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Application of linear forms in logarithms to the Zeckendorf decomposition of the Tribonacci-Lucas sequence: The Cases $k = 3$ and $k = 4$

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Abstract. It has been proved by the mathematician Zeckendorf that every positive integer has a unique decomposition into a sum of distinct and nonconsecutive Fibonacci numbers $\{F_n\}_{n\geq 0}$, known as the Zeckendorf partition. In this paper, we applied the linear forms in logarithms to the Zeckendorf decomposition of the Tribonacci-Lucas Sequence $\{S_m\}_{m\geq 0}$. We give effective computable bounds for the decomposition. As applications, we explicitly determine all the Tribonacci-Lucas numbers that admit a Zeckendorf partition into exactly three terms or exactly four terms. We found that in the case of three terms, there are exactly two, namely S_7 and S_8, and in the case of four terms, there are also two namely S_{10} and S_{13}.

Keywords: Diophantine equations, Fibonacci sequence,Tribonacci-Lucas sequence, Zeckendorf partition, linear forms in logarithms, reduction method.
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1 Introduction

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ defined as $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$ with initial conditions $F_0 = 0,\ F_1 = 1$, occupies a central place in number theory and combinatorics. A fundamental result, known as Zeckendorf's theorem [12], states that every non-zero natural number has a unique representation as a sum of non-consecutive Fibonacci numbers, the smallest index of which is at least 2. The Fibonacci sequence has been the subject of much research regarding its divisibility properties and its intersections with other families of numbers.

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The characteristic polynomial of the Fibonacci sequence is $X^2 - X - 1$, whose roots are γ and δ with

$$\gamma = \frac{1+\sqrt{5}}{2} \quad and \quad \delta = \frac{1-\sqrt{5}}{2}.$$

Meanwhile, the Tribonacci-Lucas $\{S_m\}_{m\geq 0}$, a variant of the classical Tribonacci sequence, is defined by the third-order recurrence $S_{m+3} = S_{m+2} + S_{m+1} + S_m$, $\forall\, m \geq 0$ with the initial conditions $S_0 = S_2 = 3$ and $S_1 = 1$. It is the sequence defined in OEIS [9] with the first few terms

$$\{S_m\}_{m\geq 0} = 3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071, 9327, 17155, 31553, ...$$

The characteristic polynomial of the Tribonacci-Lucas sequence is $X^3 - X^2 - X - 1$, whose roots are α, β, and $\bar{\beta}$, where

$$\alpha = \frac{1+r_1+r_2}{3}, \quad \beta = \frac{2-(r_1+r_2)+i\sqrt{3}(r_1-r_2)}{6},$$

with

$$r_1 = (19+3\sqrt{33})^{1/3}, \quad r_2 = (19-3\sqrt{33})^{1/3}.$$

The study of intersections between different recursive sequences, or the representation of one sequence by combinations of another, has been the subject of significant work in recent decades.

More recently, interest has shifted to more complex Diophantine equations involving products or sums of terms or powers of integers. In the literature, several works have been done on the study of Tribonacci sequences as a sum of Fibonacci sequences as well as a products of Fibonacci numbers and other sequences (see [5, 8, 11]). Concerning Tribonacci-Lucas studies, the recent works deal with Tribonacci-Lucas numbers that are Palindromic concatenations of two distinct repdigits [3] and common terms of Tribonacci and Tribonacci–Lucas sequences [1].

Zeckendorf's theorem [12] has been applied to Narayana numbers and successful results obtained [7].

In this paper, we propose to find the Zeckendorf partition in k terms of Tribonacci-Lucas sequence by solving the following equation

$$S_m = F_{n_1} + F_{n_2} + \cdots + F_{n_k}, \tag{1}$$

in positive integer unknowns m, n_i, $i \in \{1, 2, \cdots, k\}$. The solutions of this equation, such that $n_1 \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i \in \{1, 2, \cdots, k-1\}$, give the Tribonacci-Lucas numbers that admit the Zeckendorf partition into exactly k terms.

Here, we give effective computable bounds for solutions of Equation (1) by using the linear forms in logarithms of algebraic numbers. In explicit cases, the bounds can be reduced by reduction procedure. We have precisely obtained the following result concerning (1).

Theorem 1. *Let $(m, n_1, \ldots, n_k)$, with $n_1 \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i \in \{1, 2, \ldots, k-1\}$, be a solution of the Diophantine equation (1). Then we have*

$$n_k < 2^k C_k \left(\log C_k\right)^k,$$

where C_k is the last term of the finite sequence $\{C_i\}_{1\leq i\leq k}$ defined by:

$$\begin{cases} C_1 = 3.74 \cdot 10^{14}, \\ C_{\ell+1} = 9.16 \cdot 10^{14} \cdot (\ell+2) \cdot C_\ell + 1.8, \quad for \;\; 1 \leq \ell \leq k-2, \\ C_k = 1.91 \cdot 10^{15} \cdot (k+1) \cdot C_{k-1} + 3.74. \end{cases}$$

To demonstrate our results, we use Baker's method on linear forms of logarithms combined with Baker-Davenport reduction to bound the indices and identify the complete set of solutions. The numerical calculations have been done with the help of SageMath.

The paper is organized as follows. In Section 2, we gather the tools we need to prove Theorem 1. Section 3 deals with the proof of Theorem 1. We apply Theorem 1 with the values $k \in \{3, 4\}$ in Section 4 which allow us to determine all the Tribonacci-Lucas numbers that admit a Zeckendorf partition into exactly three or exactly four terms. Finally, the conclusion is provided in the final section.

2 Preliminaries

In this section, we present a collection of known results on Binet formulas, height properties, Matveev's theorem and reduction lemma, that we need to prove our results.

2.1 Recalls on Tribonacci-Lucas and Fibonacci sequences

In this subsection, we recall some properties of Tribonacci-Lucas sequence and Fibonacci sequence.

The Binet formula for the Tribonacci-Lucas sequence is

$$S_m = \alpha^m + \beta^m + \bar{\beta}^m \quad for \;\; all \;\; m \geq 0. \tag{2}$$

We have the numerical estimates

$$\begin{aligned} &1.83 < \alpha < 1.84, \\ &0.73 < |\beta| = \alpha^{-1/2} < 0.74. \end{aligned} \tag{3}$$

It is easy to see that the contribution of the complex conjugate roots β and $\bar{\beta}$, to the right-hand side of (2), is very small. Particularly, we set

$$e(m) := S_m - \alpha^m = \beta^m + \bar{\beta}^m. \tag{4}$$

Then

$$|e(m)| \leq 2\alpha^{-m/2} \quad for \quad all \;\; m \geq 1. \tag{5}$$

The above comes from the fact that $|\beta| = \left|\bar{\beta}\right| = \alpha^{-1/2}$. Then for any $m \geq 1$,

$$|e(m)| = \left|\beta^m + \bar{\beta}^m\right| \leq |\beta|^m + \left|\bar{\beta}\right|^m = \alpha^{-m/2} + \alpha^{-m/2} = 2\alpha^{-m/2}.$$

We also have the following estimate.

Lemma 1 ([1], Lemma 2.1)**.** *Let $m \geq 1$ be a positive integer. Then*

$$\alpha^{m-1} \leq S_m < \alpha^{m+1}. \tag{6}$$

The Binet formula of the Fibonacci sequence is given by

$$F_n = \frac{\gamma^n - \delta^n}{\sqrt{5}} \quad \text{for} \quad n \geq 0,$$

Lemma 2. *For integer $n \geq 1$, the inequalities below hold*

$$0.27\gamma^n \leq F_n \leq 0.62\gamma^n. \tag{7}$$

Proof. See Lemma 2 from [7].

□

Lemma 3. *Let k be a positive integer. For positive integers $\ell_1 < \ell_2 < \cdots < \ell_k$, we have*

$$\left|\gamma^{-\ell_1} + \gamma^{-\ell_2} + \cdots + \gamma^{-\ell_k}\right| < \frac{\gamma^{-\ell_1}}{1-\gamma^{-1}}, \tag{8}$$

$$\left|\delta^{\ell_1} + \delta^{\ell_2} + \cdots + \delta^{\ell_k}\right| < \frac{\gamma^{-\ell_1}}{1-\gamma^{-1}}. \tag{9}$$

Proof.

$$\begin{aligned}
\left|\gamma^{-\ell_1} + \gamma^{-\ell_2} + \cdots + \gamma^{-\ell_k}\right| = \gamma^{-\ell_1} + \gamma^{-\ell_2} + \cdots + \gamma^{-\ell_k} &\leq \sum_{i=\ell_1}^{\ell_k} \gamma^{-i} \\
&\leq \gamma^{-\ell_1}\left(\sum_{i=0}^{\ell_k-\ell_1} \gamma^{-i}\right) = \gamma^{-\ell_1}\left(\frac{1-\gamma^{\ell_1-\ell_k-1}}{1-\gamma^{-1}}\right) \\
&= \frac{\gamma^{-\ell_1} - \gamma^{-\ell_k-1}}{1-\gamma^{-1}}, \\
&< \frac{\gamma^{-\ell_1}}{1-\gamma^{-1}}, \quad \text{since} \quad \gamma^{-\ell_k-1} > 0.
\end{aligned}$$

Since $\gamma\delta = -1$, we have

$$\left|\delta^{\ell_1} + \delta^{\ell_2} + \cdots + \delta^{\ell_k}\right| \leq \left|\delta^{\ell_1}\right| + \left|\delta^{\ell_2}\right| + \cdots + \left|\delta^{\ell_k}\right| = \gamma^{-\ell_1} + \gamma^{-\ell_2} + \cdots + \gamma^{-\ell_k} < \frac{\gamma^{-\ell_1}}{1-\gamma^{-1}}.$$

□

Similarly, as in [1, 3, 7], let us note that for the algebraic numbers α (in the above Binet formula of $\{S_m\}$) and γ (in the above Binet formula of $\{F_n\}$), we have

$$\mathbb{Q}(\alpha) \cap \mathbb{Q}(\gamma) = \mathbb{Q} \quad \text{and} \quad 3 = [\mathbb{Q}(\alpha) : \mathbb{Q}] \neq [\mathbb{Q}(\gamma) : \mathbb{Q}] = 2.$$

Then the numbers α and γ are positive elements of the real field $\mathbb{K} = \mathbb{Q}(\alpha, \gamma)$ of degree $d_{\mathbb{K}/\mathbb{Q}} = 6$. The sppliting fields of the polynomials $X^3 - X^2 - X - 1$ and $X^2 - X - 1$ over $\mathbb{Q}$ are $\mathbb{Q}(\alpha, \beta)$ and $\mathbb{Q}(\gamma)$, respectively. They are Galois extensions of $\mathbb{Q}$. Then the fields $\mathbb{L} := \mathbb{Q}(\alpha, \beta, \gamma)$ is Galois extension of $\mathbb{Q}$ and a $\mathbb{Q}$-automorphism of $\mathbb{L}$ is for example

$$\sigma\colon\ \alpha \mapsto \beta,\ \beta \mapsto \alpha,\ \bar{\beta} \mapsto \bar{\beta} \ \text{ and } \ x \mapsto x, \ \text{ for } \ x \in \mathbb{Q}(\gamma). \tag{10}$$

Lemma 4. *The algebraic numbers α and γ are multiplicatively independent, and consequently, $\dfrac{\log \alpha}{\log \gamma}$ is irrational.*

Proof. See Lemma 3 from [7]. □

2.2 Auxiliary results on linear forms in logarithms of algebraic numbers

In this subsection, we point out some useful results from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. Let $\eta \neq 0$ be an algebraic number of degree d and let

$$a_0(X - \eta^{(1)}) \cdots (X - \eta^{(d)}) \in \mathbb{Z}[X]$$

be the minimal polynomial of $\eta = \eta^{(1)}$. Then the absolute logarithmic Weil height is defined by

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \max\{0, \log|\eta^{(i)}|\}\right).$$

It is easy to check that following basic properties of the height hold:

Lemma 5. *For algebraic numbers η, γ, and $s \in \mathbb{Z}$, we have*

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta). \end{aligned}$$

In the case of a rational number, say $p/q \in \mathbb{Q}$ with p, q integers such that $\gcd(p, q) = 1$, we have $h(p/q) = \max\{\log|p|, \log|q|\}$.

Let $\mathbb{K}$ be a real number field of degree $d_{\mathbb{K}}$, $\eta_1, \ldots, \eta_t \in \mathbb{K}$ and $b_1, \ldots, b_t \in \mathbb{Z} \setminus \{0\}$. Let $B \geq \max\{|b_1|, \ldots, |b_t|\}$ and

$$\Lambda = \eta_1^{b_1} \cdots \eta_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers with

$$A_i \geq \max\{d_{\mathbb{K}} h(\eta_i), |\log \eta_i|, 0.16\}, \quad i = 1, 2, \ldots, t.$$

With these basic notations we have the following result which is Bugeaud et al.'s version of lower bounds for linear forms in logarithms due to Matveev [6].

Theorem 2 ([2], Theorem 9.4)**.** *Assume that* $\Lambda \neq 0$. *Then*

$$\log|\Lambda| > -1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot d_{\mathbb{K}}^2 \cdot (1+\log d_{\mathbb{K}}) \cdot (1+\log B) \cdot A_1 \cdots A_t.$$

We also need the following lemma due to Guzmán and Luca.

Lemma 6 ([10], Lemma 7)**.** *If* $l \geq 1$, $H > \left(4l^2\right)^l$ *and* $H > L/(\log L)^l$, *then*

$$L < 2^l H (\log H)^l.$$

Let x be a real number, the application $\|x\| = \min\{|x-n|, n \in \mathbb{Z}\}$ is the distance from x to the nearest integer. We present herein a result from Dujella and Pethő [4] which is a variant of the Baker and Davenport reduction method. This will be used to reduce the bounds of ours solution.

Lemma 7. *Let* M *be a positive integer,* p/q *be a convergent of the continued fraction expansion of the irrational number* τ *such that* $q > 6M$, *and* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *If*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\| > 0,$$

then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v *and* w *with*

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3 Proof of Theorem 1

Let $(m, n_1, \cdots, n_k)$ be a solution of the Diophantine equation (1) with $n_1 \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i \in \{1, 2, \cdots, k-1\}$. Necessarily we have $n_k \geq 4$ and from Lemmas 2 and 3, we have:

$$\begin{aligned} F_{n_1} + F_{n_2} + \cdots + F_{n_k} &< 0.62\left(\gamma^{n_k} + \gamma^{n_{k-1}} + \cdots + \gamma^{n_1}\right) \\ &< 0.62\gamma^{n_k}\left(1 + \gamma^{n_{k-1}-n_k} + \cdots + \gamma^{n_1-n_k}\right) \\ &< 0.62\gamma^{n_k}\left(\frac{1}{1-\gamma^{-1}}\right) = \frac{0.62}{1-\gamma^{-1}}\gamma^{n_k} \\ &< 1.63\gamma^{n_k}. \\ F_{n_1} + F_{n_2} + \cdots + F_{n_k} &> F_{n_k} > 0.27\gamma^{n_k}. \end{aligned}$$

Hence, Equation (1) implies using Lemma 1, that

$$0.27\gamma^{n_k} < \alpha^{m+1} \quad \text{and} \quad \alpha^{m-1} < 1.63\gamma^{n_k}.$$

Taking logarithms of these inequalities and arranging, we obtain

$$n_k \frac{\log\gamma}{\log\alpha} - 1 + \frac{\log 0.27}{\log\alpha} < \quad m \quad < n_k \frac{\log\gamma}{\log\alpha} + 1 + \frac{\log 1.63}{\log\alpha}$$

$$\begin{aligned} n_k\left(\frac{\log\gamma}{\log\alpha}-\frac{3.15}{n_k}\right) < \quad m \quad < n_k\left(\frac{\log\gamma}{\log\alpha}+\frac{1.81}{n_k}\right) \\ n_k\left(\frac{\log\gamma}{\log\alpha}-\frac{3.15}{4}\right) < \quad m \quad < n_k\left(\frac{\log\gamma}{\log\alpha}+\frac{1.81}{4}\right), \end{aligned}$$

since $n_k \geq 4$. Finally we obtain

$$0.0024n_k < m < 2n_k. \tag{11}$$

3.1 Linear forms in logarithms from equation (1)

We construct k linear forms in logarithms from the Diophantine Equation (1) and we give an upper bound for each.

Let $(m, n_1, \cdots, n_k)$ be a solution of the Diophantine equation (1) with $n_1 \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i \in \{1, 2, \cdots, k-1\}$. To avoid permutations between index values, we assume that $n_k > n_{k-1}$. Furthermore, when looking for the solutions of equation (1) in the condition $k \geq 2$, the first possibility of combination of Fibonacci numbers is $F_2 + F_4$. So, at last, we assume $n_k \geq 4$.

Using the Binet formula of the sequences and rewriting Equation (1), we have:

$$\sqrt{5}\alpha^m = \gamma^{n_k} + \gamma^{n_{k-1}} + \cdots + \gamma^{n_1} - (\delta^{n_k} + \delta^{n_{k-1}} + \cdots + \delta^{n_1}) - \sqrt{5}e(m). \tag{12}$$

This implies that

$$\begin{aligned} \left|\sqrt{5}\alpha^m - \gamma^{n_k}\right| &= \left|\gamma^{n_{k-1}} + \cdots + \gamma^{n_1} - (\delta^{n_k} + \delta^{n_{k-1}} + \cdots + \delta^{n_1}) - \sqrt{5}e(m)\right| \\ &< \gamma^{n_{k-1}}\left|1 + \gamma^{n_{k-2}-n_{k-1}} + \cdots + \gamma^{n_1-n_{k-1}}\right| + \left|\delta^{n_k} + \cdots + \delta^{n_1}\right| + \sqrt{5}\left|e(m)\right| \\ &< \frac{\gamma^{n_{k-1}}}{1-\gamma^{-1}} + \frac{\gamma^{-2}}{1-\gamma^{-1}} + 2\sqrt{5}\alpha^{-m/2} \\ &< \left(\frac{1+\gamma^{-2}}{1-\gamma^{-1}} + 2\sqrt{5}\alpha^{-1}\right)\gamma^{n_{k-1}} \\ &< 6.05\gamma^{n_{k-1}}. \end{aligned}$$

Let set $C_0 := 6.05$. By dividing above inequality by γ^{n_k}, we get

$$\left|\sqrt{5}\alpha^m\gamma^{-n_k} - 1\right| < C_0\gamma^{n_{k-1}-n_k}. \tag{13}$$

We recall that for a positive real x, if $|x-1| < 0.5$, then $|\log x| < 1.5|x-1|$ (see [[8], Lemma 4]). The right side of inequality (13) would indeed be smaller than 0.5 if $n_k - n_{k-1} \geq 6$. We then obtain

$$\left|m\log\alpha - n_k\log\gamma + \log\left(\sqrt{5}\right)\right| < 9.08\gamma^{n_{k-1}-n_k}. \tag{14}$$

Let $1 \leq \ell \leq k-2$. From Equation (1), we have

$$\left|\sqrt{5}\alpha^m - \sum_{i=0}^{\ell}\gamma^{n_{k-i}}\right| = \left|\sum_{i=1}^{k-\ell-1}\gamma^{n_i} - \sum_{i=1}^{k}\delta^{n_i} - \sqrt{5}e(m)\right|$$

$$\begin{aligned}
&< \gamma^{n_{k-\ell-1}} \left| 1 + \sum_{i=1}^{k-\ell-2} \gamma^{n_i - n_{k-\ell-1}} \right| + \left| \sum_{i=1}^{k} \delta^{n_i} \right| + \sqrt{5}\,|e(m)| \\
&< \frac{\gamma^{n_{k-\ell-1}}}{1-\gamma^{-1}} + \frac{\gamma^{-n_1}}{1-\gamma^{-1}} + 2\sqrt{5}\alpha^{-m/2} \\
&< C_0 \gamma^{n_{k-\ell-1}}.
\end{aligned}$$

Dividing by $\gamma^{n_k}\left(1 + \sum_{i=1}^{\ell} \gamma^{n_{k-i}-n_k}\right)$, we get

$$\left| \frac{\sqrt{5}}{1 + \sum_{i=1}^{\ell} \gamma^{n_{k-i}-n_k}} \alpha^m \gamma^{-n_k} - 1 \right| < C_0 \gamma^{n_{k-\ell-1}-n_k}. \tag{15}$$

We also have $n_{k-\ell-1} - n_k \geq 6$ and we get

$$\left| m \log \alpha - n_k \log \gamma + \log \left(\frac{\sqrt{5}}{1 + \sum_{i=1}^{\ell} \gamma^{n_{k-i}-n_k}} \right) \right| < 9.08 \gamma^{n_{k-\ell-1}-n_k}. \tag{16}$$

A final linear form in logarithms is obtained as follows.

$$\left| \sqrt{5}\alpha^m - \sum_{i=1}^{k} \gamma^{n_i} \right| \leq \left| \sum_{i=1}^{k} \delta^{n_i} \right| + \sqrt{5}\,|e(m)| < C_0.$$

Dividing by $\gamma^{n_k}\left(1 + \sum_{i=1}^{k-1} \gamma^{n_i-n_k}\right)$, we get

$$\left| \frac{\sqrt{5}}{1 + \sum_{i=1}^{k-1} \gamma^{n_i-n_k}} \alpha^m \gamma^{-n_k} - 1 \right| < C_0 \gamma^{-n_k}. \tag{17}$$

Finally, we get

$$\left| m \log \alpha - n_k \log \gamma + \log \left(\frac{\sqrt{5}}{1 + \sum_{i=1}^{k-1} \gamma^{n_i-n_k}} \right) \right| < 9.08 \gamma^{-n_k}. \tag{18}$$

3.2 Bounds for the solutions

Using Lemma 5, we compute the absolute logarithmic Weil height of the algebraic numbers in presence as

$$\begin{aligned}
&h(\alpha) = \frac{1}{3}\log\alpha, \quad h(\gamma) = \frac{1}{2}\log\gamma, \quad h\left(\sqrt{5}\right) = \frac{1}{2}\log 5,\\
&h\left(1+\sum_{i=1}^{\ell}\gamma^{n_{k-i}-n_k}\right) \leq 2(\ell+1)(n_k-n_{k-\ell})\log\gamma,\\
&h\left(x_\ell\right) \leq 2\left(\ell+2\right)\left(n_k-n_{k-\ell}\right)\log\gamma,
\end{aligned}$$

with

$$x_\ell := \frac{\sqrt{5}}{1+\sum_{i=1}^{\ell}\gamma^{n_{k-i}-n_k}}, \text{ for } 1 \leq \ell \leq k-1.$$

We set now

$$\begin{aligned}
\Lambda_0 &:= \sqrt{5}\alpha^m\gamma^{-n_k} - 1,\\
\Lambda_\ell &:= \frac{\sqrt{5}}{1+\sum_{i=1}^{\ell}\gamma^{n_{k-i}-n_k}}\alpha^m\gamma^{-n_k} - 1, \text{ for } 1 \leq \ell \leq k-2,\\
\Lambda_{k-1} &:= \frac{\sqrt{5}}{1+\sum_{i=1}^{k-1}\gamma^{n_i-n_k}}\alpha^m\gamma^{-n_k} - 1.
\end{aligned}$$

One can prove easily that each of these linear forms in logarithms is non-zero. We can therefore determine the bounds for each of them by applying Theorem 2 and using the different absolute logarithmic Weil height of the algebraic numbers.

Let us note that for each linear forms in logarithms $\Lambda_i,\ i = 0, 1, \ldots, k-1$ above, the form $\Lambda_i + 1$ has three algebraic numbers factors with distinct exponent: $m,\ -n_k$ and 1. From (11), we have $2n_k > \max\{|m|,\ |-n_k|, 1\}$ and $1+\log(2n_k) < 3\log n_k$ for $n_k \geq 4$.

From bounds for Λ_0: We compute upper bounds on the absolute logarithmic Weil height of each of the above algebraic numbers by using Lemma 5 and we get:

$$t := 3 \quad \textit{and} \quad (\eta_1, b_1) := (\alpha, m), (\eta_2, b_2) := (\gamma, -n_k) \quad \textit{and} \quad (\eta_3, b_3) := (\sqrt{5}, 1).$$

$$\begin{aligned}
h(\eta_1) &= h(\alpha) < 0.204,\\
h(\eta_2) &= h(\gamma) < 0.241,\\
h(\eta_3) &= h(\sqrt{5}) = \log\sqrt{5} = 0.81.
\end{aligned}$$

We have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad A_3 = 4.86 \ \ and \ \ B = 2n_k.$$

We then get by Theorem (2),

$$\begin{aligned} \log|\Lambda_1| &> -7.63 \cdot 10^{13} A_3 \log n_k \\ &> 3.74 \cdot 10^{14} \log n_k. \end{aligned}$$

Combining this inequality with the logarithm of (13), we get

$$(n_k - n_{k-1}) \log \gamma < 3.74 \cdot 10^{14} \log n_k. \tag{19}$$

From bounds for Λ_ℓ, $1 \leq \ell \leq k-2$: Let us first give bounds for Λ_1, i.e. the case $\ell = 1$. From (15), we have

$$\log|\Lambda_1| < (n_{k-2} - n_k) \log \gamma + \log C_0. \tag{20}$$

We choose

$$t := 3, \qquad (\eta_1, b_1) := (\alpha, m), (\eta_2, b_2) := (\gamma, -n_k) \quad and \quad (\eta_3, b_3) = \left(\frac{\sqrt{5}}{1 + \sum_{i=1}^{\ell} \gamma^{n_{k-i} - n_k}}, 1 \right).$$

We have

$$\begin{aligned} h(\eta_1) &= h(\alpha) < 0.204, \\ h(\eta_2) &= h(\gamma) < 0.241, \\ h(\eta_3) &< h(x_\ell) \leq 6(n_k - n_{k-1}) \log \gamma. \end{aligned}$$

We have $d_{\mathbb{K}} = 6$ and we can choose

$$A_1 = 1.23, \quad A_2 = 1.45, \quad A_3 = 36(n_k - n_{k-1}) \log \gamma \ \ and \ \ B = 2n_k.$$

$$\begin{aligned} \log|\Lambda_1| &> -7.63 \cdot 10^{13} A_3 \log n_k \\ &> -2.75 \cdot 10^{15} (n_k - n_{k-1}) \log \gamma \log n_k. \end{aligned}$$

So, combining the above inequality and (20), we get

$$(n_k - n_{k-2}) \log \gamma - \log C_0 < 2.75 \cdot 10^{15} (n_k - n_{k-1}) \log \gamma \log n_k. \tag{21}$$

Considering the above inequality and using the upper bound of $(n_k - n_{k-1}) \log \gamma$ from (19), we get

$$(n_k - n_{k-2}) \log \gamma < 1.03 \cdot 10^{30} (\log n_k)^2.$$

We see that the lower bound for $\log|\Lambda_1|$ depend on the upper bound of $(n_k - n_{k-1})\log\gamma$ which is obtained from bounds study of $|\Lambda_0|$. This fact will be recursive in the remain cases.

Form (15), we have for $2 \leq \ell \leq k-2$,

$$\log|\Lambda_\ell| < (n_{k-\ell-1} - n_k)\log\gamma + \log C_0,$$

and

$$\begin{aligned}\log|\Lambda_\ell| &> -12 \cdot 7.63 \cdot 10^{13}(\ell+2)\cdot(n_k - n_{k-\ell})\log\gamma\log n_k \\ &> -9.16\cdot 10^{14}(\ell+2)\cdot(n_k - n_{k-\ell})\log\gamma\log n_k.\end{aligned}$$

So, there are effectively computable constants $C_1, \ldots, C_{\ell+1},\ 1 \leq \ell \leq k-2$, such that

$$(n_k - n_{k-\ell-1})\log\gamma < C_{\ell+1}\left(\log n_k\right)^{\ell+1}. \tag{22}$$

The constants $C_i,\ i \geq 1$ are recursively defined and depend on ℓ and/or k. More precisely, we have

$$\begin{cases} C_1 = 3.74\cdot 10^{14}, \\ C_{\ell+1} = 9.16\cdot 10^{14}\cdot(\ell+2)\cdot C_\ell + \log C_0, \quad \text{for } \ 1 \leq \ell \leq k-2. \end{cases} \tag{23}$$

From bounds for Λ_{k-1}: From (17), we have

$$\log|\Lambda_{k-1}| < -n_k\log\gamma + \log C_0,$$

and

$$\log|\Lambda_{k-1}| > -9.16\cdot 10^{14}\cdot(k+1)\cdot(n_k - n_1)\log\gamma\log n_k.$$

Combining these two inequalities, we get

$$n_k\log\gamma < 9.16\cdot 10^{14}\cdot(k+1)\cdot(n_k - n_1)\log\gamma\cdot\log n_k + \log C_0.$$

From (22) with $\ell = k-2$, we have $(n_k - n_1)\log\gamma < C_{k-1}\left(\log n_k\right)^{k-1}$ and then we obtain

$$n_k\log\gamma < 9.16\cdot 10^{14}\cdot(k+1)\cdot C_{k-1}\left(\log n_k\right)^k + \log C_0.$$

By applying Lemma 6, we obtain

$$n_k < 2^k C_k\left(\log C_k\right)^k, \tag{24}$$

where $C_k = \left(9.16\cdot 10^{14}\cdot(k+1)\cdot C_{k-1} + \log C_0\right)/\log\gamma$.

Because $m \leq 2n_k$, we also have from (24)

$$m < 2^{k+1}C_k\left(\log C_k\right)^k. \tag{25}$$

This completes the proof of Theorem 1.

4 Some applications

Let us consider the Diophantine equation (1). For explicit value of k we obtain, using Theorem 1, explicit bounds for the solutions. One can reduce the bounds using Lemma 7 and explicitly solve the equation. We have checked the cases $k \in \{3, 4\}$.

4.1 Tribonacci-Lucas numbers that admit the Zeckendorf partition into exactly three terms

For $k = 3$, the Diophantine equation (1) becomes

$$S_m = F_{n_1} + F_{n_2} + F_{n_3}, \tag{26}$$

and we have the following result:

Theorem 3. *The solutions of the Diophantine equation (26) such that $2 \leq n_1$ and $n_3 \geq n_2+2 \geq n_1+4$ are*

$$S_7 = F_4 + F_7 + F_{10} = 71 \quad \text{and} \quad S_8 = F_6 + F_9 + F_{11} = 131.$$

That are the only Tribonacci-Lucas numbers that admit the Zeckendorf partition into exactly three terms.

Proof. Let (m, n_1, n_2, n_3) be a solution of the Diophantine equation (26). From Theorem 1, we have:

Table 1: Table summarizing the constants C_0, C_1, C_2, C_3, and n_3 for $k = 3$.

$k = 3$	$C_0 = 6.05$
	$C_1 = 3.74 \cdot 10^{14}$
	$C_2 = 1.03 \cdot 10^{30}$
	$C_3 = 7.87 \cdot 10^{45}$
	$n_3 < 2^3 C_3 \left(\log C_3\right)^3 < 7.43 \cdot 10^{52}$

We then have

$$0.0024 \cdot n_3 < m < 2n_3$$
$$0.0024 \cdot n_3 < m < 1.49 \cdot 10^{53}.$$

We consider three inequalities namely (14), (16), and (18), with $k = 3$ in each ones.
From (14), we have

$$\left| m\frac{\log \alpha}{\log \gamma} - n_3 + \frac{\log(\sqrt{5})}{\log \gamma} \right| < \frac{9.08}{\log \gamma}\gamma^{n_2-n_3}. \tag{27}$$

Setting data

$$M_1 := 1.49 \cdot 10^{53}, \quad \tau = \frac{\log \alpha}{\log \gamma}, \quad A = \frac{9.08}{\log \gamma} \text{ and } B = \gamma, \tag{28}$$

we therefore apply Lemma 7 several times.
First we apply Lemma 7 to (27) with additional data

$$w = n_3 - n_2,\ \mu = \frac{\log\left(\sqrt{5}\right)}{\log\gamma}.$$

With the help of SageMath, we find the 104-th convergent of τ

$$\frac{p_{104}}{q_{104}} = \frac{201200139798966751193574147435929776297154141211196697 83}{158883169678965009201330599350659934100235723795751379 64},$$

which satisfies $q_{104} > 6M$ and $\varepsilon = 0.474047610213760 > 0$. Thus, inequality (27) has no solution due to the Lemma 7 for

$$n_3 - n_2 \geq \frac{\log(A \cdot q_{104}/\varepsilon)}{\log\gamma} > 271.79,$$

which implies that

$$n_3 - n_2 \leq 271. \tag{29}$$

From (16), we have

$$\left| m\frac{\log\alpha}{\log\gamma} - n_3 + \frac{\log\left(\sqrt{5}/(1+\gamma^{n_2-n_3})\right)}{\log\gamma} \right| < \frac{9.08}{\log\gamma}\gamma^{n_1-n_3}. \tag{30}$$

Then we apply Lemma 7 to (30) with the data in (28) and the additional data

$$w = n_3 - n_1,\ \mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_2-n_3})\right)}{\log\gamma}.$$

With the help of SageMath, and using (29) we find that the 104-th convergent of τsatisfies $q_{104} > 6M$ and $\varepsilon = 0.000695763156361009 > 0$. Thus, inequality (30) has no solution due to the Lemma 7 for

$$n_3 - n_1 \geq \frac{\log(A \cdot q_{104}/\varepsilon)}{\log\gamma} > 285.34,$$

which implies that

$$n_3 - n_1 \leq 285. \tag{31}$$

At last with inequality (18), we have

$$\left| m\frac{\log\alpha}{\log\gamma} - n_3 + \frac{\log\left(\sqrt{5}/(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})\right)}{\log\gamma} \right| < \frac{9.08}{\log\gamma}\gamma^{-n_3}. \tag{32}$$

We now apply Lemma 7 to (32) with the data in (28) and

$$w = n_3,\ \mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3})\right)}{\log\gamma}.$$

Using (29) and (31), we find with a SageMath calculations program that the 104-th convergent of τ satisfies $q_{104} > 6M$ and $\varepsilon = 7.25655391147981e - 6 > 0$. Thus, inequality (32) has no solution due to the Lemma 7 for

$$n_3 \geq \frac{\log(Aq_{104}/\varepsilon)}{\log \gamma} > 294,83,$$

which implies that

$$n_3 \leq 294. \tag{33}$$

Table 2: Table summarizing M, τ, A, B, μ and ω for $k = 3$ for each step of the reduction procedure.

	Step 1	Step 2	Step 3
$k = 3$ $M = 1.49 \cdot 10^{53}$ $\tau = \frac{\log \alpha}{\log \gamma}$ $A = \frac{9.08}{\log \gamma}$ $B = \gamma$	$\mu = \frac{\log(\sqrt{5})}{\log \gamma}$ $\omega = n_3 - n_2$	$\mu = \frac{\log(\sqrt{5}/(1+\gamma^{n_2-n_3}))}{\log \gamma}$ $\omega = n_3 - n_1$	$\mu = \frac{\log(\sqrt{5}/(1+\gamma^{n_1-n_3}+\gamma^{n_2-n_3}))}{\log \gamma}$ $\omega = n_3$

We now check for the solutions of equation (26) for $2 \leq n_1 + 4 \leq n_2 + 2 \leq n_3 \leq 294$ and $m \leq 588$. We use a small program in SageMath. We exhaustively get

$$(m, n_1, n_2, n_3) \in \{(7, 4, 7, 10), (8, 6, 9, 11)\},$$

corrresponding to the Tribonacci-Lucas numbers

$$S_7 = F_4 + F_7 + F_{10} = 71 \quad \textit{and} \quad S_8 = F_6 + F_9 + F_{11} = 131,$$

that are the only Tribonacci-Lucas numbers that admit a Zeckendorf partition into exactly three terms listed in Theorem 3. This completes the proof. □

4.2 Tribonacci-Lucas numbers that admit the Zeckendorf partition into exactly four terms

For $k = 4$, the Diophantine equation (1) becomes

$$S_m = F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4}, \tag{34}$$

and we have the following result:

Theorem 4. *The solutions of the Diophantine equation (34) such that $2 \leq n_1$ and $n_{i+1} \geq 2 + n_i$ for $i \in \{1, 2, 3\}$, are*

$$S_{10} = F_4 + F_6 + F_{10} + F_{14} = 443 \quad \textit{and} \quad S_{13} = F_6 + F_8 + F_{12} + F_{18} = 2757.$$

That are the only Tribonacci-Lucas numbers that admit the Zeckendorf partition into exactly four terms.

Proof. Let $(m, n_1, n_2, n_3, n_4))$ be a solution of the Diophantine equation (34). From Theorem 1, we have

Table 3: Table summarizing the constants C_0, C_1, C_2, C_3, C_4 and n_4 for $k = 4$.

$k = 4$	$C_0 = 6.05$
	$C_1 = 3.74 \cdot 10^{14}$
	$C_2 = 1.03 \cdot 10^{30}$
	$C_3 = 3.78 \cdot 10^{45}$
	$C_4 = 3.61 \cdot 10^{61}$
	$n_4 < 2^4 C_4 (\log C_4)^4 < 2.331 \cdot 10^{71}$

We then have

$$0.0024 \cdot n_4 < m < 2n_4$$
$$0.0024 \cdot n_4 < m < 4.67 \cdot 10^{71}.$$

Considering the same inequalities namely (14), (16), and (18), with $k = 4$, we respectively apply Lemma 7.

From (14),

$$\left| m\frac{\log \alpha}{\log \gamma} - n_4 + \frac{\log(\sqrt{5})}{\log \gamma} \right| < \frac{9.08}{\log \gamma}\gamma^{n_3 - n_4}, \tag{35}$$

with

$$M_2 := 4.67 \cdot 10^{71}, \quad \tau = \frac{\log \alpha}{\log \gamma}, \quad A = \frac{9.08}{\log \gamma} \text{ and } B = \gamma. \tag{36}$$

We apply Lemma 7 several times.

First we apply it to (35) with additional data

$$w = n_4 - n_3, \ \mu = \frac{\log(\sqrt{5})}{\log \gamma},$$

and we get (with the help of SageMath) that the 131-th convergent of τ

$$\frac{p_{131}}{q_{131}} = \frac{3887848094007753904801597458193467462311966784551656229542391114670820748}{3070145125264206658740034208230548645046580879204317312576627319870788909},$$

satisfies $q_{131} > 6M$ and $\varepsilon = 0.279180610419272 > 0$. Thus, inequality (35) has no solution for

$$n_4 - n_3 \geq \frac{\log(A \cdot q_{131}/\varepsilon)}{\log \gamma} > 355.60.$$

Thus,

$$n_4 - n_3 \leq 355. \tag{37}$$

Now from (16), we have the following first ($\ell = 1$) inequality

$$\left| m\frac{\log \alpha}{\log \gamma} - n_4 + \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4})\right)}{\log \gamma} \right| < \frac{9.08}{\log \gamma}\gamma^{n_2-n_4}. \tag{38}$$

We secondly apply Lemma 7 to (38) with the data in (36) and

$$w = n_4 - n_2,\ \mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4})\right)}{\log \gamma}.$$

Using (37), we find with the help of SageMath that the 131-th convergent of τ satisfies $q_{131} > 6M$ and $\varepsilon = 0.00174731838213157 > 0$. So, we obtain

$$n_4 - n_2 \leq 366. \tag{39}$$

From (16), we have for $\ell = 2$ the inequality

$$\left| m\frac{\log \alpha}{\log \gamma} - n_4 + \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4})\right)}{\log \gamma} \right| < \frac{9.08}{\log \gamma}\gamma^{n_1-n_4}. \tag{40}$$

Again we apply Lemma 7 here to (40) with the data in (36) and

$$w = n_4 - n_1,\ \mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4})\right)}{\log \gamma}.$$

With the help of SageMath and using (37), and (39), we find that the 131-th convergent of τ satisfies $q_{131} > 6M$ and $\varepsilon = 7.31621325368581e-6 > 0$. Therefore, we obtain

$$n_4 - n_1 \leq 377. \tag{41}$$

Considering inequality (18), we have

$$\left| m\frac{\log \alpha}{\log \gamma} - n_4 + \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4}+\gamma^{n_1-n_4})\right)}{\log \gamma} \right| < \frac{9.08}{\log \gamma}\gamma^{-n_4}. \tag{42}$$

So, we apply Lemma 7 one last times, to (42) with the data from (36) and

$$w = -n_4,\ \mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4}+\gamma^{n_1-n_4})\right)}{\log \gamma}.$$

With the help of SageMath and using (37), and (39), and (41), we find that the 131-th convergent of τ satisfies $q_{131} > 6M$ and $\varepsilon = 9.95874753911830e-8 > 0$. Then we obtain

$$n_4 \leq 386.$$

We now check for the solutions of equation (34) for $2 \leq n_1+6 \leq n_2+4 \leq n_3+2 \leq n_4 \leq 386$ and $m \leq 772$. We use a small program in SageMath. We get

$$(m, n_1, n_2, n_3, n_4) \in \{(10, 4, 6, 10, 14), (13, 6, 8, 12, 18)\}$$

corrresponding to the Tribonacci-Lucas numbers

$$S_{10} = F_4 + F_6 + F_{10} + F_{14} = 443 \quad \textit{and} \quad S_{13} = F_6 + F_8 + F_{12} + F_{18} = 2757$$

that are the only Tribonacci-Lucas numbers that admit a Zeckendorf partition into exactly four terms listed in Theorem 4. This completes the proof. □

Table 4: Table summarizing M, τ, A, B, μ and ω for $k = 4$ for each step of the reduction procedure.

	Step 1	Step 2
$k = 4$ $M = 4.67 \cdot 10^{71}$ $\tau = \frac{\log \alpha}{\log \gamma}$ $A = \frac{9.08}{\log \gamma}$ $B = \gamma$	$\mu = \frac{\log(\sqrt{5})}{\log \gamma}$ $\omega = n_4 - n_3$	$\mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4})\right)}{\log \gamma}$ $\omega = n_4 - n_2$

	Step 3	Step 4
$k = 4$ $M = 4.67 \cdot 10^{71}$ $\tau = \frac{\log \alpha}{\log \gamma}$ $A = \frac{9.08}{\log \gamma}$ $B = \gamma$	$\mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4})\right)}{\log \gamma}$ $\omega = n_4 - n_1$	$\mu = \frac{\log\left(\sqrt{5}/(1+\gamma^{n_3-n_4}+\gamma^{n_2-n_4}+\gamma^{n_1-n_4})\right)}{\log \gamma}$ $\omega = n_4$

5 Conclusion

In this study, we have examined the way to find Zeckendorf partition in exactly k terms of the Tribonacci-Lucas sequence. The problem consisted of solving the Diophantine equation

$$S_m = F_{n_1} + F_{n_2} + F_{n_3} + ... + F_{n_k},$$

in positive integer unknowns $m, n_1, n_2, n_3, ..., n_k$, with the conditions $n_1 \geq 2$ and $n_{i+1} \geq n_i + 2$ for $i \in \{1, 2, \cdots, k-1\}$. We estabilish a recurrence relation to obtain the integer unknowns $m, n_1, n_2, n_3, ..., n_k$. As applications, we have checked the cases $k \in \{3, 4\}$ corrresponding to the Tribonacci-Lucas numbers that admit a Zeckendorf partition into three and four terms. The study pointed out that for $k = 3$, only the Fibonacci-Lucas numbers S_7 and S_8 admit a Zeckendorf partition in three terms and for $k = 4$, only the Fibonacci-Lucas numbers S_{10} and S_{13} admit a Zeckendorf partition in four terms.

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Appendix 1: SageMath code used to search the solutions in the case $k = 3$

Below is the complete script used to find solutions for the equation $S_m = F_{n_1} + F_{n_2} + F_{n_3}$:

```
#!/usr/bin/env sage
% -*- coding: utf-8 -*-

from sage.all import *
import json
import os
import sys
from datetime import datetime

% --- DISPLAY CONFIGURATION ---
def print_log(message):
    print(message)
    sys.stdout.flush()

% --- RESULTS INITIALIZATION ---
```

```
results = {
    "date": str(datetime.now()),
    "sections": {}
}
solutions_list = []

% Output directory
output_dir = os.path.dirname(os.path.abspath(__file__))

print_log("="*70)
print_log("START OF MATHEMATICAL ANALYSIS (TRIBONACCI-LUCAS)")
print_log("="*70)

% --- SECTION 1 & 2: ROOTS AND CONSTANTS ---
print_log("\n[1-2] Calculating roots and constants...")
x = var('x')
% Characteristic equation associated with Tribonacci-Lucas: x^3 - x^2 - x -
    1 = 0
poly_eq = x^3 - x^2 - x - 1
roots_all = poly_eq.roots(x, ring=CC)

% Extract and sort roots by decreasing magnitude
roots_sorted = sorted(roots_all, key=lambda r: abs(r), reverse=True)
alpha_val = roots_sorted
beta_val = roots_sorted

gamma = (1 + sqrt(5)) / 2  % The golden ratio for Fibonacci

results["sections"]["constants"] = {
    "roots": str(roots_all),
    "alpha": float(N(alpha_val)),
    "abs_beta": float(N(abs(beta_val))),
    "gamma": float(N(gamma))
}

% --- SECTION 3: SEQUENCE AND CACHE GENERATION ---
print_log("\n Generating Tribonacci-Lucas sequence...")
S =
for i in range(3, 598):
    S.append(S[i-1] + S[i-2] + S[i-3])

tl_cache = {m: S[m] for m in range(598)}
fib_cache = {n: fibonacci(n) for n in range(400)}

% --- SECTION 11-12: CONVERGENCE AND BOUNDS ---
print_log("\n[11-12] Convergence analysis (Continued fractions)")
prec = 600
M = 1.49 * 10^(53)
A = N(9.08 / log(gamma))
```

```
B = N(gamma)
mu = N(log(a * sqrt(5)) / log(gamma), prec)
tau = N(log(alpha_val) / log(gamma), prec)

% --- FINAL SECTION: TRIPLE SEARCH LOOP ---
print_log("\n[Final] Searching for solutions S_m = F_n1 + F_n2 + F_n3")
print_log("Targets: m [1..598], n3 < 294")

count = 0
for m in range(1, 588):
    if m % 100 == 0:
        print_log(f"   Progress m : {m}/588")
    T = tl_cache[m]

    for n3 in range(2, 294):
        F3 = fib_cache[n3]
        if F3 >= T:
            break

        for n2 in range(2, n3-1):
            F2 = fib_cache[n2]
            if F2 + F3 >= T:
                break

            for n1 in range(2, n2-1):
                F1 = fib_cache[n1]
                if F1 + F2 + F3 == T:
                    count += 1
                    sol = {"m": int(m), "n1": int(n1), "n2": int(n2), "n3": int(n3), "val": int(T)}
                    solutions_list.append(sol)
                    print_log(f"   SOL : m={m}, n1={n1}, n2={n2}, n3={n3} | {T} = {F1}+{F2}+{F3}")

results["solutions"] = solutions_list
print_log(f"\nTotal found: {count} solutions.")

% --- DATA EXPORTS ---
print_log("\n[Saving] Writing output files...")

with open(os.path.join(output_dir, "analysis_results.json"), "w") as f:
    json.dump(results, f, indent=4)

with open(os.path.join(output_dir, "solutions.csv"), "w") as f:
    f.write("m,n1,n2,n3,Tribonacci_Lucas_m\n")
    for s in solutions_list:
        f.write(f"{s['m']},{s['n1']},{s['n2']},{s['n3']},{s['val']}\n")

with open(os.path.join(output_dir, "analysis_results.txt"), "w") as f:
```

```
    f.write(f"Analysis completed on {datetime.now()}\n")
    f.write(f"Total solutions: {count}\n")
    f.write("-" * 30 + "\n")
    for s in solutions_list:
        f.write(f"m={s['m']}: F_{s['n1']} + F_{s['n2']} + F_{s['n3']} = {s['val']}\n")

print_log("\n" + "="*70)
print_log("SCRIPT EXECUTED SUCCESSFULLY")
print_log("="*70)
```

Listing 1: 3-Term Tribonacci-Lucas Zeckendorf Decomposition script

Appendix 2: Sagemath code used to search the solutions in the case $k = 4$

Below is the complete script used to find solutions for the 4-term equation $S_m = F_{n_1} + F_{n_2} + F_{n_3} + F_{n_4}$:

```
#!/usr/bin/env sage
% -*- coding: utf-8 -*-

from sage.all import *
import json
import os
import sys
from datetime import datetime

% --- DISPLAY CONFIGURATION ---
def print_log(message):
    print(message)
    sys.stdout.flush()

% --- RESULTS INITIALIZATION ---
results = {
    "date": str(datetime.now()),
    "sections": {}
}
solutions_list = []

% Output directory
output_dir = os.path.dirname(os.path.abspath(__file__))

print_log("="*70)
print_log("START OF MATHEMATICAL ANALYSIS (4-TERM DECOMPOSITION)")
print_log("="*70)

% --- SECTION 1 & 2: ROOTS AND CONSTANTS ---
```

```
print_log("\n[1-2] Calculating roots and constants...")
x = var('x')
% Characteristic equation associated with Tribonacci-Lucas: x^3 - x^2 - x -
    1 = 0
poly_eq = x^3 - x^2 - x - 1
roots_all = poly_eq.roots(x, ring=CC)

% Sort roots by decreasing magnitude
roots_sorted = sorted(roots_all, key=lambda r: abs(r), reverse=True)
alpha_val = roots_sorted
beta_val = roots_sorted

gamma = (1 + sqrt(5)) / 2  % The golden ratio for Fibonacci

results["sections"]["constants"] = {
    "roots": str(roots_all),
    "alpha": float(N(alpha_val)),
    "abs_beta": float(N(abs(beta_val))),
    "gamma": float(N(gamma))
}

% --- SECTION 3: SEQUENCE AND CACHE GENERATION ---
print_log("\n Generating Tribonacci-Lucas sequence (up to m=772)...")
S =
for i in range(3, 772):
    S.append(S[i-1] + S[i-2] + S[i-3])

tl_cache = {m: S[m] for m in range(772)}
fib_cache = {n: fibonacci(n) for n in range(400)}

% --- SECTION 11-12: CONVERGENCE AND BOUNDS ---
print_log("\n[11-12] Convergence analysis (Continued fractions)")
prec = 600
M = 4.67 * 10^(71)
A = N(9.08 / log(gamma))
B = N(gamma)
mu = N(log(a * sqrt(5)) / log(gamma), prec)
tau = N(log(alpha_val) / log(gamma), prec)

% --- FINAL SECTION: QUADRUPLE SEARCH LOOP ---
print_log("\n[Final] Searching for solutions S_m = F_n1 + F_n2 + F_n3 +
    F_n4")
print_log("Targets: m [1..772], n4 < 386")

count = 0
for m in range(1, 772):
    if m % 100 == 0:
        print_log(f"   Progress m : {m}/772")
    T = tl_cache[m]
```

```
77
78      for n4 in range(2, 386):
79          F4 = fib_cache[n4]
80          if F4 >= T:
81              break
82
83          for n3 in range(2, n4-1):
84              F3 = fib_cache[n3]
85              if F4 + F3 >= T:
86                  break
87
88              for n2 in range(2, n3-1):
89                  F2 = fib_cache[n2]
90                  if F4 + F3 + F2 >= T:
91                      break
92
93                  for n1 in range(2, n2-1):
94                      F1 = fib_cache[n1]
95                      if F1 + F2 + F3 + F4 == T:
96                          count += 1
97                          sol = {"m": int(m), "n1": int(n1), "n2": int(n2), "
    n3": int(n3), "n4": int(n4), "val": int(T)}
98                          solutions_list.append(sol)
99                          print_log(f"   SOL : m={m}, n1={n1}, n2={n2}, n3={
    n3}, n4={n4} | {T} = {F1}+{F2}+{F3}+{F4}")
100
101 results["solutions"] = solutions_list
102 print_log(f"\nTotal found: {count} solutions.")
103
104 % --- DATA EXPORTS ---
105 print_log("\n[Saving] Writing output files...")
106
107 with open(os.path.join(output_dir, "analysis_results.json"), "w") as f:
108     json.dump(results, f, indent=4)
109
110 with open(os.path.join(output_dir, "solutions.csv"), "w") as f:
111     f.write("m,n1,n2,n3,n4,Tribonacci_Lucas_m\n")
112     for s in solutions_list:
113         f.write(f"{s['m']},{s['n1']},{s['n2']},{s['n3']},{s['n4']},{s['val
    ']}\n")
114
115 with open(os.path.join(output_dir, "analysis_results.txt"), "w") as f:
116     f.write(f"Analysis completed on {datetime.now()}\n")
117     f.write(f"Total solutions: {count}\n")
118     f.write("-" * 30 + "\n")
119     for s in solutions_list:
120         f.write(f"m={s['m']}: F_{s['n1']} + F_{s['n2']} + F_{s['n3']} + F_{
    s['n4']} = {s['val']}\n")
121
```

```
print_log("\n" + "="*70)
print_log("SCRIPT EXECUTED SUCCESSFULLY")
print_log("="*70)
```

Listing 2: 4-Term Tribonacci-Lucas Zeckendorf Decomposition script