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A relation between cofiniteness and minimaxness for generalized local cohomology modules

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Abstract. Let R be a commutative Noetherian ring with non-zero identity and let $\mathfrak{a}$ be an ideal of R. Consider M as a finitely generated R-module and X as an arbitrary R-module. Let n be a non-negative integer. This paper explores a relation between the properties of cofiniteness and minimaxness for generalized local cohomology modules. We establish that if $\operatorname{Ext}^n_R(M/\mathfrak{a}M, X)$ is a finitely generated R-module, $\operatorname{H}^i_{\mathfrak{a}}(M, X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < n$, and $\operatorname{H}^n_{\mathfrak{a}}(M, X)$ is a locally minimax R-module, then $\operatorname{H}^n_{\mathfrak{a}}(M, X)$ is an $\mathfrak{a}$-cofinite minimax R-module.

Keywords: Cofinite modules, Generalized local cohomology modules, Local cohomology modules, Locally minimax modules, Minimax modules.
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1 Introduction

Throughout this paper, R denotes a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ represents an ideal of R, and X and Y denote arbitrary R-modules, which are not necessarily finite (i.e., finitely generated). Furthermore, M and N are finite R-modules, while n represents a non-negative integer. For foundational results, notations, and terminology not explicitly addressed herein, readers are referred to [4, 5, 18].

In [11], Hartshorne introduced the notion of cofinite modules, providing a negative resolution to a conjecture previously posed by Grothendieck [10, Expose XIII, Conjecture 1.1]. He introduced the notion of an R-module X being $\mathfrak{a}$-torsion as $\mathfrak{a}$-*cofinite* (alternatively termed *cofinite with respect to* $\mathfrak{a}$) under the condition that $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for every integer i. Subsequently, Hartshorne articulated the following fundamental inquiry (cf. [11, First Question]):

Question 1. Is $\operatorname{H}^i_{\mathfrak{a}}(N)$ an $\mathfrak{a}$-cofinite R-module for every integer i?

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This question has been extensively studied, with affirmative results established under specific conditions (for instance, refer to [2,3,6,11,14–17]). An R-module X is classified as a *minimax R-module* if there exists a finite R-submodule X' of X such that the quotient R-module X/X' is Artinian [21]. Aghapournahr and Melkersson, in [1, Corollary 2.7], studied the relationship between cofiniteness and minimaxness for local cohomology modules. They demonstrated that if X is a finite R-module for which $\mathrm{H}^i_{\mathfrak{a}}(X)$ is $\mathfrak{a}$-cofinite for all $i < n$ and $\mathrm{H}^n_{\mathfrak{a}}(X)$ is locally minimax, then $\mathrm{H}^n_{\mathfrak{a}}(X)$ is indeed an $\mathfrak{a}$-cofinite minimax R-module.

Extensions of ordinary local cohomology theory have been proposed, beginning with Herzog [13], who introduced the *ith generalized local cohomology module of Y and X with respect to $\mathfrak{a}$*:

$$\mathrm{H}^i_{\mathfrak{a}}(Y,X) \cong \varinjlim_{j\in\mathbb{N}} \mathrm{Ext}^i_R(Y/\mathfrak{a}^jY, X).$$

Note that $\mathrm{H}^i_{\mathfrak{a}}(R,X)$ coincides with the conventional local cohomology module $\mathrm{H}^i_{\mathfrak{a}}(X)$ of X with respect to $\mathfrak{a}$. In continuation of aforementioned question, we propose the following inquiry regarding generalized local cohomology (cf. [20, Question 2.7]):

Question 2. Is $\mathrm{H}^i_{\mathfrak{a}}(M,N)$ an $\mathfrak{a}$-cofinite R-module for every integer i?

This question has been extensively investigated, with affirmative results established in various cases (for example, refer to [7,8,12,20]).

In this paper, we describe a relation between the properties of cofiniteness and minimaxness for generalized local cohomology modules. We extend and refine [1, Corollary 2.7], demonstrating that if $\mathrm{Ext}^n_R(M/\mathfrak{a}M,X)$ is finite, $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite for all $i<n$, and $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is locally minimax, then $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is indeed an $\mathfrak{a}$-cofinite minimax R-module.

2 Main results

In [1, Corollary 2.7], it is shown that if X is a finite R-module such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is $\mathfrak{a}$-cofinite for all $i<n$ and $\mathrm{H}^n_{\mathfrak{a}}(X)$ is locally minimax, then $\mathrm{H}^n_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-cofinite minimax R-module. The following theorem, the main result of this paper, generalizes and improves this earlier result.

Theorem 1. *Let n be a non-negative integer, M a finite R-module, and X an arbitrary R-module such that $\mathrm{Ext}^n_R(M/\mathfrak{a}M,X)$ is finite. Assume that $\mathrm{H}^i_{\mathfrak{a}}(M,X)$ is $\mathfrak{a}$-cofinite for all $i<n$ and that $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is locally minimax. Then $\mathrm{H}^n_{\mathfrak{a}}(M,X)$ is an $\mathfrak{a}$-cofinite minimax R-module.*

Proof. We begin by selecting an injective resolution of the module X, represented by the exact sequence

$$E^\bullet : 0 \longrightarrow X \longrightarrow E^0 \longrightarrow \cdots \longrightarrow E^j \longrightarrow \cdots .$$

Next, we apply the functor $\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(-))$ to its deletion, denoted as $E^{\bullet X}$, resulting in the complex

$$\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{\bullet X})) : 0 \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^0)) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^j)) \longrightarrow \cdots .$$

Referring to [18, Theorem 10.45], we can choose a Cartan–Eilenberg injective resolution of this complex, of the form

$$0 \longrightarrow \mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(E^{\bullet X})) \longrightarrow T^{\bullet,0} \longrightarrow \cdots \longrightarrow T^{\bullet,j} \longrightarrow \cdots ,$$

which provides a suitable framework for our analysis. Our focus will be on the third quadrant bicomplex $\mathcal{T} = \{\mathrm{Hom}_R(R/\mathfrak{a}, T^{p,q})\}$. Denote its total complex by $\mathrm{Tot}(\mathcal{T})$.

We proceed to compute the first iterated cohomology of $\mathcal{T}$. Based on [4, Proposition 2.1.4] and [18, Corollary 10.63 and Theorem 7.1], the following isomorphisms hold:

$$\begin{aligned}\mathrm{H}''^{p,q}(\mathcal{T}) &\cong \mathrm{H}^q(\mathrm{Hom}_R(R/\mathfrak{a}, T^{p,\bullet}))\\ &\cong \mathrm{Ext}^q_R(R/\mathfrak{a}, \mathrm{Hom}_R(M, \Gamma_\mathfrak{a}(E^p)))\\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_q(R/\mathfrak{a}, M), \Gamma_\mathfrak{a}(E^p)).\end{aligned}$$

Applying [9, Lemma 2.1(i)] yields the isomorphism

$$\mathrm{H}'^p\,\mathrm{H}''^{p,q}(\mathcal{T}) \cong \mathrm{H}^p(\mathrm{Hom}_R(\mathrm{Tor}^R_q(R/\mathfrak{a}, M), \Gamma_\mathfrak{a}(E^{\bullet X}))) \cong \mathrm{H}^p_\mathfrak{a}(\mathrm{Tor}^R_q(R/\mathfrak{a}, M), X).$$

Consequently, by [19, Lemma 2.5(c)], this leads to the third quadrant spectral sequence:

$$^{I}\mathrm{E}_2^{p,q} := \mathrm{Ext}^p_R(\mathrm{Tor}^R_q(R/\mathfrak{a}, M), X) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}(\mathrm{Tot}(\mathcal{T})). \tag{1}$$

Similarly, to analyze the second iterated cohomology of $\mathcal{T}$, note that any short exact sequence of injective modules splits, and this splitting property is preserved under the functor $\mathrm{Hom}_R(R/\mathfrak{a}, -)$. Using this fact together with the Cartan–Eilenberg injective resolution $T^{\bullet,\bullet}$ of $\mathrm{Hom}_R(M, \Gamma_\mathfrak{a}(E^{\bullet X}))$, we obtain:

$$\mathrm{H}'^{q,p}(\mathcal{T}) \cong \mathrm{H}^q(\mathrm{Hom}_R(R/\mathfrak{a}, T^{\bullet,p})) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^q(T^{\bullet,p})) \cong \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{q,p}).$$

Applying [9, Lemma 2.1(i)] then yields:

$$\mathrm{H}''^p\,\mathrm{H}'^{q,p}(\mathcal{T}) \cong \mathrm{H}^p(\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{q,\bullet})) \cong \mathrm{Ext}^p_R(R/\mathfrak{a}, \mathrm{H}^q_\mathfrak{a}(M, X)).$$

This leads to the second third quadrant spectral sequence:

$$^{II}\mathrm{E}_2^{p,q} := \mathrm{Ext}^p_R(R/\mathfrak{a}, \mathrm{H}^q_\mathfrak{a}(M, X)) \underset{p}{\Longrightarrow} \mathrm{H}^{p+q}(\mathrm{Tot}(\mathcal{T})). \tag{2}$$

We shall now present an inductive argument with respect to n to establish that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ constitutes a finite R-module for all indices $i < n$. There is no argument required for the case when $n = 0$. Assume $n > 0$ and that the case for $n-1$ has been confirmed. According to the induction hypothesis, $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is a finite R-module for all $i < n-1$. We prove that $\mathrm{Ext}^{n-1}_R(M/\mathfrak{a}M, X)$ is also a finite R-module. Consider the third quadrant spectral sequence (2). For all $i \leq n-1$, we can affirm that $^{II}\mathrm{E}_\infty^{n-1-i,i} = {}^{II}\mathrm{E}_{n+1}^{n-1-i,i}$ due to the fact that $^{II}\mathrm{E}_j^{n-1-i-j,i-1+j} = 0 = {}^{II}\mathrm{E}_j^{n-1-i+j,i+1-j}$ for all $j \geq n+1$. This ensures that $^{II}\mathrm{E}_\infty^{n-1-i,i}$ is a finite R-module, given that $^{II}\mathrm{E}_{n+1}^{n-1-i,i}$ is a subquotient of $^{II}\mathrm{E}_2^{n-1-i,i}$, which is a finite R-module by assumption. Additionally, there exists a finite filtration represented as:

$$0 = \psi^n\,\mathrm{H}^{n-1} \subseteq \psi^{n-1}\,\mathrm{H}^{n-1} \subseteq \cdots \subseteq \psi^1\,\mathrm{H}^{n-1} \subseteq \psi^0\,\mathrm{H}^{n-1} = \mathrm{H}^{n-1}(\mathrm{Tot}(\mathcal{T}))$$

such that $^{II}\mathrm{E}_\infty^{n-1-i,i} \cong \psi^{n-1-i}\,\mathrm{H}^{n-1}\,/\psi^{n-i}\,\mathrm{H}^{n-1}$ holds true for all $i \leq n-1$. Thus, the following short exact sequences:

$$0 \longrightarrow \psi^{n-i}\,\mathrm{H}^{n-1} \longrightarrow \psi^{n-1-i}\,\mathrm{H}^{n-1} \longrightarrow {}^{II}\mathrm{E}_\infty^{n-1-i,i} \longrightarrow 0,$$

for all $i \leq n-1$, demonstrate that $\mathrm{H}^{n-1}(\mathrm{Tot}(\mathcal{T}))$ is indeed a finite R-module. Now, consider the third quadrant spectral sequence (1). There exists a finite filtration defined as follows:

$$0 = \phi^n \mathrm{H}^{n-1} \subseteq \phi^{n-1} \mathrm{H}^{n-1} \subseteq \cdots \subseteq \phi^1 \mathrm{H}^{n-1} \subseteq \phi^0 \mathrm{H}^{n-1} = \mathrm{H}^{n-1}(\mathrm{Tot}(\mathcal{T}))$$

such that ${}^I\mathrm{E}_\infty^{n-1-i,i} \cong \phi^{n-1-i}\mathrm{H}^{n-1}/\phi^{n-i}\mathrm{H}^{n-1}$ for all $i \leq n-1$. Since $\mathrm{H}^{n-1}(\mathrm{Tot}(\mathcal{T}))$ is a finite R-module, it follows that $\phi^{n-1}\mathrm{H}^{n-1}$ is also a finite R-module. Consequently, we have ${}^I\mathrm{E}_\infty^{n-1,0} \cong \phi^{n-1}\mathrm{H}^{n-1}/\phi^n\mathrm{H}^{n-1}$ as a finite R-module. Thus, ${}^I\mathrm{E}_{n+1}^{n-1,0}$ is also a finite R-module because ${}^I\mathrm{E}_{n+1}^{n-1,0} = {}^I\mathrm{E}_\infty^{n-1,0}$, given that ${}^I\mathrm{E}_j^{n-1-j,j-1} = 0 = {}^I\mathrm{E}_j^{n-1+j,1-j}$ for all $j \geq n+1$. For all $r \geq 2$, we define ${}^I\mathrm{Z}_r^{n-1,0} := \mathrm{Ker}({}^I\mathrm{E}_r^{n-1,0} \longrightarrow {}^I\mathrm{E}_r^{n-1+r,1-r})$ and ${}^I\mathrm{B}_r^{n-1,0} := \mathrm{Im}({}^I\mathrm{E}_r^{n-1-r,r-1} \longrightarrow {}^I\mathrm{E}_r^{n-1,0})$. We have the following short exact sequences:

$$0 \longrightarrow {}^I\mathrm{Z}_r^{n-1,0} \longrightarrow {}^I\mathrm{E}_r^{n-1,0} \longrightarrow {}^I\mathrm{E}_r^{n-1,0}/{}^I\mathrm{Z}_r^{n-1,0} \longrightarrow 0$$

and

$$0 \longrightarrow {}^I\mathrm{B}_r^{n-1,0} \longrightarrow {}^I\mathrm{Z}_r^{n-1,0} \longrightarrow {}^I\mathrm{E}_{r+1}^{n-1,0} \longrightarrow 0.$$

Since ${}^I\mathrm{E}_2^{n-1+r,1-r} = 0$ and ${}^I\mathrm{E}_2^{n-1-r,r-1}$ is a finite R-module by [12, Proposition 3.4(i)], it follows that ${}^I\mathrm{E}_r^{n-1+r,1-r} = 0$ and ${}^I\mathrm{E}_r^{n-1-r,r-1}$ is a finite R-module. Consequently, this implies that ${}^I\mathrm{E}_r^{n-1,0} = {}^I\mathrm{Z}_r^{n-1,0}$ and that ${}^I\mathrm{B}_r^{n-1,0}$ is a finite R-module. This confirms that ${}^I\mathrm{E}_r^{n-1,0}$ is a finite R-module whenever ${}^I\mathrm{E}_{r+1}^{n-1,0}$ is also a finite R-module. Therefore, we conclude that ${}^I\mathrm{E}_2^{n-1,0} = \mathrm{Ext}_R^{n-1}(M/\mathfrak{a}M, X)$ is a finite R-module. This completes the inductive proof.

According to [1, Theorem 2.6], it suffices to verify that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_\mathfrak{a}(M,X))$ is a finite R-module. Consider the third quadrant spectral sequence (1). For all $i \leq n$, we observe that ${}^I\mathrm{E}_\infty^{n-i,i} = {}^I\mathrm{E}_{n+2}^{n-i,i}$, since the terms ${}^I\mathrm{E}_j^{n-i-j,i-1+j}$ and ${}^I\mathrm{E}_j^{n-i+j,i+1-j}$ are zero for all $j \geq n+2$. This indicates that ${}^I\mathrm{E}_\infty^{n-i,i}$ is a finite R-module, since it is a subquotient of ${}^I\mathrm{E}_2^{n-i,i}$, which is known to be finite over R by the previous paragraph, the given assumption, and [12, Proposition 3.4(i)]. Moreover, there exists a finite filtration of the form

$$0 = \phi^{n+1}\mathrm{H}^n \subseteq \phi^n\mathrm{H}^n \subseteq \cdots \subseteq \phi^1\mathrm{H}^n \subseteq \phi^0\mathrm{H}^n = \mathrm{H}^n(\mathrm{Tot}(\mathcal{T})),$$

such that for each $i \leq n$, ${}^I\mathrm{E}_\infty^{n-i,i} \cong \phi^{n-i}\mathrm{H}^n/\phi^{n-i+1}\mathrm{H}^n$. The short exact sequences

$$0 \longrightarrow \phi^{n-i+1}\mathrm{H}^n \longrightarrow \phi^{n-i}\mathrm{H}^n \longrightarrow {}^I\mathrm{E}_\infty^{n-i,i} \longrightarrow 0,$$

for all $i \leq n$, establish that $\mathrm{H}^n(\mathrm{Tot}(\mathcal{T}))$ is finite over R. Now, consider the third quadrant spectral sequence (2). There exists a finite filtration

$$0 = \psi^{n+1}\mathrm{H}^n \subseteq \psi^n\mathrm{H}^n \subseteq \cdots \subseteq \psi^1\mathrm{H}^n \subseteq \psi^0\mathrm{H}^n = \mathrm{H}^n(\mathrm{Tot}(\mathcal{T}))$$

such that for all $i \leq n$, ${}^{II}\mathrm{E}_\infty^{n-i,i} \cong \psi^{n-i}\mathrm{H}^n/\psi^{n-i+1}\mathrm{H}^n$. Since $\mathrm{H}^n(\mathrm{Tot}(\mathcal{T})) = \psi^0\mathrm{H}^n$ is finite over R, it follows that ${}^{II}\mathrm{E}_\infty^{0,n} \cong \psi^0\mathrm{H}^n/\psi^1\mathrm{H}^n$ is also finite over R. Furthermore, since ${}^{II}\mathrm{E}_{n+2}^{0,n} = {}^{II}\mathrm{E}_\infty^{0,n}$, it too is a finite R-module. For all $r \geq 2$, define ${}^{II}\mathrm{Z}_r^{0,n} := \mathrm{Ker}({}^{II}\mathrm{E}_r^{0,n} \longrightarrow {}^{II}\mathrm{E}_r^{r,n+1-r})$ and ${}^{II}\mathrm{B}_r^{0,n} := \mathrm{Im}({}^{II}\mathrm{E}_r^{-r,n-1+r} \longrightarrow {}^{II}\mathrm{E}_r^{0,n})$. There exist the following short exact sequences:

$$0 \longrightarrow {}^{II}\mathrm{Z}_r^{0,n} \longrightarrow {}^{II}\mathrm{E}_r^{0,n} \longrightarrow {}^{II}\mathrm{E}_r^{0,n}/{}^{II}\mathrm{Z}_r^{0,n} \longrightarrow 0,$$

and

$$0 \longrightarrow {}^{II}\mathrm{B}_r^{0,n} \longrightarrow {}^{II}\mathrm{Z}_r^{0,n} \longrightarrow {}^{II}\mathrm{E}_{r+1}^{0,n} \longrightarrow 0.$$

Since ${}^{II}\mathrm{E}_2^{r,n+1-r}$ is assumed to be a finite R-module, and given that ${}^{II}\mathrm{E}_2^{-r,n-1+r} = 0$, it follows that ${}^{II}\mathrm{E}_r^{r,n+1-r}$ is also a finite R-module and ${}^{II}\mathrm{E}_r^{-r,n-1+r} = 0$. This implies that the quotient ${}^{II}\mathrm{E}_r^{0,n}/{}^{II}\mathrm{Z}_r^{0,n}$ is a finite R-module and that ${}^{II}\mathrm{B}_r^{0,n} = 0$. Consequently, ${}^{II}\mathrm{E}_r^{0,n}$ is finite over R whenever ${}^{II}\mathrm{E}_{r+1}^{0,n}$ is finite. From this, we deduce that ${}^{II}\mathrm{E}_2^{0,n} = \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^n_\mathfrak{a}(M,X))$ is a finite R-module, thereby completing the proof. □

As applications of the aforementioned theorem, the subsequent corollaries are presented.

Corollary 1. *Let n be a non-negative integer, M a finite R-module, and X an arbitrary R-module such that $\mathrm{Ext}^i_R(M/\mathfrak{a}M, X)$ is finite for all $i \geq n$. Assume that $\mathrm{H}^i_\mathfrak{a}(M,X)$ is $\mathfrak{a}$-cofinite for all $i < n$ and that $\mathrm{H}^i_\mathfrak{a}(M,X)$ is locally minimax for all $i \geq n$. Then $\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$-cofinite R-module for all i.*

Corollary 2. *Let n be a non-negative integer and let M and X be finite R-modules. Then the following statements hold:*

(i) *$\mathrm{H}^n_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$-cofinite minimax R-module, provided that for all integers $i < n$, $\mathrm{H}^i_\mathfrak{a}(M,X)$ is $\mathfrak{a}$-cofinite and that $\mathrm{H}^n_\mathfrak{a}(M,X)$ is locally minimax;*

(ii) *$\mathrm{H}^i_\mathfrak{a}(M,X)$ is an $\mathfrak{a}$-cofinite R-module for all integers i, assuming that for all $i < n$, $\mathrm{H}^i_\mathfrak{a}(M,X)$ is $\mathfrak{a}$-cofinite and that $\mathrm{H}^i_\mathfrak{a}(M,X)$ is locally minimax for all $i \geq n$.*

By substituting $M = R$ into Theorem 1 and Corollaries 1 and 2, we obtain the ensuing results concerning ordinary local cohomology modules.

Corollary 3. *Let n be a non-negative integer and let X be an arbitrary R-module such that $\mathrm{Ext}^n_R(R/\mathfrak{a}, X)$ is finite. Assume that $\mathrm{H}^i_\mathfrak{a}(X)$ is $\mathfrak{a}$-cofinite for all $i < n$ and that $\mathrm{H}^n_\mathfrak{a}(X)$ is locally minimax. Then $\mathrm{H}^n_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite minimax R-module.*

Corollary 4. *Let n be a non-negative integer and let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is finite for all $i \geq n$. Assume that $\mathrm{H}^i_\mathfrak{a}(X)$ is $\mathfrak{a}$-cofinite for all $i < n$ and that $\mathrm{H}^i_\mathfrak{a}(X)$ is locally minimax for all $i \geq n$. Then $\mathrm{H}^i_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite R-module for all i.*

Corollary 5. (see [1, Corollary 2.7]) *Let n be a non-negative integer and let X be a finite R-module. Then the following statements hold:*

(i) *$\mathrm{H}^n_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite minimax R-module, provided that for all integers $i < n$, $\mathrm{H}^i_\mathfrak{a}(X)$ is $\mathfrak{a}$-cofinite and that $\mathrm{H}^n_\mathfrak{a}(X)$ is locally minimax;*

(ii) *$\mathrm{H}^i_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite R-module for all integers i, assuming that for all $i < n$, $\mathrm{H}^i_\mathfrak{a}(X)$ is $\mathfrak{a}$-cofinite and that $\mathrm{H}^i_\mathfrak{a}(X)$ is locally minimax for all $i \geq n$.*

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