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On homological classification of monoids by condition $(EP)_E$

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Abstract. Condition (EP), introduced by Golchin and Mohammadzadeh in (*On Condition* (EP), International Mathematical Forum, 2 (2007), no. 19, 911–918), is a generalization of Conditions (E) and (P). They demonstrated that Condition (EP) does not imply torsion freeness and so it does not imply flatness properties that lead to torsion freeness. Furthermore, they characterized monoids based on this condition in relation to other properties of acts. In this paper, we first introduce Condition $(EP)_E$, which generalizes Condition (EP), and then provide a characterization of monoids based on this condition of their (cyclic, Rees factor) acts.

Keywords: Condition $(EP)_E$; Rees factor acts; $(EP)_E$-left annihilating.
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1 Introduction

In [8], Kilp and Knauer presented Conditions (E) and (P). Subsequently, in 2006, Condition (EP) was introduced by Golchin and Mohammadzadeh as an extension of Conditions (E) and (P) [3]. They asserted that the necessary and sufficient condition for all right S-acts satisfy Condition (EP) is that S is either a group or a group with a zero adjoined.
This paper introduces Condition $(EP)_E$ for S-acts, positioning it within the classical hierarchy of flatness properties. It is established as the weakest property in the following canonical, non-reversible chains:

$$\mathrm{SF} \Rightarrow \mathrm{EF} \Rightarrow (E) \Rightarrow (EP) \Rightarrow (EP)_E, \quad \mathrm{SF} \Rightarrow \mathrm{WPF} \Rightarrow (P) \Rightarrow (EP) \Rightarrow (EP)_E,$$

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where EF denotes *equalizer-flatness*. This positioning reveals $(EP)_E$ as a natural common generalization of classical concepts. The utility of this condition is demonstrated through its connections to fundamental results established in Section 4. Theorem 25 shows that every P-regular act satisfies $(EP)_E$. Furthermore, Theorem 26 shows that, for injective and weakly injective right S-acts, Condition $(EP)_E$ is equivalent to a cofreeness condition, which is explicitly formulated in the theorem. Additional results, including Theorems 13 and 16, provide further insights into the implications and applications of Condition $(EP)_E$, contributing to a more comprehensive understanding of its role in the theory of S-acts. Consequently, $(EP)_E$ emerges as a potent tool for classifying monoids and analyzing act-theoretic properties.

Throughout this paper, we refer to a monoid as S, with 1 representing its identity element. A nonempty set A is called a right S-act, usually denoted by A_S (or simply A) if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A,\ (a, s) \mapsto as$, satisfying the conditions

$$(as)t = a(st) \text{ and } a1 = a, \text{ for all } a \in A \text{ and all } s, t \in S.$$

Left S-act can be defined dually. Hereafter, when we mention S-act, we are referring to a right S-act. For further details and definitions related to semigroups and acts over monoids, we recommend referring to ([6, 8]).

For $x, y, s, t \in S$, let $\rho(xs, yt)$ denote the right congruence on S generated by the pair (xs, yt). Let $w \in S$, denote by $\rho(w, 1)$ the right congruence on S generated by the pair $(w, 1)$. Also, for $s \in S$, let $\rho_s : S \to S$ be the right translation defined by $\rho_s(x) = xs$. For $z \in S$, let $\lambda_z : S \to S$ be the left translation defined by $\lambda_z(s) = zs$ for all $s \in S$. We denote by $\ker \lambda_z$ the kernel of this map, i.e., $\ker \lambda_z = \{(x, y) \in S \times S \mid zx = zy\}$. Let K_S be a right ideal of S. The right congruence ρ_K on S is defined by

$$(x, y) \in \rho_K \iff x = y \ \text{ or } \ (x, y \in K_S).$$

In other words, $\rho_K = \Delta_S \cup (K_S \times K_S)$, where $\Delta_S = \{(x, x) \mid x \in S\}$.

Recall from ([3, 4, 7, 8]) that an S-act A satisfies Condition (P) if for all $a, a' \in A$ and all $s, s' \in S$, $as = a's'$ implies that there exist $a'' \in A,\ u, u' \in S$ such that $a = a''u,\ a' = a''u'$ and $us = u's'$. A satisfies Condition (E) if for all $a \in A$ and all $s, s' \in S$, $as = as'$ implies there exist $a' \in A,\ u \in S$ such that $a = a'u$ and $us = us'$. A satisfies Condition (PWP) if for all $a, a' \in A$ and all $s \in S$, $as = a's$ implies there exist $a'' \in A,\ u, v \in S$ such that $a = a''u,\ a' = a''v$ and $us = vs$. A satisfies Condition (E_E) if for all $a \in A$ and all $s, s' \in S$, $as = as'$ implies that there exist $a' \in A,\ u \in S$ and $e \in E(S)$ such that $ae = a'ue,\ es = s,\ es' = s'$, and $us = us'$. A satisfies Condition (EP) if for all $a \in A$ and all $s, s' \in S$, $as = as'$ implies that there exist $a' \in A,\ u, v \in S$ such that $a = a'u = a'v$ and $us = vs'$. A satisfies Condition (P_E) if for all $a, a' \in A$ and all $s, s' \in S$, $as = a's'$ implies there exist $a'' \in A,\ u, v \in S$ and $e, f \in E(S)$ such that $ae = a''ue,\ a'f = a''vf,\ es = s,\ fs' = s'$, and $us = vs'$.

Recall from ([1, 8, 9]) that an act A is called *flat* if the functor $A \otimes_S -$ preserves monomorphisms. An act A is called *weakly flat* WF if the functor $A \otimes_S -$ preserves all embeddings of left ideal into S. An act A is called *principally weakly flat* PWF if the functor $A \otimes_S -$ preserves all embeddings of principal left ideal into S. An act A is *weakly pullback flat* WPF if the corresponding ϕ is bijective for every pullback diagram $P(I, I, f, g, S)$, where I is a left ideal

of S. An act A is *weakly homoflat* WP if the corresponding ϕ is surjective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S. An act A is *weakly kernel flat* WKF if the corresponding ϕ is bijective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S. An act A is *principally weakly kernel flat* $PWKF$ if the corresponding ϕ is bijective for every pullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$. An act A is *translation kernel flat* TKF if the corresponding ϕ is bijective for every pullback diagram $P(S, S, f, f, S)$.

2 General properties

In this section, we present Condition $(EP)_E$ and show that this condition for acts can be transferred to their coproduct and vice versa. Additionally, we show that a retract or coproduct of any act satisfying Condition $(EP)_E$ also satisfies Condition $(EP)_E$. We also show that Condition $(EP)_E$ does not imply Condition $(E)_E$.

Definition 1. *A right S-act A satisfies Condition $(EP)_E$ if for all $a \in A$ and all $s, t \in S, as = at$ implies that there exist $a' \in A$ and $u, v \in S$, $e, f \in E(S)$ such that $ae = a'ue$, $af = a'vf, es = s, ft = t$ and $us = vt$.*

It is clear that Condition (EP) implies Condition $(EP)_E$, but the converse is not true, as illustrated in Example 5. Additionally, it is evident that both Conditions (E_E) and (P_E) separately imply Condition $(EP)_E$. However, we will demonstrate that the converse of these implications is generally false.

Lemma 1. *A right S-act A satisfies Condition $(EP)_E$ if and only if for all $a \in A$ and all $s, t \in S, as = at$ implies that there exist $a' \in A$ and $u, v \in S$, $e, f \in E(S)$ such that $ae = a'u$, $af = a'v, es = s, ft = t$ and $us = vt$.*

Proof. Suppose A satisfies Condition $(EP)_E$ and there exist $a \in A$, $s, t \in S$ such that $as = at$. Hence, there exist $a' \in A$, $u_1, v_1 \in S$, $e, f \in E(S)$ such that $ae = a'u_1e$, $af = a'v_1f$, $es = s$, $ft = t$ and $u_1s = v_1t$. Now, let $u = u_1e$, $v = v_1f$. Hence, we have:

$$us = (u_1e)s = u_1(es) = u_1s;\ vt = (v_1f)t = v_1(ft) = v_1t.$$

On the other hand, suppose for all $a \in A$ and all $s, t \in S, as = at$ implies that there exist $a' \in A$ and $u, v \in S$, $e, f \in E(S)$ such that $ae = a'u$, $af = a'v, es = s, ft = t$ and $us = vt$. Note that $ae = ae^2 = a'ue$, $af = af^2 = a'vf$. Therefore, A satisfies Condition $(EP)_E$. □

In the following, we present example that satisfies Condition $(EP)_E$.

Example 1. Let A, B be non-empty sets and set $S = A \times B$. Define a multiplication on S by

$$(a, b)(c, d) = (a, d);$$

for $a, c \in A$ and $b, d \in B$. This semigroup is called a rectangular band. Every element of S^1 is idempotent. We will show that every $(a, b)S^1$ for all $(a, b) \in S$ satisfies Condition $(EP)_E$.
Note that for all $(x_1, x_2) \in S$, we have $(a, x_2) = (a, b)(x_1, x_2) \in (a, b)S$. Thus, we denote the

elements of $(a,b)S$ as $(a,z),\ z\in B$. For all $s=(s_1,s_2),t=(t_1,t_2)\in S$ such that $(a,x_2)(s_1,s_2)=(a,x_2)(t_1,t_2)$, it follows that $(a,s_2)=(a,t_2)$ and consequently, $s_2=t_2$. Now, let

$$a'=(a,x_2),u=v=(x_1,x_2),e=s=(s_1,s_2),f=t=(t_1,t_2)\in S=E(S).$$

Then:

$$\begin{aligned} ae &= a'ue;\\ af &= a'vf;\\ es &= s;\\ ft &= t;\\ us &= vt. \end{aligned}$$

If $(a,x_2).1=(a,x_2)(t_1,t_2)$, therefore, $(a,x_2)=(a,t_2)$ and consequently, $t_2=x_2$. Now, let

$$a'=(a,x_2),u=v=(x_1,x_2),e=1,f=t=(t_1,t_2)\in S=E(S).$$

Thus $(a,b)S^1$ satisfies Condition $(EP)_E$ for all $(a,b)\in S$.

Let G be a group, A,B be non-empty sets, and $P:B\times A\to G^0$ a mapping. Recall from [6],

$$\mathcal{M}^0(G,A,B,P)=\{(a,g,b)|a\in A,g\in G,b\in B\}\cup\{0\}$$

where 0 is an element different from all triples (a,g,b). Define a multiplication on $\mathcal{M}^0(G,A,B,P)$ so that

$$0(a,g,b)=(a,g,b)0=00=0;$$

and for any triples $(a,g,b),(c,h,d)$ we have

$$(a,g,b)(c,h,d)=\begin{cases}(a,gP(b,c)h,d); & P(b,c)\neq 0\\ 0; & otherwise.\end{cases}$$

$\mathcal{M}^0(G,A,B,P)$ is called a *Rees matrix semigroup with zero* with sandwich matrix P.

Theorem 1. *Let G be a group, A,B be non-empty sets, and $P:B\times A\to G$ a mapping. Let $S=\mathcal{M}(G,A,B,P)$ and $a\in S$, then aS^1 satisfies Condition $(EP)_E$ as a S^1-right act.*

Proof. First note that if $a=(i,\alpha,\lambda)\in S$, then

$$aS^1=\{(i,\beta,\gamma)|\forall\beta\in G,\forall\gamma\in B\}.$$

Now, suppose $s=(i_s,\alpha_s,\lambda_s),\ t=(i_t,\alpha_t,\lambda_t)$. Then:

$$(i,\beta,\gamma)s=(i,\beta,\gamma)(i_s,\alpha_s,\lambda_s)=(i,\beta P(\gamma,i_s)\alpha_s,\lambda_s);$$

and $(i,\beta,\gamma)t = (i,\beta,\gamma)(i_t,\alpha_t,\lambda_t) = (i,\beta P(\gamma,i_t)\alpha_t,\lambda_t)$;
Therefore $(i,\beta,\gamma)s = (i,\beta,\gamma)t$ if and only if

$$P(\gamma,i_s)\alpha_s = P(\gamma,i_t)\alpha_t \text{ and } \lambda_s = \lambda_t.$$

On the other hand, $e = (j,b,\mu) \in E(S)$ if and only if $b = (P(\mu,j))^{-1}$. So let

$$e = (i_s,(P(\gamma,i_s))^{-1},\gamma),\ f = (i_t,(P(\gamma,i_t))^{-1},\gamma) \in E(S).$$

Then we have $es = s,\ ft = t$ and $a' = (i,(P(\gamma,i))^{-1},\gamma)$

$$\begin{aligned}(i,\beta,\gamma)e &= (i,\beta P(\gamma,i_s)(P(\gamma,i_s))^{-1},\gamma)\\ &= (i,\beta,\gamma)\\ &:= u;\\ (i,\beta,\gamma)f &= (i,\beta P(\gamma,i_t)(P(\gamma,i_t))^{-1},\gamma)\\ &= (i,\beta,\gamma)\\ &:= v.\end{aligned}$$

and obviously $us = vt$. Thus, aS^1 satisfies Condition $(EP)_E$. □

Recall from [8], a monoid S is called *left collapsible* if for every $s,t \in S$, there exist $u \in S$ such that $us = ut$ and S is called *right reversible* if for every $s,s' \in S$ there exist $u,v \in S$ such that $us = vs'$. In the following theorem, all statements are straightforward consequences of the definition.

Theorem 2. *The following statements are true:*

(1) S_S satisfies $(EP)_E$.

(2) Θ_S satisfies Condition $(EP)_E$ if and only if S is right reversible.

(3) Let $\{B_i \mid i \in I\}$ be a collection of subacts of A and every B_i, $i \in I$, satisfies Condition $(EP)_E$, then $\bigcup_{i\in I} B_i$ satisfies Condition $(EP)_E$ as a subact of A.

(4) Let $A = \coprod_{i\in I} A_i$, where each A_i is an S-act, then A satisfies Condition $(EP)_E$ if and only if each A_i satisfies Condition $(EP)_E$.

(5) If A satisfies Condition $(EP)_E$, then every retract of A satisfies Condition $(EP)_E$.

Corollary 1. *Let A be an S-act. For every $a \in A_S$, if aS satisfies Condition $(EP)_E$ then A satisfies Condition $(EP)_E$.*

Proof. Let $as = at$ for $a \in A$ and $s,t \in S$. Since aS satisfies Condition $(EP)_E$, there exist $z,u_1,v_1 \in S$ and $e,f \in E(S)$ such that $ae = (az)u_1,\ af = (az)v_1,\ es = s,\ ft = t$ and $u_1 s = v_1 t$. Thus, let $u = zu_1,\ v = zv_1$ and hence, A satisfies Condition $(EP)_E$, as required. □

Now, in the following example, we demonstrate that Condition $(EP)_E$ does not imply Condition $(E)_E$.

Example 2. Let S be a nontrivial group. Then, S is right reversible. Thus, Θ_S satisfies Condition $(EP)_E$ by Theorem 2. However, it does not satisfy Condition (E_E) because S is not left collapsible, as stated in [7, Theorem 2.3].

3 Cyclic and rees factor acts satisfying condition $(EP)_E$

In this section, we give a necessary and sufficient condition for cyclic and Rees factor acts of S that satisfy Condition $(EP)_E$. Here we give a criterion for a cyclic right S-act to satisfy Condition $(EP)_E$. We also show that Condition $(EP)_E$ does not imply Condition (EP).

Theorem 3. *Let ρ be a right congruence on S. Then the right S-act S/ρ satisfies Condition $(EP)_E$ if and only if for all $x, s, t \in S$ with $(xs)\rho(xt)$, there exist $u, v \in S,\ e, f \in E(S)$ such that $(xe)\rho(u),\ (xf)\rho(v),\ es = s,\ ft = t$ and $us = vt$.*

Proof. Necessity. Let $(xs)\rho(xt)$ for $x, s, t \in S$, then $[x]_\rho s = [x]_\rho t$. Therefore there exist $u', v', z \in S,\ e, f \in E(S)$, such that $[x]_\rho e = [z]_\rho u' = [zu']_\rho$, $[x]_\rho f = [z]_\rho v' = [zv']_\rho,\ es = s,\ ft = t$ and $u's = v't$. By letting $u = zu'$ and $v = zv'$, we get $(xe)\rho(u),\ (xf)\rho(v)$ and $us = vt$, as required. Sufficiency. Suppose that $[x]_\rho s = [x]_\rho t$ for $x, s, t \in S$, then $(xs)\rho(xt)$. By the assumption, there exist $u, v \in S,\ e, f \in E(S)$, such that $(xe)\rho(u),\ (xf)\rho(v),\ es = s,\ ft = t$ and $us = vt$. Then we have $[x]_\rho e = [1]_\rho u$ and $[x]_\rho f = [1]_\rho v$, Therefore, S/ρ satisfies Condition $(EP)_E$. □

Corollary 2. *For $a \in S$, the principal right ideal aS as a right S-act satisfies Condition $(EP)_E$ if and only if $axs = axt$, for $x, s, t \in S$, implies that the existence of $u, v \in S,\ e, f \in E(S)$ such that $axe = au,\ axf = av,\ es = s,\ ft = t$ and $us = vt$.*

Proof. Since $aS \cong S/\ker \lambda_a$, by Theorem 3 it suffices to take $\rho = \ker \lambda_a$. □

Example 3. Let

$$S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \mid a, b \in \mathbb{Z} \right\},$$

with matrices multiplication. We will show that αS satisfies Condition $(EP)_E$ for all $\alpha \in S$. For this purpose, let $\alpha = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ be an arbitrary element of S. Then:

$$\alpha S = \left\{ \begin{bmatrix} an & 0 \\ m & 1 \end{bmatrix} \mid n, m \in \mathbb{Z} \right\},$$

because

$$\begin{bmatrix} an & 0 \\ m & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} n & 0 \\ m - bn & 1 \end{bmatrix} \in \alpha S.$$

Let $(\alpha x)s = (\alpha x)t$, where

$$\alpha x = \begin{bmatrix} an & 0 \\ m & 1 \end{bmatrix},\ s = \begin{bmatrix} s_1 & 0 \\ s_2 & 1 \end{bmatrix},\ t = \begin{bmatrix} t_1 & 0 \\ t_2 & 1 \end{bmatrix}.$$

The equality $(\alpha x)s = (\alpha x)t$ implies that

$$\begin{bmatrix} ans_1 & 0 \\ ms_1 + s_2 & 1 \end{bmatrix} = \begin{bmatrix} ant_1 & 0 \\ mt_1 + t_2 & 1 \end{bmatrix}.$$

If $an \neq 0$, then $s_1 = t_1$ and therefore $s = t$. So, let $e = f = I_{2\times 2},\ u, v = \begin{bmatrix} n & 0 \\ m - bn & 1 \end{bmatrix}$. Thus, $\alpha xe = \alpha u,\ \alpha xf = \alpha v,\ es = s,\ ft = t$ and $us = vt$.
If $an = 0$, $(\alpha x)s = (\alpha x)t$ implies that

$$\begin{bmatrix} 0 & 0 \\ ms_1 + s_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ mt_1 + t_2 & 1 \end{bmatrix}.$$

Therefore, $ms_1 + s_2 = mt_1 + t_2$.

So, let $e = f = I_{2\times 2},\ u, v = \begin{bmatrix} 0 & 0 \\ m & 1 \end{bmatrix}$. Thus, $\alpha xe = \alpha u,\ \alpha xf = \alpha v,\ es = s,\ ft = t$ and

$$us = \begin{bmatrix} 0 & 0 \\ m & 1 \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ s_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ms_1 + s_2 & 1 \end{bmatrix},$$

$$vt = \begin{bmatrix} 0 & 0 \\ m & 1 \end{bmatrix} \begin{bmatrix} t_1 & 0 \\ t_2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ mt_1 + t_2 & 1 \end{bmatrix}.$$

Therefore, αS satisfies Condition $(EP)_E$, by Corollary 2.

Theorem 4. *For any monoid S the following statements are equivalent:*

(1) All right S-acts satisfy Condition $(EP)_E$.

(2) All cyclic right S-acts satisfy Condition $(EP)_E$.

(3) All monocyclic right S-acts satisfy Condition $(EP)_E$.

(4) $(\forall x, s, t \in S)(\exists u, v \in S,\ \exists e, f \in E(S),\ us = vt,\ es = s,\ ft = t,$
$(xe)\rho(xs, xt)(u),\ (xf)\rho(xs, xt)(v))$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). Since, by assumption, for any $x, s, t \in S,\ S/\rho(xs, xt)$ satisfies Condition $(EP)_E$ and $(xs)\rho(xs, xt)(xt)$, then by Theorem 3, there exist $u, v \in S$ and $e, f \in E(S)$ such that $us = vt$, $(xe)\rho(xs, xt)(u),\ (xf)\rho(xs, xt)(v),\ es = s,\ ft = t$.
(4) $\Rightarrow$ (1). First, we demonstrate that every cyclic right S-act satisfies Condition $(EP)_E$. To do this, we assume ρ is a right congruence on S, $(xs)\rho(xt)$ for $x, s, t \in S$. By assumption there exist $u, v \in S,\ e, f \in E(S)$ such that $es = s,\ ft = t,\ us = vt$ and $(xe)\rho(xs, xt)(u),\ (xf)\rho(xs, xt)(v)$. Since $(xs)\rho(xt)$, we have $\rho(xs, xt) \subseteq \rho$ and so $(xe)\rho(u),\ (xf)\rho(v)$. Therefore, by Theorem 3, S/ρ satisfies Condition $(EP)_E$ as required. Now, we consider A as a right S-act, where $as = at$ for $a \in A$ and $s, t \in S$. Since $aS \cong S/\ker \lambda_a$ satisfies Condition $(EP)_E$, it follows that A satisfies Condition $(EP)_E$ as required, by Corollary 1. □

Theorem 5. *Let $w \in S$ and $\rho = \rho(w, 1)$. The cyclic S-act S/ρ satisfies Condition $(EP)_E$ if and only if for every $x, s, t \in S, m, n \in \mathbb{N}_0$*

$$(w^m xs = w^n xt) \Rightarrow (\exists p_1, q_1, p_2, q_2 \in \mathbb{N}_0)(\exists u, v \in S,\ e, f \in E(S))$$
$$(w^{p_1} xe = w^{q_1} u \wedge w^{p_2} xf = w^{q_2} v \wedge us = vt,\ es = s,\ ft = t).$$

Proof. Necessity. Since $w^m xs = w^n xt$ by [8, III, Corollary 8.7], we have $(xs)\rho(xt)$, for $x, s, t \in S$. By Theorem 3, there exist $u, v \in S,\ e, f \in E(S)$ such that $us = vt,\ es = s,\ ft = t$ and $(xe)\rho(u),\ (xf)\rho(v)$. By [8, III, Corollary 8.7], we have $w^{p_1} xe = w^{q_1} u,\ w^{p_2} xf = w^{q_2} v$ for some $p_1, q_1, p_2, q_2 \in \mathbb{N}_0$.
Sufficiency. Let $(xs)\rho(xt)$, for $x, s, t \in S$. By [8, III, Corollary 8.7], $w^m xs = w^n xt$, for $m, n \in \mathbb{N}_0$. By assumption, there exist $u, v \in S,\ e, f \in E(S)$ and $p_1, q_1, p_2, q_2 \in \mathbb{N}_0$ such that $us = vt,\ es = s,\ ft = t$ and $w^{p_1} xe = w^{q_1} u,\ w^{p_2} xf = w^{q_2} v$. Hence $(xe)\rho(u),\ (xf)\rho(v)$, by [8, III, Corollary 8.7], and so S/ρ satisfies Condition $(EP)_E$, by Theorem 3. □

Theorem 6. *Let S be a monoid such that every element of $E(S) \setminus \{1\}$ is a right zero and let ρ be a right congruence on S. Then S/ρ satisfies Condition (EP) if and only if it satisfies $(EP)_E$.*

Proof. It is clear that Condition (EP) implies Condition $(EP)_E$. Now, suppose that S/ρ satisfies Condition $(EP)_E$ and let $(xs)\rho(xt)$, for $x, s, t \in S$. By assumption, there exist $u, v \in S,\ e, f \in E(S)$ such that $(xe)\rho(u),\ (xf)\rho(v),\ es = s,\ ft = t$ and $us = vt$.
If $e = f = 1$, we have nothing to prove. If $e = 1, f \neq 1$, then f is a right zero, so for all $z \in S,\ zf = f$. Thus, $zt = zft = ft = t$. Therefore t is a right zero. Note that $x\rho u,\ x\rho x$ and $us = vt = xt$.
If $e \neq 1, f = 1$, a similar argument as the previous argument shows the result.
If $e, f \neq 1$, then both e, f are right zeros, and so are s, t. Note that $s = us = vt = t$. Then $x\rho x,\ x\rho x$ and $xs = xt$, thus S/ρ satisfies Condition (EP). □

Example 4. Let

$$S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} | a, b \in \mathbb{Z} \right\},$$

with matrices multiplication. Then $e = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \in E(S)$ if and only if $e^2 = e$ and therefore:

$$\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ b(a+1) & 1 \end{bmatrix} \Longrightarrow \begin{cases} a^2 = a \\ b(a+1) = b \end{cases}$$

Thus if $a = 1$, then $e = I_{2\times 2} = 1_S$. Otherwise, if $a = 0$, then:

$$e = \begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix};\ b \in \mathbb{Z}.$$

Note that for all $s \in S$, we have

$$se = \begin{bmatrix} c & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix} = e.$$

So e is a right zero of S. Therefore, the conditions of Theorem 6 are satisfied.

Proposition 1. *Let S be a monoid such that every idempotent element of S is left identity in $S \setminus \{1\}$. Then, for every cyclic right S-act, Conditions $(EP)_E$ and (P_E) coincide.*

Proof. Let ρ be a right congruence on S and let $[x]_\rho s = [y]_\rho t$, for $x, y, s, t \in S$. Then $[1]_\rho xs = [1]_\rho yt$. Since S/ρ satisfies Condition $(EP)_E$, there exist $u, v, z \in S, e, f \in E(S)$ such that $[1]_\rho e = [z]_\rho u$, $[1]_\rho f = [z]_\rho v$, $exs = xs$, $fyt = yt$ and $uxs = vyt$. Now, $[x]_\rho e = [1]_\rho xe$. Since $e \in E(S)$, so by assumption, $[x]_\rho e = [1]_\rho xe = [1]_\rho exe = [z]_\rho uxe$. Similarly we have:

$$[y]_\rho f = [1]_\rho yf = [1]_\rho fyf = [z]_\rho vyf$$

and $es = s$, $ft = t$, $(ux)s = (vy)t$. Hence, the right S-act S/ρ satisfies Condition (P_E), by [4, Theorem 2.5]. It is clear that the converse is always true, i.e., the Condition (P_E) implies Condition $(EP)_E$ for every right S-act. Therefore, the result follows. □

Remark 1. *In a right zero semigroup S, every element is right zero and left identity. Hence, one could add an element 1 to S, resulting in S^1, which is a monoid such that every idempotent element $e \in E(S)\backslash\{1\}$ is a left identity in $S \setminus \{1\}$.*

Definition 2. *A right ideal K_S of S is called $(EP)_E$-left annihilating if for all $x, s, t \in S$ with $xs \neq xt$, at least one of the following conditions holds:*

(1) $x \in K_S$;

(2) $xs \notin K_S$ or $xt \notin K_S$;

(3) there exist $u, v \in S$ and $e, f \in E(S)$ such that $es = s$, $ft = t$, $us = vt$, and the following implications hold:

- *if $xe \neq u$, then $xe, u \in K_S$;*
- *if $xf \neq v$, then $xf, v \in K_S$.*

We recall from [8] that a right ideal K_S of S is called *left stabilizing* or satisfies Condition (LU) if for every $k \in K_S$, there exists $l \in K_S$ such that $lk = k$.

Lemma 2. *Let K_S be a right ideal of S. If K_S is $(EP)_E$-left annihilating, then K_S is left stabilizing.*

Proof. Suppose that K_S is $(EP)_E$-left annihilating. If $K_S = S$, then it is obvious that K_S is left stabilizing. Therefore, we assume $K_S \neq S$, which implies $1 \notin K_S$. Let $k \in K_S$. If there exists an element $l \in K_S$, such that $lk = k$, then we are done. Otherwise, for every $l \in K_S$, we have $lk \neq k$. Consequently, $1 \cdot lk \neq 1 \cdot k$ and $1 \notin K_S$, $1 \cdot k, 1 \cdot lk \in K_S$. Since K_S is $(EP)_E$-left annihilating, there exist $u, v \in S$, $e, f \in E(S)$, $elk = lk$, $fk = k$, $ulk = vk$. If $f \neq v$, then $f, v \in K_S$. Therefore, based on the equality $fk = k$, we reach a contradiction, and so, $f = v$. If $e \neq u$, then $e, u \in K_S$, which implies $ulk = vk = fk = k$. Since $ul \in K_S$, we reach a contradiction based on our assumption. Thus, we conclude that $e = u$ and $f = v$. Therefore, we have

$$lk = elk = ulk = vk = fk = k,$$

which is a contradiction. Hence, K_S is left stabilizing. □

Here we give a criterion for a Rees factor acts of S that satisfy Condition $(EP)_E$.

Theorem 7. *Let K_S be a nonempty right ideal of S. Then S/K_S satisfies Condition $(EP)_E$ if and only if S is right reversible and K_S is $(EP)_E$-left annihilating right ideal.*

Proof. Necessity. Suppose that S/K_S satisfies condition $(EP)_E$. If $K_S = S$, then $S/K_S \cong \Theta_S$ and so S is right reversible, by Theorem 2. Moreover, since $K_S = S$ is $(EP)_E$-left annihilating, the result follows.
Now, suppose $K_S \neq S$. Note that if $x \in K_S$, for all $s, t \in S$, we have $(xs)\rho_K(xt)$. Since S/K_S satisfies condition $(EP)_E$, there exists $a' \in S/K_S$ and exist $u, v \in S,\ e, f \in E(S)$ such that

$$xe = a'u,\ xf = a'v,\ es = s,\ ft = t,\ \text{and}\ us = vt.$$

Therefore, S is right reversible and K_S is $(EP)_E$-left annihilating right ideal, by Definition 2.

Now let $xs \neq xt$ for $x \in S \setminus K$ and $s, t \in S$. If $xs, xt \in K_S$, then $(xs)\rho_K(xt)$; hence, by Theorem 3, there exist $u, v \in S,\ e, f \in E(S)$ such that $(xe)\rho_K(u),\ (xf)\rho_K(v),\ es = s,\ ft = t$ and $us = vt$. If $xe \neq u$, then $xe, u \in K_S$. Similarly, if $xf \neq v$, then $xf, v \in K_S$ and thus K_S is $(EP)_E$- left annihilating.

Sufficiency. Let $K = S$ and S is right reversible. Then $S/K_S \cong \Theta_S$ satisfies Condition $(EP)_E$ by Theorem 2. Now, suppose that K_S be an arbitrary $(EP)_E$-left annihilating proper right ideal of S and $(xs)\rho_K(xt)$ for $x, s, t \in S$. Then, three possibilities may arise as follows:

1. If $xs = xt$, let $u = v = x$ and $e = f = 1$;

2. If $xs \neq xt$ and $x \in K_S$. Since S is right reversible, then there exist $u, v \in S$ such that $us = vt$. On the other hand, since $xs, xt \in K_S$, we have $(xs)\rho_K(xt)$. Therefore, let $e = f = 1$ and $a' = x$. Therefore:

$$xe = xf = x \in K,\ xu, xv \in K,\ es = s,\ ft = t,\ \text{and}\ us = vt.$$

3. If $xs \neq xt$ and $x \notin K_S$. Since $(xs)\rho_K(xt)$, thus $xs, xt \in K_S$. Since K_S is $(EP)_E$-left annihilating, there exist $u, v \in S, e, f \in E(S)$ such that $es = s, ft = t, us = vt$. Note that if $xe = u,\ xf = v$, then $xs = xt$, which is a contradiction. Therefore, $xs \neq xt$ implies that $xe \neq u$ or $xf \neq v$. Thus, we have three cases:

   i. If $xe \neq u$ and $xf \neq v$, then $xe, u \in K_S$ and $xf, v \in K_S$. Thus $(xe)\rho_K(u)$ and $(xf)\rho_K(v),\ es = s, ft = t,\ us = vt$;

   ii. If $xe \neq u$ and $xf = v$, then $xe, u \in K_S$ and therefore, $(xe)\rho_K(u)$ and $(xf)\rho_K(v)$;

   iii. If $xe = u$ and $xf \neq v$, then $xf, v \in K_S$ and therefore, $(xe)\rho_K(u)$ and $(xf)\rho_K(v)$.

Therefore, S/K_S satisfies Condition $(EP)_E$, as required. □

The following example shows that Condition $(EP)_E$ does not imply Condition (EP).

Example 5. Let $S = \{1, a, 0\}$ where $a^2 = a$, and $K_S = \{a, 0\}$. We claim that the right Rees factor act S/K_S satisfies Condition $(EP)_E$. To this end, it suffices to show that K_S is an $(EP)_E$-left annihilating proper right ideal of S. Let $x, s, t \in S$ with $xs \neq xt$. If $x \in K_S$, then the condition holds. Hence, assume that $x \notin K_S$, so that $x = 1$. Then $xs = s$ and $xt = t$, and $xs \neq xt$ yields $s \neq t$. If $s = 1$ or $t = 1$, then $xs \notin K_S$ or $xt \notin K_S$, and again the condition holds.

The only remaining case is $\{s,t\} = \{a,0\}$. Without loss of generality, take $x = 1$, $s = a$, and $t = 0$. Choose

$$e = f = a \in E(S), \qquad u = v = 0 \in S.$$

Then

$$es = a \cdot a = a = s, \qquad ft = a \cdot 0 = 0 = t, \qquad us = 0 \cdot a = 0 = 0 \cdot 0 = vt.$$

Moreover,

$$xe = 1 \cdot a = a \neq 0 = u, \qquad xf = 1 \cdot a = a \neq 0 = v,$$

and clearly $xe, xf, u, v \in K_S$. Hence K_S is $(EP)_E$-left annihilating. Therefore, by Theorem 7, the right Rees factor act S/K_S satisfies Condition $(EP)_E$. However, by [3], Conditions (P) and (EP) coincide for cyclic acts. Consequently, by [8, III, Proposition 13.9], S/K_S does not satisfy Condition (EP), since $|K_S| = 2$.

Theorem 8. *Every right Rees factor act of S satisfying Condition $(EP)_E$ satisfies Condition (EP) if and only if there are no $(EP)_E$-left annihilating proper right ideals K_S of S with $|K_S| > 1$.*

Proof. If every right Rees factor S-act satisfying Condition $(EP)_E$ also satisfies Condition (EP), and let K_S be an $(EP)_E$-left annihilating proper right ideal of S. Then by Theorem 7, S/K_S satisfies Condition $(EP)_E$. Thus, by assumption, S/K_S satisfies Condition (EP). Since Condition (P) and (EP) coincide over cyclic acts by [3], and therefore, $|K_S| = 1$ by [8, III, Proposition 13.9].
Conversely, suppose there exist no $(EP)_E$-left annihilating proper right ideal K_S of S with $|K_S| > 1$ and let S/K_S satisfies Condition $(EP)_E$. If $K_S = S$, then by Theorem 7, S is right reversible. Thus $S/K_S = S/S_S \cong \Theta_S$ satisfies Condition (P). Thus, $S/K_S \cong \Theta_S$ satisfies Condition (EP). If $K_S \neq S$, then by Theorem 7, K_S is $(EP)_E$-left annihilating proper right ideal. Therefore, $|K_S| = 1$ and so $S/K_S \cong S_S$, satisfies Condition (EP). □

4 Classification of monoids by condition $(EP)_E$

In this section, we present some results on homological classifications. Firstly, we determine which monoids satisfy Condition $(EP)_E$ for their finitely generated, cyclic, and monocyclic acts. Secondly, a classification of monoids for which acts with this property have some other flatness properties will be provided.
Recall from [8], If B_S is a proper subact of A_S and $x \neq y$ are the elements not belonging to A_S, then

$$A_S \overset{B_S}{\amalg} A_S = \{(a,x) \mid a \in A \setminus B\} \dot\cup B_S \dot\cup \{(a,y) \mid a \in A \setminus B\}$$

with

$$(a,z)s = \begin{cases} (as,z); & as \notin B_S \\ as; & otherwise. \end{cases}$$

for every $a \in A \setminus B, s \in S$ and $z \in \{x,y\}$. Note that

$$\{(a,x) \mid a \in A \setminus B\} \dot\cup B_S \cong A_S \cong \{(b,y) \mid b \in A \setminus B\} \dot\cup B_S.$$

Theorem 9. *Let A_S be an act satisfying Condition $(EP)_E$. If B_S is a proper subact of A_S, then the amalgam $A_S \overset{B_S}{\amalg} A_S$ satisfies Condition $(EP)_E$.*

Proof. Based on the aforementioned discussions, we have, $A_S \overset{B_S}{\amalg} A_S = C_S \cup D_S$, where

$$C_S = \{(c, x) \mid c \in C \setminus B_S \} \dot\cup B_S \cong A_S$$

and

$$D_S = \{(d, y) \mid d \in D \setminus B_S\} \dot\cup B_S \cong A_S.$$

By assumption and part (3) of Theorem 2, the result follows. □

Corollary 3. *Let K_S be a proper right ideal of S. Then $S \overset{K_S}{\amalg} S$ satisfies Condition $(EP)_E$.*

The right S-act $(S \times S)_S$, equipped with the right S-action $(s, t)u = (su, tu)$ for $s, t, u \in S$, is called the *diagonal act of S* and is denoted by $D(S)$.

Let S be a monoid and $s, t \in S$. Define

$$L(s, t) = \{(u, v) \in D(S) \mid us = vt\}.$$

It is a routine matter to show that $L(s, t) = \emptyset$ or $L(s, t)$ is a left S-act.

Theorem 10. *Let S be a monoid, the following statements are equivalent:*
(1) *$S_S^I = \prod_I S_S$ satisfy Condition $(EP)_E$, for $I \neq \emptyset$.*
(2) *For all $s, t \in S$, $L(s, t) \cap \Delta_S = \emptyset$ or there exist $e, f \in E(S)$ such that:*

$$(\exists u, v \in S)\ (L(s, t) \cap \Delta_S \subseteq L(e, u) \circ L(v, f) \wedge es = s \wedge ft = t \wedge us = vt).$$

Proof. (1) ⇒ (2). Suppose $s, t \in S$ and $L(s, t) \cap \Delta_S \neq \emptyset$. We put $L(s, t) \cap \Delta_S = \{(x_i, x_i) | i \in I\}$. Because $L(s, t) \cap \Delta_S \neq \emptyset$, thus $I \neq \emptyset$. Hence, for all $i \in I,\ x_i s = x_i t$, then $(x_i)_I s = (x_i)_I t$. By reason of assumption, S_S^I satisfies Condition $(EP)_E$. It follows from the above equation, there exist $(y_i)_I \in S_S^I,\ e, f \in E(S),\ u, v \in S$ such that $(x_i)_I e = (y_i)_I u,\ (x_i)_I f = (y_i)_I v$, and $es = s,\ ft = t,\ us = vt$. Then, $(x_i)_I e = (y_i)_I u$ implies that for all $i \in I,\ x_i e = y_i u$. Therefore, for all $i \in I,\ (x_i, y_i) \in L(e, u)$ and similarly the equality $(x_i)_I f = (y_i)_I v$ concludes that for all $i \in I,\ x_i f = y_i v$ and consequently for all $i \in I,\ (y_i, x_i) \in L(v, f)$. Thus, for all $i \in I,\ (x_i, x_i) \in L(e, u) \circ L(v, f)$. Hence, $es = s,\ ft = t,\ us = vt$ and

$$L(s, t) \cap \Delta_S \subseteq L(e, u) \circ L(v, f).$$

(2) ⇒ (1). We show that for $I \neq \emptyset,\ S_S^I$ satisfies Condition $(EP)_E$. For this aim, suppose for $(x_i)_I \in S_S^I,\ s, t \in S$, we have $(x_i)_I s = (x_i)_I t$. So, $\forall i \in I, x_i s = x_i t$, then for all $i \in I, (x_i, x_i) \in L(s, t) \cap \Delta_S$. Therefore, $L(s, t) \cap \Delta_S \neq \emptyset$. By assumption, there exist $e, f \in E(S),\ u, v \in S$ such that,

$$L(s, t) \cap \Delta_S \subseteq L(e, u) \circ L(v, f),\ es = s,\ ft = t,\ us = vt.$$

We have for all $i \in I,\ (x_i, x_i) \in L(e,u) \circ L(v,f)$. Thus, for all $i \in I$, there exist $y_i \in S,\ (x_i, y_i) \in L(e,u),\ (y_i, x_i) \in L(v,f)$. Therefore, we have for all $i \in I,\ x_i e = y_i u$ and $x_i f = y_i v$. Thus, $(x_i)_I e = (y_i)_I u,\ (x_i)_I f = (y_i)_I v$ and $es = s,\ ft = t,\ us = vt$. Hence, S_S^I satisfies Condition $(EP)_E$. □

We recall from [8] that an S-act A is called *torsion free* if for any $a, b \in A$ and for any right cancellable element $u \in S$, the equality $au = bu$ implies $a = b$.

Example 6. Let $S = (\mathbb{N}, .)$ be the monoid of natural numbers with multiplication, and let $A_{\mathbb{N}} = \mathbb{N} \overset{\mathbb{N}\setminus\{1\}}{\amalg} \mathbb{N}$. Then, by Theorem 9, $A_{\mathbb{N}}$ satisfies Condition $(EP)_E$. However, $A_{\mathbb{N}}$ is not torsion free because $(1,x)2 = 2 = (1,y)2$, but $(1,x) \neq (1,y)$.

As shown in Example 6, Condition $(EP)_E$ does not imply torsion freeness. Therefore, Condition $(EP)_E$ does not imply the flatness properties that lead to torsion freeness. Thus, it is natural to inquire about those monoids over which all right S-acts satisfying Condition $(EP)_E$ have other flatness properties. Now, we begin our investigation with the weakest of flatness property.

Theorem 11. *The following statements are equivalent:*

(1) All right S-acts are torsion free.

(2) All right S-acts satisfying Condition $(EP)_E$ are torsion free.

(3) All finitely generated right S-acts satisfying Condition $(EP)_E$ are torsion free.

(4) All right S-acts generated by at most two elements satisfying Condition $(EP)_E$ are torsion free.

(5) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are torsion free.

(6) All indecomposable right S-acts satisfying Condition $(EP)_E$ are torsion free.

(7) All finitely generated indecomposable right S-acts satisfying Condition $(EP)_E$ are torsion free.

(8) All indecomposable right S-acts generated by at most two elements satisfying Condition $(EP)_E$ are torsion free.

(9) All indecomposable right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are torsion free.

(10) Every right cancellable element of S is right invertible.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5), (1) ⇒ (6) ⇒ (7) ⇒ (8) ⇒ (9) and (5) ⇒ (9), are obvious.

(9) ⇒ (10). Let $c \in S$ be a right cancellable element of S such that $cS \neq S$ and let $A = S \overset{cS}{\amalg} S$. Obviously, A is an indecomposable right S-act generated by two elements $(1,x), (1,y)$

and satisfying Condition $(EP)_E$, by Corollary 3. Thus, A is torsion free, and the equality $(1,x)c = c = (1,y)c$ implies $(1,x) = (1,y)$, which leads to a contradiction. Therefore, $cS = S$ and so c is right invertible, as required.

(10) $\Rightarrow$ (1). It is obvious, by [8, IV, Theorem 6.1]. □

Theorem 12. *The following statements are equivalent:*

(1) All right S-acts are WPF.

(2) All right S-acts satisfying Condition $(EP)_E$ are WPF.

(3) All right S-acts satisfying Condition $(EP)_E$ satisfy Condition (P).

(4) All right S-acts satisfying Condition $(EP)_E$ are WKF.

(5) All right S-acts satisfying Condition $(EP)_E$ are $PWKF$.

(6) All right S-acts satisfying Condition $(EP)_E$ are TKF.

(7) All right S-acts satisfying Condition $(EP)_E$ satisfy Condition (WP).

(8) All right S-acts satisfying Condition $(EP)_E$ satisfy Condition (PWP).

(9) S is a group.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (7) $\Rightarrow$ (8) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (8) follow directly from the definitions of the respective conditions.
(9) $\Rightarrow$ (1). It is a direct consequence of [1, Proposition 2.9].
(8) $\Rightarrow$ (9). Suppose that S is not a group. Then there exists an element $s \in S$ such that $sS \neq S$. Consider $A = S \overset{sS}{\amalg} S$. By Corollary 3, this act A satisfies Condition $(EP)_E$. Therefore, by our assumption (8), A must satisfy Condition (PWP).

The act A is generated by the two elements $(1,x)$ and $(1,y)$. In this act, we have the equality

$$(1,x)s = (1,y)s.$$

Since A satisfies Condition (PWP), there must exist an element $a \in A$ and elements $u, v \in S$ such that:

$$(1,x) = au, \quad (1,y) = av \quad \text{and} \quad us = vs.$$

From the equation $(1,x) = au$, and by the structure of the $A = S \overset{sS}{\amalg} S$, we deduce that the element a must be of the form $a = (t,x)$ for some $t \in S \setminus sS$. Similarly, from the equation $(1,y) = av$, we deduce that the same element a must also be of the form $a = (t',y)$ for some $t' \in S \setminus sS$. However, this implies that the components of a are equal, meaning $x = y$. This is a contradiction, as x and y represent distinct generators in the pushout construction. Therefore, we must have $sS = S$ for every $s \in S$, which means every element $s \in S$ has a right inverse. Consequently, S is a group. □

The validity of the above theorem extends to finitely generated S-acts and S-acts generated by at most (exactly) two elements.

Theorem 13. *The following statements are equivalent:*

(1) All right S-acts are PWF.

(2) All right S-acts satisfying Condition $(EP)_E$ are PWF.

(3) All finitely generated right S-acts satisfying Condition $(EP)_E$ are PWF.

(4) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are PWF.

(5) S is regular.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5). Let $s \in S$. If $sS = S$, then s is obviously regular. Thus, we suppose that $sS \neq S$. Then the amalgam $S \overset{sS}{\amalg} S$ satisfies Condition (EP_E) by Corollary 3 and so by assumption, $S \overset{sS}{\amalg} S$ is PWF. Therefore, there exists $j \in sS$ such that $js = s$ by [8, III, Proposition 12.19], and so s is regular. Hence S is regular, as required.
(5) $\Rightarrow$ (1). By [8, IV, Theorem 6.6], as required. □

In Theorem 13, it is claimed that the necessary and sufficient condition for all S-acts satisfying Condition $(EP)_E$ to be PWF is the regularity of S. In the following, we will determine a condition on S such that all right S-acts satisfying Condition $(EP)_E$ are flat:

Theorem 14. *Let S be a monoid such that every idempotent element of S is a left identity in $S \setminus \{1\}$. Then the following statements are equivalent:*

(1) All right S-acts satisfying Condition $(EP)_E$ are flat.

(2) All right S-acts satisfying Condition $(EP)_E$ are WF.

(3) All right S-acts satisfying Condition $(EP)_E$ are PWF.

(4) S is regular.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). This follows directly from Theorem 13.
(4) $\Rightarrow$ (1). Assume that S is a regular monoid in which every idempotent element of $S \setminus \{1\}$ acts as a left identity. Let A be a right S-act satisfying Condition $(EP)_E$, and suppose that $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_SM$, for some $a, a' \in A_S$ and $m, m' \in {}_SM$. We will verify that this equality also holds in $A_S \otimes_S (Sm \cup Sm')$. For this aim, from the equality $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_SM$, we obtain the following S-tossing:

$$\begin{array}{cc} & s_1n_1 = m \\ as_1 = b_1t_1 & s_2n_2 = t_1n_1 \\ b_1s_2 = b_2t_2 & s_3n_3 = t_2n_2 \\ \ldots & \ldots \\ b_{k-1}s_k = a't_k & m' = t_kn_k \end{array}$$

of length k, where $s_1, \cdots, s_k, t_1, \cdots, t_k \in S$, $b_1, \cdots, b_{k-1} \in A_S$, $n_1, \cdots, n_k \in {}_SM$. If $k = 1$, then we have:

$$\begin{array}{cc} & s_1n_1 = m \\ as_1 = a't_1 & m' = t_1n_1. \end{array}$$

Since S is regular, there exists $s_1' \in V(s_1)$ such that, from the equality $as_1 = a't_1$, we obtain $a't_1 = a't_1s_1's_1$. Since A_S satisfies Condition $(EP)_E$, there exist $a'' \in A_S$, $u, v \in S$, and $e, f \in E(S)$ such that:

$$\begin{aligned} a'e &= a''u, \\ a'f &= a''v, \\ et_1 &= t_1, \\ ft_1s_1's_1 &= t_1s_1's_1, \\ ut_1 &= vt_1s_1's_1. \end{aligned}$$

From the last equality and the assumption, we obtain $um' = ut_1n_1 = vt_1s_1's_1n_1 = vt_1s_1'm$, and $em' = et_1n_1 = t_1n_1 = m'$. Since $m = s_1n_1$, it follows that $s_1s_1'm = m$. Thus, we have:

$$\begin{aligned} a \otimes m &= a \otimes s_1s_1'm \quad (\text{since } s_1s_1'm = m) \\ &= as_1 \otimes s_1'm \quad (\text{by the right } S\text{-act structure}) \\ &= a't_1 \otimes s_1'm \\ &= a'et_1 \otimes s_1'm \quad (\text{since } et_1 = t_1) \\ &= a''ut_1 \otimes s_1'm \\ &= a'' \otimes ut_1s_1'm \\ &= a'' \otimes vt_1s_1's_1s_1'm \quad (\text{from } ut_1 = vt_1s_1's_1) \\ &= a'' \otimes vt_1s_1'm \quad (\text{simplifying}) \\ &= a'' \otimes um' \quad (\text{from } um' = vt_1s_1'm) \\ &= a''ue \otimes m' \quad (\text{since } em' = m') \\ &= a'e \otimes m' \\ &= a' \otimes em' \\ &= a' \otimes m' \quad (\text{since } em' = m') \end{aligned}$$

This shows that the required equality holds in $A_S \otimes_S (Sm \cup Sm')$. We proceed by induction on the length k of the tossing. Let $k \geq 2$, and suppose the required equality holds for tossing of length less than k. From $as_1 = b_1t_1$, regularity gives us:

$$b_1t_1 = b_1t_1s_1's_1 \quad \text{for some } s_1' \in V(s_1),$$

and

$$as_1 = as_1t_1't_1 \quad \text{for some } t_1' \in V(t_1).$$

Applying Condition $(EP)_E$ yields elements $a_1'', a_2'' \in A_S$, $u_1, u_2, v_1, v_2 \in S$ and $e_1, e_2, f_1, f_2 \in E(S)$ such that

$$\begin{cases} ae_1 = a_1''u_1, \\ af_1 = a_1''v_1, \\ e_1s_1 = s_1, \\ f_1s_1t_1't_1 = s_1t_1't_1, \\ u_1s_1 = v_1s_1t_1't_1, \end{cases} \qquad \begin{cases} b_1e_2 = a_2''u_2, \\ b_1f_2 = a_2''v_2, \\ e_2t_1 = t_1, \\ f_2t_1s_1's_1 = t_1s_1's_1, \\ u_2t_1 = v_2t_1s_1's_1. \end{cases}$$

Note that

$$b_1t_1 = b_1e_2t_1 = a_2''u_2t_1, \quad as_1 = ae_1s_1 = a_1''u_1s_1.$$

Therefore, we have the following tossing:

$$\begin{array}{cc} & u_1s_1n_1 = u_1m \\ a_1''u_1s_1 = a_2''u_2t_1 & u_2s_2n_2 = u_2t_1n_1 \end{array}$$

of length 1. Also, $b_1s_2 = b_1s_2s_2's_2$ where $s_2' \in V(s_2)$ and $s_2s_2' \in E(S)$. Since every idempotent element of S is a left identity, we have $e_2s_2s_2' = s_2s_2'$. Thus, we can write

$$b_1s_2s_2's_2 = b_1e_2s_2s_2's_2 = a_2''u_2s_2s_2's_2 = a_2''u_2s_2.$$

Thus, we obtain the following tossing

$$\begin{array}{rcccccl} & & & & u_2s_2n_2 & = & u_2t_1n_1 \\ a_2''u_2s_2 & = & b_2t_2 & & s_3n_3 & = & t_2n_2 \\ & \ldots & & & & \ldots & \\ b_{k-1}s_k & = & a't_k & & m' & = & t_kn_k \end{array}$$

of length $k-1$. From the tossing of length 1, we have $a_1'' \otimes u_1m = a_2'' \otimes u_2s_2n_2 \in A_S \otimes {}_SM$. By the inductive hypothesis, this equality holds in

$$A_S \otimes {}_S \left(Su_1m \cup Su_2s_2n_2\right).$$

Since $u_2s_2n_2 = u_2t_1n_1$ and $u_2t_1n_1 = v_2t_1s_1's_1n_1 = v_2t_1s_1'm \in Sm$, we have $a_1'' \otimes u_1m = a_2'' \otimes u_2s_2n_2 \in A_S \otimes_S (Sm \cup Sm')$. Moreover, from the tossing of length $k-1$, we have $a_2'' \otimes u_2t_1n_1 = a' \otimes m' \in A_S \otimes_S (Sm \cup Sm')$. Thus, we have

$$a \otimes m = a \otimes e_1m = ae_1 \otimes m = a_1''u_1 \otimes m = a_1'' \otimes u_1m = a_2'' \otimes u_2s_2n_2 = a_2'' \otimes u_2t_1n_1 = a' \otimes m'$$

in $A_S \otimes {}_S(Sm \cup Sm')$, as required.

□

The validity of the above theorem extends to finitely generated S-acts and S-acts generated by at most (exactly) two elements.

Theorem 15. *Let S be a right cancellative monoid. Then the following statements are equivalent:*

(1) All right S-acts are flat.

(2) All right S-acts satisfying Condition $(EP)_E$ are flat.

(3) All right S-acts satisfying Condition $(EP)_E$ are WF.

(4) All right S-acts satisfying Condition $(EP)_E$ are PWF.

(5) S is a group.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5). Suppose that all right S-acts satisfying Condition $(EP)_E$ are PWF. Then, by Theorem 13, S must be regular. Under the additional assumption that S is right cancellative, we observe that every element of S has a right inverse, which makes S a group. The result follows immediately from this conclusion.
(5) $\Rightarrow$ (1). It follows from [8, IV, Theorem 9.10]. □

The conclusion reached by the proof of Theorem 15 affirms the validity of the theorem for finitely generated S-acts and S-acts generated by at most (exactly) two elements.

Theorem 16. *Let S be a monoid such that every idempotent element of S is a left identity in $S \setminus \{1\}$. Then the following statements are equivalent:*

(1) All right S-acts satisfying Condition $(EP)_E$ satisfy Condition (P_E).

(2) All finitely generated right S-acts satisfying Condition $(EP)_E$ satisfy Condition (P_E).

(3) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ satisfy Condition (P_E).

(4) S is regular.

Proof. Implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (4). Regularity of S follows from Theorem 13, since by [4, Theorem 2.3] Condition (P_E) implies PWF.
(4) $\Rightarrow$ (1). Let A_S satisfies Condition $(EP)_E$. By Theorem 14, A_S is flat. Since every regular monoid is left PP, it follows that by [4, Theorem 2.5], A_S satisfies Condition (P_E). □

Theorem 17. *The following statements are equivalent:*

(1) All right S-acts satisfying Condition $(EP)_E$ are free.

(2) All right S-acts satisfying Condition $(EP)_E$ are projective generators.

(3) All right S-acts satisfying Condition $(EP)_E$ are projective.

(4) All right S-acts satisfying Condition $(EP)_E$ are strongly flat.

(5) $S = \{1\}$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious.
(4) $\Rightarrow$ (5). Since strong flatness implies Condition (P), all right S-acts satisfying Condition $(EP)_E$ also satisfy Condition (P). Therefore, by Theorem 12, S is a group. Furthermore, [8, IV, Theorem 9.10] establishes that for group S, all right S-acts satisfy Condition (P). As we know, Condition (P) follows (EP) and Condition (EP) follows $(EP)_E$, hence all right S-acts satisfy Condition $(EP)_E$. Consequently, by assumption, all right S-acts are strongly flat, and thus, by [8, IV, Theorem 10.5], we conclude that $S = \{1\}$.
(5) $\Rightarrow$ (1). If $S = \{1\}$, then all S-acts are free, and so the result follows. □

As per [10], an S-act A is considered GP-flat if $a \otimes s = a' \otimes s$ in $A \otimes_S S$, for $s \in S$ and $a, a' \in A$ implies the existence of a natural number n such that $a \otimes s^n = a' \otimes s^n$ in $A \otimes_S Ss^n$. It is evident that every PWF act is GP-flat, and one could prove that every GP-flat act is torsion free, but the converse of both implications are not generally true.
As mentioned in ([2, 6, 10]), the monoid S is called *right(left) generally regular* if, for every $s \in S$, there exist $n \in \mathbb{N}$ and $x \in S$ such that $s^n = sxs^n$ ($s^n = s^nxs$).

Theorem 18. *The following statements are equivalent:*

(1) All right S-acts are GP-flat.

(2) All right S-acts satisfying Condition $(EP)_E$ are GP-flat.

(3) All finitely generated right S-acts satisfying Condition $(EP)_E$ are GP-flat.

(4) All right S-acts generated by at most two elements satisfying Condition $(EP)_E$ are GP-flat.

(5) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are GP-flat.

(6) S is right generally regular.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.
(5) $\Rightarrow$ (6). Let $s \in S$. If $sS = S$, then it is obvious that s is right generally regular. Thus, we suppose that $sS \neq S$ and let $A = S \overset{sS}{\amalg} S$. By Corollary 3, A satisfies Condition $(EP)_E$, and so, by assumption, A is GP-flat. Now, for $s \in S$ and $1 \in S \setminus sS$, since $1 \cdot s \in sS$, hence there exist $n \in \mathbb{N}$ and $j \in sS$ such that $s^n = js^n$, by [11, Lemma 2.4]. Therefore, there exists $x \in S$ such that $j = sx$ and so $s^n = sxs^n$, as required.
(6) $\Rightarrow$ (1). By [11, Theorem3.4], it is obvious. □

We recall from [8] that a right S-act A is *divisible* if for every element $a \in A_S$ and any left cancellable element $c \in S$ there exists $b \in A_S$ such that $a = bc$.

Theorem 19. *The following statements are equivalent:*

(1) All right S-acts are divisible.

(2) All right S-acts satisfying Condition $(EP)_E$ are divisible.

(3) All left cancellable elements of S are left invertible.

Proof. The implication (1) ⇒ (2) is obvious.

(2) ⇒ (3). By Theorem 2, the right S-act S_S satisfies Condition $(EP)_E$. Therefore, by assumption, S_S is divisible. Consequently, all left cancellable elements of S are left invertible, as stated in [8, III,Proposition 2.2].

(3) ⇒ (1). It is evident by ([8], III, Proposition 2.2). □

The conclusion reached by the proof of Theorem 19 affirms the validity of the theorem for finitely generated S-acts and cyclic (monocyclic) S-acts. We recall from [8] that a right S-act A is called *simple* if it contains no subacts other than itself, and A is called *completely reducible* if it is a disjoint union of simple subacts.

Theorem 20. *The following statements are equivalent:*

(1) All right S-acts are completely reducible.

(2) All right S-acts satisfying Condition $(EP)_E$ are completely reducible.

(3) All finitely generated right S-acts satisfying Condition $(EP)_E$ are completely reducible.

(4) All right S-acts generated by at most two elements satisfying Condition $(EP)_E$ are completely reducible.

(5) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are completely reducible.

(6) S is a group.

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) are obvious.

(5) ⇒ (6). Let $s \in S$. If $sS \neq S$, then $A = S \overset{sS}{\amalg} S$ satisfies Condition $(EP)_E$. Therefore, by assumption, A is completely reducible. Hence, A is a disjoint union of its simple subacts. However, A is indecomposable, which implies that A is simple, where sS is a subact different from A, leading to a contradiction. Thus, we conclude that $sS = S$ and therefore S is a group, as required.

(6) ⇒ (1). It is obvious by [8, I, Proposition 5.34]. □

We recall from [12] that a right S-act A is called *strongly torsion free* if the equality $as = bs$ for all $a, b \in A$ and all $s \in S$, implies that $a = b$. So, Θ_S is strongly torsion free for any monoid S, while it satisfies Condition $(EP)_E$ if and only if S is right reversible, by Theorem 2. So strong torsion freeness does not imply Condition $(EP)_E$. Also, recall from [12] that for any monoid S and $x, y \in S$, $\rho_{STF}(x, y)$ denotes the smallest right congruence on S containing (x, y) such that the cyclic right S-act $S/\rho_{STF}(x, y)$ is strongly torsion free.

Theorem 21. *The following statements are equivalent:*

(1) All right S-acts are strongly torsion free.

(2) All right S-acts satisfying Condition $(EP)_E$ are strongly torsion free.

(3) All finitely generated right S-acts satisfying Condition $(EP)_E$ are strongly torsion free.

(4) All right S-acts generated by at most two elements satisfying Condition $(EP)_E$ are strongly torsion free.

(5) All right S-acts generated by exactly two elements satisfying Condition $(EP)_E$ are strongly torsion free.

(6) S is a group.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious.

(5) $\Rightarrow$ (6). Let $s \in S$ be such that $sS \neq S$ and suppose $A = S \overset{sS}{\amalg} S$. By Corollary 3, A satisfies Condition $(EP)_E$, and so by assumption, A is strongly torsion free. Now let

$$B = \{(l, x) | l \in S \setminus sS\} \dot{\cup} sS \cong S_S \cong \{(t, y) | t \in S \setminus sS\} \dot{\cup} sS = C.$$

Then we have

$$B, C \leq A_S,\ A_S = B \cup C,\ B = (1, x)S,\ C = (1, y)S.$$

By [12, Proposition 2.1], B as a subact of A_S is strongly torsion free, and thus S_S is strongly torsion free, as well. Hence, S is right cancellative by [12, Proposition 2.1]. However, in this case, strong torsion freeness and torsion freeness coincide, and by Theorem 11, every right cancellable element of S is right invertible, which implies $sS = S$ and it is a contradiction by assumption. Thus, for every $s \in S$, $sS = S$, and therefore S is a group, as required.

(6) $\Rightarrow$ (1). It is true by [12, Theorem 6.1]. □

Lemma 3. *If all cyclic strongly torsion free right S-acts satisfy Condition $(EP)_E$ then S is right reversible.*

Proof. The one-element right S-act Θ_S is strongly torsion free by [12, Proposition 2.1], and so by assumption it satisfies Condition $(EP)_E$. Thus, S is right reversible by Theorem 2. □

Theorem 22. *For any monoid S the following statements are equivalent:*

(1) All strongly torsion free right S-acts satisfy Condition $(EP)_E$.

(2) All strongly torsion free finitely generated right S-acts satisfy Condition $(EP)_E$.

(3) All cyclic strongly torsion free right S-acts satisfy Condition $(EP)_E$.

(4) $(\forall x, s, t \in S)(\exists u, v \in S)(\exists e, f \in E(S))$
$(us = vt,\ es = s,\ ft = t, (xe)\rho_{STF}(xs, xt)(u),\ (xf)\rho_{STF}(xs, xt)(v))$.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.
(3) $\Rightarrow$ (1). Let the right S-act A_S be strongly torsion free. It is obvious that every subact of A_S is also strongly torsion free. Thus, by assumption, aS satisfies Condition $(EP)_E$ for every $a \in A_S$. Therefore, A_S satisfies Condition $(EP)_E$ by Corollary 1.
(3) $\Rightarrow$ (4). Let $x, s, t \in S$. Then by [12, Theorem 3.3], $S/\rho_{STF}(xs, xt)$ is strongly torsion free and so by assumption, it satisfies Condition $(EP)_E$. Thus by Theorem 3, $(xt)\rho_{STF}(xs, xt)(xt)$ implies that there exist $u, v \in S$, $e, f \in E(S)$ such that $(xe, u), (xf, v) \in \rho_{STF}(xs, xt)$ and $es = s, ft = t,\ us = vt$.

(4) $\Rightarrow$ (3). Let S/ρ be strongly torsion free for the right congruence ρ on S and suppose $(xt)\rho(xs)$ for $x, s, t \in S$. Then by [12, Theorem 3.3], there exist $u, v \in S,\ e, f \in E(S)$ such that $(xe, u), (xf, v)\rho_{STF}(xt, xs)$ and $us = vt, es = s, ft = t$. Since $\rho_{STF}(xt, xs) \subseteq \rho$, then

$$(xe)\rho(u), \quad (xf)\rho(v).$$

Thus, by Theorem 3, S/ρ satisfies Condition $(EP)_E$, as required.

□

Corollary 4. *Let S be an idempotent monoid. Then the following statements are equivalent:*

(1) All strongly torsion free right S-acts satisfy Condition $(EP)_E$.

(2) All strongly torsion free finitely generated right S-acts satisfy Condition $(EP)_E$.

(3) All cyclic strongly torsion free right S-acts satisfy Condition $(EP)_E$.

(4) S is right reversible.

Proof. Implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear and implication (3) $\Rightarrow$ (4) is true by Lemma 3. (4) $\Rightarrow$ (1). Suppose that the right S-act A_S is strongly torsion free. Then the equality $ae = (ae)e$, for $a \in A_S, e \in E(S) = S$, implies that $ae = a$. Hence, $aS = \{a\}$, for every $a \in A_S$. Therefore, $as = at = a$ for all $a \in A$ and all $s, t \in S$. Since S is right reversible, there exist $u, v \in S$ such that $us = vt$. Let $a' = a, a = a'u = a'v, es = s, ft = t$ for $e = f = 1$ and so A_S satisfies Condition $(EP)_E$ as required. □

As mentioned in [13], the S-act A is called $\mathfrak{R}$-torsion free if, for any $a, b \in A$ and $c \in S$, where c is right cancellable, $ac = bc$ and $a\mathfrak{R}b$ (recall from [8], $\mathfrak{R}$ is Green's equivalence, which is $a\mathfrak{R}b$ if and only if $aS = bS$), imply that $a = b$. It is clear that torsion freeness implies $\mathfrak{R}$-torsion freeness. However, in general, the converse is not true.

Theorem 23. *All right S-act satisfying Condition $(EP)_E$ are $\mathfrak{R}$-torsion free.*

Proof. Let $ac = a'c$, $a\mathfrak{R}a'$, where $a, a' \in A$, $c \in S$ is right cancellable element. Since $a\mathfrak{R}a'$, there exist $s, t \in S$ such that $a = a's, a' = at$. Thus, we have $ac = atc$. Since A satisfies Condition $(EP)_E$, there exist $b \in A, u, v \in S$ and $e, f \in E(S)$ such that $ae = bu,\ af = bv,\ ec = c,\ ftc = tc$ and $uc = vtc$. Since $ec = c$ and c is right cancellable, it follows that $e = 1,\ a = bu,\ af = bv,\ ft = t$ and $u = vt$. Thus, from $af = bv$, we get $at = aft = bvt$. Hence, $a' = at = bvt = bu$. However, since $a = bu$,we conclude that $a = a'$, as required. □

Recall from [8] that A_S is a *(strongly) faithful* right S-act if for $s, t \in S$ the equality $as = at$ for (some) all $a \in A$ implies $s = t$. It is straightforward that every strongly faithful right S-act is faithful, but the converse is not true in general.

Example 7. Every right S-act satisfying Condition $(EP)_E$ is not necessarily faithful in general. If S is a right reversible and nontrivial monoid, it is evident that the right S-act $\Theta_S \cong S/S_S$ satisfies Condition $(EP)_E$. However, since $|S| > 1$, it is not faithful. It is also important to note that every nontrivial group S is right reversible nontrivial monoid.

Theorem 24. *Every cyclic strongly faithful right S-act satisfies Condition $(EP)_E$.*

Proof. Let $A = aS$ be a cyclic strongly faithful right S-act. Define the mapping $\Psi : aS \longrightarrow S_S$ by $\Psi(as) = s$. It is clear that Ψ is an isomorphism, and thus we have $aS \cong S_S$. Consequently, A satisfies Condition $(EP)_E$, as required. □

Recall from [5] that the right S-act A is called *P-regular* if all cyclic subacts of A satisfy Condition (P).

Theorem 25. *Let A be a P-regular right S-act. Then A satisfies Condition $(EP)_E$.*

Proof. Let $as = at$ for $a \in A$ and $s, t \in S$. By assumption aS satisfies Condition (P). Therefore, there exist $w_1, w_2, l \in S$, such that $a = (al)w_1,\ \ a = (al)w_2$ and $w_1 s = w_2 t$. Put $u = lw_1$ and $v = lw_2$. Thus we have

$$a = (al)w_1 = a(lw_1) = au$$
$$a = (al)w_2 = a(lw_2) = av$$

Let $e = f = 1$. Note that $ae = au,\ af = av,\ es = s,\ ft = t$ and $us = vt$. Thus, A satisfies Condition $(EP)_E$. □

A binary relation $P_{u,v}$ can be defined for fixed elements $u, v \in S$ as follows:

$$(x, y) \in P_{u,v} \Leftrightarrow ux = vy\ (x, y \in S).$$

Theorem 26. *The following statements are equivalent:*

(1) All fg-weakly injective right S-acts satisfy Condition $(EP)_E$.

(2) All weakly injective right S-acts satisfy Condition $(EP)_E$.

(3) All injective right S-acts satisfy Condition $(EP)_E$.

(4) All cofree right S-acts satisfy Condition $(EP)_E$.

(5) $(\forall s, t \in S)(\exists u, v \in S)(\exists e_1, e_2 \in E(S)); e_1 s = s, e_2 t = t, us = vt$ and the following conditions hold:

1. *$ker\lambda_u \cap (e_1 S \times e_1 S) \subseteq \rho(s, t)$.*
2. *$ker\lambda_v \cap (e_2 S \times e_2 S) \subseteq \rho(s, t)$.*
3. *$P_{u,v} \cap (e_1 S \times e_2 S) \subseteq \rho(s, t)$.*

Proof. Implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious, because cofree ⇒ injective ⇒ weakly injective ⇒ fg-weakly injective.
(4) ⇒ (5). Let $s, t \in S$. Put $X = S/\rho(s, t)$ and define the mapping

$$f : S \to X$$
$$x \mapsto [x]_{\rho(s,t)}.$$

Let $(sx, tx) \in \rho(s,t)$ for $x \in S$. Thus, we have $[sx]_{\rho(s,t)} = [tx]_{\rho(s,t)}$. Therefore, $f(sx) = f(tx)$, which implies $fs = ft$ in X^S. Since X^S is cofree, it follows that X^S satisfies Condition $(EP)_E$, by assumption. Thus $fs = ft$ implies that X^S satisfies Condition $(EP)_E$. Consequently, from $fs = ft$ in X^S, there exists a map $g : S \to X, u, v \in S$ and $e_1, e_2 \in E(S)$ such that $fe_1 = gue_1, fe_2 = gve_2$, $e_1 s = s, e_2 t = t, us = vt$. Now, we will demonstrate the correctness of Condition (i). For this purpose, assume $(l_1, l_2) \in ker\lambda_u \cap (e_1 S \times e_1 S)$. Therefore, there exist $r_1, r_2 \in S$ such that $l_1 = e_1 r_1, l_2 = e_1 r_2, ul_1 = ul_2$. Thus we have $ue_1 r_1 = ue_1 r_2$. Then, according to the definition of f, we have $f(l_1) = f(e_1 r_1) = [e_1 r_1]_{\rho(s,t)} = [l_1]_{\rho(s,t)}$. We have:

$$f(e_1 r_1) = g(ue_1 r_1) = g(ul_1) = g(ul_2) = g(ue_1 r_2) = (fe_1)(r_2) = f(e_1 r_2).$$

So, $[e_1 r_1]_{\rho(s,t)} = [e_1 r_2]_{\rho(s,t)}$. Then, $[l_1]_{\rho(s,t)} = [l_2]_{\rho(s,t)}$. Hence, $(l_1, l_2) \in \rho(s,t)$, as a result, $\ker \lambda_u \cap (e_1 S \times e_1 S) \subseteq \rho(s,t)$. Therefore condition (i) is proved.
In a similar way, since $fe_2 = gve_2$, we can prove condition (ii).
For (iii), suppose $(l_1, l_2) \in P_{u,v} \cap (e_1 S \times e_2 S)$. So, there exist $r_1, r_2 \in S$ such that

$$ul_1 = vl_2,\ l_1 = e_1 r_1,\ l_2 = e_2 r_2.$$

Therefore we have:

$$f(e_1 r_1) = (fe_1)(r_1) = g(ue_1) r_1 = g(ue_1 r_1) = g(ul_1) = g(vl_2) = g(ve_2 r_2) = (gve_2)(r_2) = f(e_2)(r_2).$$

Therefore, $[e_1 r_1]_{\rho(s,t)} = [e_2 r_2]_{\rho(s,t)}$. Thus, $(e_1 r_1, e_2 r_2) \in \rho(s,t)$. Hence, $(l_1, l_2) \in \rho(s,t)$. As a result, $P_{u,v} \cap (e_1 S \times e_1 S) \subseteq \rho(s,t)$.
(5) $\Rightarrow$ (1). Suppose that A is fg-weakly injective right S- act and $as = at$ for $a \in A$. By assumption, there exist $u, v \in S$ and $e_1, e_2 \in E(S)$ such that $e_1 s = s, e_2 t = t$ and $us = vt$ and condition $(i), (ii)$ and (iii) are correct.
Define

$$f : ue_1 S \cup ve_2 S \to A$$
$$x \mapsto \begin{cases} ae_1 p, & \exists p \in S :\ x = ue_1 p \\ \\ ae_2 q, & \exists q \in S :\ x = ve_2 q \end{cases}$$

First, we must show that f is well-defined. Let $ue_1 p_1 = ue_1 p_2$ for $p_1, p_2 \in S$. Then,

$$(e_1 p_1, e_1 p_2) \in ker\lambda_u \cap (e_1 S \times e_1 S) \subseteq \rho(s,t).$$

By [8, I, Lemma 4.37], there exist $n \in \mathbb{N}$ and $y_1, \cdots, y_n \in S$ such that

$$\begin{array}{ccccc} e_1 p_1 = sy_1 & ty_2 = sy_3 & \cdots & ty_{n-1} = sy_n \\ ty_1 = sy_2 & \cdots & & ty_n = e_1 p_2. \end{array}$$

Because $as = at$, therefore $asy_i = aty_i$ for every $i \in \{1, 2, \cdots n\}$. Thus,

$$ae_1 p_1 = asy_1 = aty_1 = \cdots = aty_n = ae_1 p_2.$$

In a similar way, $ve_2q_1 = ve_2q_2$ for $q_1, q_2 \in S$, implies $ae_2q_1 = ae_2q_2$.
Now, suppose $ue_1p = ve_2q$ for $p, q \in S$. Then $(e_1p, e_2q) \in P_{u,v} \cap (e_1S \times e_2S)$, and therefore, by condition (iii), $(e_1p, e_2q) \in \rho(s, t)$.
By [8, I, Lemma 4.37], there exist $n \in \mathbb{N}$ and $y_1, \cdots, y_n \in S$ such that

$$\begin{array}{ccccc} e_1p_1 = sy_1 & ty_2 = sy_3 & \cdots & ty_{n-1} = sy_n \\ ty_1 = sy_2 & \cdots & & ty_n = e_2q. \end{array}$$

Since $as = at$, we have: $asy_i = aty_i$ for $i = 1, \ldots, n$. Therefore,

$$ae_1p = asy_1 = aty_1 = \cdots = aty_n = ae_2q.$$

Thus, f is well defined and is also a homomorphism.
Since A is fg-weakly injective, there exists an S-homomorphism $g : S \to A$ such that $g|_{ue_1S \cup ve_2S} = f$.
Let $a' = g(1)$, then

$$\begin{aligned} ae_1 = f(ue_1) = g(ue_1) = g(1)ue_1 = a'ue_1, \\ ae_2 = f(ve_2) = g(ve_2) = g(1)ve_2 = a've_2. \end{aligned}$$

Thus, A satisfies Condition $(EP)_E$. □

An element $a \in A_S$ is called *act-regular* if there exists a homomorphism $f : aS \to S$ such that $af(a) = a$ and A_S is called a *regular* if every $a \in A_S$ is an act-regular. Also, an element s of S is called *left (right) e-cancellable*, for an idempotent $e \in S$ if $s = se$ ($s = es$) and $ker\lambda_s \leq ker\lambda_e$ ($ker\rho_s \leq ker\rho_e$). A monoid S is called *right (left) PP* if every principal right (left) ideal of S is projective as a right (left) S-act. This is equivalent to saying that every element $s \in S$ is left (right) e-cancellable, for an idempotent $e \in S$.
Recall from [8] that a monoid S satisfies Condition (K) if every left collapsible submonoid of S contains a left zero. According to [8, III, Corollary 19.3], a monoid S satisfies Condition (K) if and only if every strongly flat cyclic right S-act is projective. We now prove a theorem which demonstrates that the regularity of all right S-acts satisfying Condition $(EP)_E$ implies that S is a right PP-monoid and satisfies Condition (K). This result provides a characterization that allows us to describe the structure of such monoids.

Theorem 27. *If all right S-acts satisfying Condition $(EP)_E$ are regular then S is right PP and satisfies Condition (K).*

Proof. Since S_S satisfies Condition $(EP)_E$ by Theorem 2, it follows that, by assumption, S_S is regular. Consequently, by [8, III, Corollary 19.3], all principal right ideal of S are projective. Thus, by [8, IV, Theorem 11.15], S is right PP. On the other hand, by [8, III, Theorem 16.7], all strongly flat cyclic right S-acts satisfy Condition (E), and hence Condition $(EP)_E$. Therefore, all strongly flat cyclic right S-acts are regular, by assumption. Hence, by [8, III, Corollary 19.3], all strongly flat cyclic right S-acts are projective, and thus, by [8, IV, Theorem 11.2], S satisfies Condition (K). □

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