_1(a,b,u),z) \\ &\quad + (-1)^{(|x|+|u|)(|a|+|b|)}\rho_2(x,u,[a,b,z]_{\mathfrak{g}}),\end{aligned} \tag{37}$$

$$\begin{aligned}\rho_1(a,b,\rho_3(u,y,z)) &= \rho_3(\rho_1(a,b,u),y,z) + (-1)^{|u|(|a|+|b|)}\rho_3(u,[a,b,y]_{\mathfrak{g}},z)\\ &\quad + (-1)^{(|y|+|u|)(|a|+|b|)}\rho_3(u,y,[a,b,z]_{\mathfrak{g}}), \qquad (38)\\ \rho_2(a,u,[x,y,z]_{\mathfrak{g}}) &= \rho_3(\rho_2(a,u,x),y,z) + (-1)^{|x|(|a|+|u|)}\rho_2(x,\rho_2(a,u,y),z)\\ &\quad + (-1)^{(|x|+|y|)(|a|+|u|)}\rho_1(x,y,\rho_2(a,u,z)), \qquad (39)\\ \rho_3(u,b,[x,y,z]_{\mathfrak{g}}) &= \rho_3(\rho_3(u,b,x),y,z) + +(-1)^{|x|(|b|+|u|)}\rho_2(x,\rho_3(u,b,y),z)\\ &\quad + (-1)^{(|x|+|y|)(|b|+|u|)}\rho_1(x,y,\rho_3(u,b,z)). \qquad (40)\end{aligned}$$

Theorem 3. *Let $T : \mathfrak{g}' \longrightarrow \mathfrak{g}$ be a nonabelian embedding tensor from a LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot]')$ to another LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot])$ with respect to a coherent action θ.*

1. *Define a linear map $[\cdot,\cdot,\cdot]_T : \mathfrak{g}' \times \mathfrak{g}' \times \mathfrak{g}' \longrightarrow \mathfrak{g}'$ by*

$$[u,v,w]_T = D(Tu,Tv)w + [u,v,w]',$$

for all $u,v,w \in \mathfrak{g}'$. Then $(\mathfrak{g}';[\cdot,\cdot,\cdot]_T)$ is a 3-Leibniz superalgebra. The 3-Leibniz superalgebra $(\mathfrak{g}';[\cdot,\cdot,\cdot]_T)$ is called the descendent 3-Leibniz superalgebra of the nonabelian embedding tensor T.

2. *Define actions*

$$\begin{aligned}\rho_1^T &: \mathfrak{g}' \otimes \mathfrak{g}' \otimes \mathfrak{g} \longrightarrow \mathfrak{g},\\ \rho_2^T &: \mathfrak{g}' \otimes \mathfrak{g} \otimes \mathfrak{g}' \longrightarrow \mathfrak{g},\\ \rho_3^T &: \mathfrak{g} \otimes \mathfrak{g}' \otimes \mathfrak{g}' \longrightarrow \mathfrak{g},\end{aligned}$$

by

$$\begin{aligned}\rho_1^T(u,v,x) &= [Tu,Tv,x],\\ \rho_2^T(u,x,v) &= [Tu,x,Tv] - TD(Tu,x)v,\\ \rho_3^T(x,u,v) &= [x,Tu,Tv] - TD(x,Tu)v,\end{aligned}$$

for $u,v \in \mathfrak{g}'$ and $x \in \mathfrak{g}$. Then $(\mathfrak{g};\rho_1^T,\rho_2^T,\rho_3^T)$ is a representation of the 3-Leibniz superalgebra $(\mathfrak{g}';[\cdot,\cdot,\cdot]_T)$.

Proof. Let $T : \mathfrak{g}' \longrightarrow \mathfrak{g}$ be a nonabelian embedding tensor from a LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot]')$ to another LSTS $(\mathfrak{g};[\cdot,\cdot,\cdot])$ with respect to a coherent action θ.

1) For any $x,y,z,w,u \in \mathfrak{g}'$, by Eqs. (25), (28), (31), (32) and (13), we have

$$\begin{aligned}&[[x,y,z]_T,w,u]_T + (-1)^{|z|(|x|+|y|)}[z,[x,y,w]_T,u]_T\\ &\quad + (-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]_T]_T - [x,y,[z,w,u]_T]_T\\ &= [D(Tx,Ty)z + [x,y,z]',w,u]_T + (-1)^{|z|(|x|+|y|)}[z,D(Tx,Ty)w + [x,y,w]',u]_T\\ &\quad + (-1)^{(|x|+|y|)(|z|+|w|)}[z,w,D(Tx,Ty)u + [x,y,u]']_T - [x,y,D(Tz,Tw)u + [z,w,u]']_T\\ &= [D(Tx,Ty)z,w,u]_T + [[x,y,z]',w,u]_T + (-1)^{|z|(|x|+|y|)}[z,D(Tx,Ty)w,u]_T\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{|z|(|x|+|y|)}[z,[x,y,w]',u]_T+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,D(Tx,Ty)u]_T\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]']_T-[x,y,D(Tz,Tw)u]_T-[x,y,[z,w,u]']_T\\
&=D(TD(Tx,Ty)z,Tw)u+[D(Tx,Ty)z,w,u]'+D(T[x,y,z]',Tw)u+[[x,y,z]',w,u]'\\
&+(-1)^{|z|(|x|+|y|)}D(Tz,TD(Tx,Ty)w)u+(-1)^{|z|(|x|+|y|)}[z,D(Tx,Ty)w,u]'\\
&+(-1)^{|z|(|x|+|y|)}D(Tz,T[x,y,w]')u+(-1)^{|z|(|x|+|y|)}[z,[x,y,w]',u]'\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}D(Tz,Tw)D(Tx,Ty)u+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,D(Tx,Ty)u]'\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}D(Tz,Tw)[x,y,u]'+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]']'\\
&-D(Tx,Ty)D(Tz,Tw)u-[x,y,D(Tz,Tw)u]'-D(Tx,Ty)[z,w,u]'-[x,y,[z,w,u]']'\\
&=\Big((-1)^{(|x|+|y|)(|z|+|w|)}D(Tz,Tw)D(Tx,Ty)u-D(Tx,Ty)D(Tz,Tw)u\\
&+D([Tx,Ty,Tz],Tw)u+(-1)^{|z|(|x|+|y|)}D(Tz,[Tx,Ty,Tw])u\Big)\ (\text{by }(28))\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}D(Tz,Tw)[x,y,u]'(\text{by }(32))\\
&-D(Tx,Ty)[z,w,u]'(\text{by }(32))\\
&+\Big([[x,y,z]',w,u]'+(-1)^{|z|(|x|+|y|)}[z,[x,y,w]',u]'\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,[x,y,u]']'-[x,y,[z,w,u]']'\Big)\ (\text{by }(25))\\
&+[D(Tx,Ty)z,w,u]'(\text{by }(31))\\
&+(-1)^{|z|(|x|+|y|)}[z,D(Tx,Ty)w,u]'(\text{by }(31))\\
&-[x,y,D(Tz,Tw)u]'(\text{by }(31))\\
&+(-1)^{(|x|+|y|)(|z|+|w|)}[z,w,D(Tx,Ty)u]'(\text{by }(31))\\
&=0.
\end{aligned}
$$

Therefore, $(\mathfrak{g}';[\cdot,\cdot,\cdot]_T)$ is a 3-Leibniz superalgebra.

2) We proceed to check that the triple $(\mathfrak{g};\rho_1^T,\rho_2^T,\rho_3^T)$ as defined above satisfies the set of axioms as in Definition 3.9.

For Eq. (36), for any $u,v,s,t\in\mathfrak{g}'$ and $x\in\mathfrak{g}$, by Eqs. (25), (13) and (1), we have

$$
\begin{aligned}
&\rho_1^T(u,v,\rho_1^T(s,t,x))-\rho_1^T([u,v,s]_T,t,x)-(-1)^{|s|(|u|+|v|)}\rho_1^T(s,[u,v,t]_T,x)\\
&\quad-(-1)^{(|s|+|t|)(|u|+|v|)}\rho_1^T(s,t,\rho_1^T(u,v,x))\\
&=[Tu,Tv,[Ts,Tt,x]]-[T[u,v,s]_T,Tt,x]-(-1)^{|s|(|u|+|v|)}[Ts,T[u,v,t]_T,x]\\
&\quad-(-1)^{(|s|+|t|)(|u|+|v|)}[Ts,Tt,[Tu,Tv,x]]\\
&=[Tu,Tv,[Ts,Tt,x]]-[[Tu,Tv,Ts],Tt,x]-(-1)^{|s|(|u|+|v|)}[Ts,[Tu,Tv,Tt],x]\\
&\quad-(-1)^{(|s|+|t|)(|u|+|v|)}[Ts,Tt,[Tu,Tv,x]]\\
&=0,
\end{aligned}
$$

so Eq. (36) holds for $(\mathfrak{g};\rho_1^T,\rho_2^T,\rho_3^T)$.

For Eq. (37), by Eqs. (25), (28), (13) and (14), we have

$$\rho_1^T(u,v,\rho_2^T(s,x,t))-\rho_2^T([u,v,s]_T,x,t)-(-1)^{|s|(|u|+|v|)}\rho_2^T(s,\rho_1^T(u,v,x),t)$$

$$
\begin{aligned}
&- (-1)^{(|s|+|x|)(|u|+|v|)}\rho_2^T(s,x,[u,v,t]_T)\\
&= [Tu,Tv,[Ts,x,Tt]] - [Tu,Tv,TD(Ts,x)t] - [T[u,v,s]_T,x,Tt] + TD(T[u,v,s]_T,x)t\\
&\quad - (-1)^{|s|(|u|+|v|)}[Ts,[Tu,Tv,x],Tt] + (-1)^{|s|(|u|+|v|)}TD(Ts,[Tu,Tv,x])t\\
&\quad - (-1)^{(|s|+|x|)(|u|+|v|)}[Ts,x,T[u,v,t]_T] + (-1)^{(|s|+|x|)(|u|+|v|)}TD(Ts,x)[u,v,t]_T\\
&= [Tu,Tv,[Ts,x,Tt]] - [Tu,Tv,TD(Ts,x)t] - [[Tu,Tv,Ts],x,Tt] + TD([Tu,Tv,Ts],x)t\\
&\quad - (-1)^{|s|(|u|+|v|)}[Ts,[Tu,Tv,x],Tt] + (-1)^{|s|(|u|+|v|)}TD(Ts,[Tu,Tv,x])t\\
&\quad - (-1)^{(|s|+|x|)(|u|+|v|)}[Ts,x,[Tu,Tv,Tt]] + TD(Ts,x)[u,v,t]_T\\
&= T(-D(Tu,Tv)D(Ts,x)t + D([Tu,Tv,Ts],x)t + (-1)^{|s|(|u|+|v|)}D(Ts,[Tu,Tv,x])t\\
&\quad + (-1)^{(|s|+|x|)(|u|+|v|)}D(Ts,x)D(Tu,Tv)t)\\
&= 0,
\end{aligned}
$$

so Eq. (37) holds for $(\mathfrak{g};\rho_1^T,\rho_2^T,\rho_3^T)$.

For Eqs. (38)-(40), similar to the proofs of Eqs. (36) and (37), it is easy to see that that Eqs. (38)-(40) holds. Therefore, we deduce that $(\mathfrak{g};\rho_1^T,\rho_2^T,\rho_3^T)$ is a representation of the 3-Leibniz superalgebra $(\mathfrak{g}';[\cdot,\cdot,\cdot]_T)$. This completes the proof. □

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