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Strongly Gorenstein n-flat modules over n-coherent rings and their dimensions

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Abstract. In this article, we investigate the strongly Gorenstein n-flat modules and their related results over different classes of rings. A module M is called strongly Gorenstein n-flat if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of n-projective R-modules with $M \cong \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, F)$ leaves the sequence exact, whenever F is an n-flat R-module. We establish equivalent conditions for strongly Gorenstein n-flat modules over a left n-coherent ring. Further, we discuss the results related to precovers and preenvelopes of the classes of strongly Gorenstein n-flat modules. Finally, we proved that for any module M with finite n-flat dimension, the strongly Gorenstein n-flat dimension of M coincides with the n-projective dimension over an n-coherent ring.

Keywords: Gorenstein n-flat modules, Strongly Gorenstein n-flat modules, Precovers, Preenvelopes,Strongly Gorenstein n-flat dimension.
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1 Introduction

Throughout this paper, R is an associative ring with identity. The notion of Gorenstein flat modules was first introduced by Enoch et al. [8] in 1993. An R-module M is called Gorenstein flat (G-flat for short) if there is an exact sequence $\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ of flat right R-modules such that $M \cong \mathrm{im}(F_0 \to F^0)$ and such that $- \otimes I$ leaves the above sequence exact whenever I is an injective left R-module. The concept of Gorenstein flat modules have been studied extensively by several authors. In 2007 Bennis D et al. [2] first introduced and

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studied the notion of strongly Gorenstein projective, injective and flat modules. A complete flat resolution of the form $\mathbf{F} : \cdots \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} \cdots$ is called a strongly complete flat resolution and denoted by $(\mathbf{F}, f)$. An R-module M is called strongly Gorenstein flat (SG-flat for short) if $M \cong \operatorname{Ker} f$ for some strongly complete flat resolution $(\mathbf{F}, f)$. The authors in [3, 12, 16–19] obtained some important results in Gorenstein and strongly Gorenstein projective, injective and flat R-modules. Later, Selvaraj et al. [15] in 2014 introduced the concept of Gorenstein n-flat modules and their properties and further they studied [14] the stability of Gorenstein n-flat R-modules in the same year and also [13] in 2015 they studied the Gorenstein n-flat dimension and Tate homology of modules. In 2009 N. Ding et al. [5] introduced the notion strongly Gorenstein flat R-modules with respect to Hom functor and studied their covers and envelopes with dimension. A right R-module M is called strongly Gorenstein flat if there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective right R-modules with $M = \ker(P^0 \to P^1)$ such that $\operatorname{Hom}_R(-, \mathcal{F}lat)$ leaves the sequence exact, where $\mathcal{F}lat$ the class of all flat R-modules. This motivate us to study the strongly Gorenstein n-flat modules with respect to the Hom functor and to establish several fundamental results concerning their structure in relation to homological properties.

The main purpose of this paper is to investigate the fundamental properties of strongly Gorenstein n-flat modules and explore their applications in homological algebra. We examine their relation to other homological classes, study their closure properties, and establish connections with precovering and preenveloping classes. The following main theorem gives the equivalent condition for strongly Gorenstein n-flat modules:

Theorem 1. *Let R be a left n-coherent ring. Then the following statements are equivalent for a right R-module M:*

(1) M is strongly Gorenstein n-flat.

(2) M has an exact right $\mathcal{F}_n$ resolution and $\operatorname{Ext}^i(M, F) = 0$ for all right R-modules F with $\mathrm{f_n d}(F) < \infty$ and all $i \geq 1$.

(3) M has an exact right $\mathcal{F}_n$ resolution and $\operatorname{Ext}^i(M, F) = 0$ for all n-flat right R-modules F and all $i \geq 1$.
Moreover, if n-FP-$\mathrm{id}({}_R R) < \infty$, then the above conditions are equivalent to

(4) $\operatorname{Ext}^i(M, F) = 0$ for all n-flat right R-modules F and all $i \geq 1$.

We prove the following theorem, for any R-module M with finite n-flat dimension, the strongly Gorenstein n-flat dimension of M coincides with the n-projective dimension.

Theorem 2. *Let R be an n-coherent ring. If $M \in \operatorname{Mod} R$ with $\mathrm{f_n d}(M) < \infty$, then $\mathrm{SGf_n d}(M) = \mathrm{p_n d}(M)$*

This paper is divided into 4 sections. In section 2, we recall some known definitions and facts which will be needed in this sequel.

In section 3, we introduce the strongly Gorenstein n-flat modules and prove our main result that a right R-module M over a left n-coherent ring R is strongly Gorenstein n-flat if and only

if M has an exact right n-flat resolution and $\mathrm{Ext}^i(M,F)=0$ for all n-flat right R-modules F and all $i\geq 1$.

In section 4, we define and investigate the strongly Gorenstein n-flat dimension for modules and rings. Given a right R-module M, the strongly Gorenstein n-flat dimension $SGf_nd(M)$ is defined to be the infimum of the set of n such that there exists an exact sequence, $0\to G_n\to\cdots\to G_1\to G_0\to M\to 0$ of right R-modules, where each G_i is strongly Gorenstein n-flat. In addition, we have shown the results related to precovers and preenvelopes by strongly Gorenstein n-flat modules. Finally, we proved our main theorem that, for any R-module M with finite n-flat dimension, the strongly Gorenstein n-flat dimension of M coincides with the n-projective dimension.

2 Preliminaries

In this section, we recall some known definitions and terminologies which we needed in the following results.

A ring R is called left n-coherent [10] (for integers $n>0$ or $n=\infty$) if every finitely generated submodule of a free left R-module whose projective dimension is $\leq n-1$ is finitely presented. A ring R is called the right perfect ring [1] if every right R-module has a projective cover. A ring R is said to be an n-FC ring [4] if R is both left and right coherent, and the FP-injective dimensions of ${}_RR$ and R_R at most n, that is $\mathrm{FP}\text{-}\mathrm{id}({}_RR)\leq n$ and $\mathrm{FP}\text{-}\mathrm{id}(R_R)\leq n$ for some integer $n\geq 0$. A ring R is called an FC ring if R is 0-FC. A right R-module M is called n-flat [10] if $\mathrm{Tor}_1^R(M,N)=0$ holds for all finitely presented left R-modules N with $\mathrm{pd}_R N\leq n$.

An R-module M is called n-projective (n a positive integer) if the functor $\mathrm{Ext}_R^{n-1}(M,-)$ is a right exact functor [11]. A left R-module A is said to be n-FP-injective if $\mathrm{Ext}_R^1(N,A)=0$ holds for all finitely presented left R-modules N of projective dimension $\leq n$. The right strongly Gorenstein flat dimension $rSGf_nd(R)$ of a ring R is defined to be the supremum of the set of $SGf_nd(M)$ such that M is any right R-module.

Let $\mathcal{C}$ be a class of R-modules and X an R-module. Following [6], we say that a homomorphism $f:C\to X$ is a $\mathcal{C}$-precover if $C\in\mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}_R(C',f):\mathrm{Hom}_R(C',C)\to\mathrm{Hom}_R(C',X)$ is surjective for each $C'\in\mathcal{C}$. A $\mathcal{C}$-precover $f:C\to X$ is called a $\mathcal{C}$-cover if every endomorphism $g:C\to C$ such that $fg=f$ is an isomorphism. Dually, we have the definition of a $\mathcal{C}$-(pre)envelope. $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism. It is known that every left R-module M has an n-flat preenvelope over the right n-coherent ring [21]. Thus, M has a right $\mathcal{F}_n$-resolution, that is, there is an exact complex $0\to M\to F_0\to F_1\to\cdots$ with each F^i n-flat such that $\mathrm{Hom}(-,\mathcal{F}_n)$ leaves the sequence exact. Let $L^0=M$, $L^1=\mathrm{coker}(M\to F^0)$, $L^i=\mathrm{coker}(F^{i-2}\to F^{i-1})$ for $i\geq 2$. The mth cokernel L^m ($m\geq 0$) is called the mth $\mathcal{F}_n$-cosyzygy of M.

Definition 1 ([15]). *A left R-module M is said to be Gorenstein n-flat, if there exists an exact sequence of n-flat left R-modules, $\cdots\to F_1\to F_0\to F^0\to F^1\to\cdots$ such that $M\cong\mathrm{im}(F_0\to F^0)$ and such that $E\otimes_R -$ leaves the sequence exact whenever E is an n-absolutely pure right R-module.*

Definition 2 ([2]). *An R-module M is called strongly Gorenstein projective if there exists a complete projective resolution of the form $P = \cdots \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \cdots$ such that $M \cong \operatorname{Ker} f$ and such that $\operatorname{Hom}_R(-, Q)$ leaves the sequence P exact whenever Q is projective.*

Let $\mathcal{F}_n$ stands for class of all n-flat modules and let $\mathcal{SGF}_n$ be the class of all strongly Gorenstein n-flat right R-modules. For any R-module M, $\mathrm{p_n d}(M)$ (resp., $\mathrm{i_n d}(M)$, $\mathrm{f_n d}(M)$) denotes the n-projective (resp., n-injective, n-flat) dimension and $wD(R)$ stands for the weak global dimension of R.

3 Strongly Gorenstein n-flat modules

In this section we introduced strongly Gorenstein n-flat modules which is related to the Hom functor is defined as follows:

Definition 3. *A left R-module M is said to be strongly Gorenstein n-flat, if there exists an exact sequence,$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of n-projective right R-modules with $M \cong \ker(P_0 \to P^0)$ such that $\operatorname{Hom}(-, \mathcal{F}_n)$ leaves the sequence exact.*

Remark 1. *(i) Let M be an n-projective R-module. We consider an exact sequence $\cdots \to 0 \to M \to M \to 0 \to \cdots$ which remains exact after applying $\operatorname{Hom}_R(-, X)$ for any right R-module X. In particular, $\operatorname{Hom}_R(-, N)$ preserves the exactness of the sequence for any R-module N with $N \in \mathcal{F}_n$. It follows that every n-projective R-module is strongly Gorenstein n-flat. Furthermore, every n-projective R-module is n-flat R-module, but the converse need not be true. This is true only when an R-module is both n-flat and strongly Gorenstein n-flat.*

(ii) Let R be an n-coherent ring. The class of strongly Gorenstein n-flat right R-modules is closed under direct sums.

(iii) Every projective R-module is n-projective (for any integer n) and by (1) it is true that every projective R-module is strongly Gorenstein n-flat.

(iv) The Gorenstein projective R-module need not be a strongly Gorenstein n-flat R-module. For the converse, consider R as a self-injective ring. Then $\operatorname{Gproj}(R) = R\text{-Mod}$ by [22, Fact 1.2(i)]. Then every strongly Gorenstein n-flat R-module is Gorenstein projective R-module over a self-injective ring.

Example 1. Let $R = k[x]/(x^2)$ where k is a field and consider the ideal $I = (x)$. This ideal is neither projective nor flat because R is a local ring and $I = (x)$ is a non-free proper ideal. Consider the periodic complex $P_\bullet = \cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots$, where each map multiplication by x. Since $x^2 = 0$, we have $\operatorname{im}(x) \cong xR = (x) = \ker(x)$ and hence the complex is exact. It is clear that the complex $P_\bullet$ is a complete projective (n-projective) resolution of an ideal $I = (x)$. Thus, $I = (x) \cong \ker(x)$. Hence, I is strongly Gorenstein projective. Since R is quasi-Frobenius local ring, projective and injective modules coincide. Therefore, $I = (x)$ is Strongly Gorenstein flat, consequently Strongly Gorenstein n-flat. This is an example of a Strongly Gorenstein n-flat module which is not flat.

Proposition 1. *Let R be an n-coherent ring. Then every strongly Gorenstein n-flat right R-module is Gorenstein n-flat.*

Proof. Let M be a strongly Gorenstein n-flat right R-module. Then there is an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of n-projective right R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}(-, \mathcal{F}_n)$ leaves the sequence exact. Let E be an n-absolutely pure left R-module. Then E^+ is n-flat. Thus, we get an exact sequence $\cdots \to \mathrm{Hom}(P^1, E^+) \to \mathrm{Hom}(P^0, E^+) \to \mathrm{Hom}(P_0, E^+) \to \mathrm{Hom}(P_1, E^+) \to \cdots$ that gives the exactness of $\cdots \to (P^1 \otimes E)^+ \to (P^0 \otimes E)^+ \to (P_0 \otimes E)^+ \to (P_1 \otimes E)^+ \to \cdots$. Therefore, we have an exact sequence, $\cdots \to P_1 \otimes E \to P_0 \otimes E \to P^0 \otimes E \to P^1 \otimes E \to \cdots$. Hence M is Gorenstein n-flat. □

Lemma 1. *Let M be a strongly Gorenstein n-flat right R-module.Then:*

(1) $\mathrm{Ext}^i(M, G) = 0$ *for all right R-modules G with* $\mathrm{f_n d}(G) < \infty$ *and all* $i \geq 1$.

(2) either M is n-projective or $\mathrm{f_n d}(M) = \infty$.

Proof. (1). By hypothesis, there is a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequences $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i n-projective. Thus, $\mathrm{Ext}^i(M, F) = 0$ for all n-flat right R-modules F and all $i \geq 1$. By dimension shifting, $\mathrm{Ext}^i(M, G) = 0$ for all right R-modules G with $\mathrm{f_n d}(G) < \infty$ and all $i \geq 1$.

(2). Suppose $\mathrm{f_n d}(M) < \infty$. We may assume that $\mathrm{f_n d}(M) \leq n$ with $1 \leq n < \infty$. Then there is $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $0 \to F_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with F_n n-flat and each P_i n-projective. It is easy to see that M is isomorphic to a direct summand of P_0 and hence M is n-projective. □

The following main theorem gives the equivalent conditions for strongly Gorenstein n-flat modules and Ext vanishes for all modules with finite n-flat dimension and n-flat modules.

Theorem 3. *Let R be a left n-coherent ring. Then the following statements are equivalent for a right R-module M:*

(1) M is strongly Gorenstein n-flat.

(2) M has an exact right $\mathcal{F}_n$ resolution and $\mathrm{Ext}^i(M, F) = 0$ *for all right R-modules F with* $\mathrm{f_n d}(F) < \infty$ *and all* $i \geq 1$.

(3) M has an exact right $\mathcal{F}_n$ resolution and $\mathrm{Ext}^i(M, F) = 0$ *for all n-flat right R-modules F and all* $i \geq 1$.
Moreover, if $n\text{-}FP\text{-}\mathrm{id}({}_R R) < \infty$, *then the above conditions are equivalent to*

(4) $\mathrm{Ext}^i(M, F) = 0$ *for all n-flat right R-modules F and all* $i \geq 1$.

Proof. (1) ⇒ (2) follows from Lemma 1(1).

(2) ⇒ (3) is trivial.

(3) ⇒ (1). Let $f : M \to F^0$ be a $\mathcal{F}_n$ preenvelope of M. Consider the short exact sequence $0 \to K^0 \to P^0 \to F^0 \to 0$ with P^0 n-projective. There exists $g : M \to P^0$ such that $\pi g = f$ since K^0 is n-flat and $\mathrm{Ext}^1(M, K^0) = 0$. It is easy to verify that $g : M \to P^0$ is a $\mathcal{F}_n$-preenvelope. Thus, for any n-flat right R-module F, there is the exact sequence $\mathrm{Hom}(P^0, F) \to$

$\mathrm{Hom}(\mathrm{im}(g), F) \to 0$. In addition, the exactness of $0 \to \mathrm{im}(g) \to P^0 \to \mathrm{coker}(g) \to 0$ yields the exact sequence

$$\mathrm{Hom}(P^0, F) \to \mathrm{Hom}(\mathrm{im}(g), F) \to \mathrm{Ext}^1(\mathrm{coker}(g), F) \to 0.$$

Hence, $\mathrm{Ext}^1(\mathrm{coker}(g), F) = 0$. Therefore, $\mathrm{coker}(g)$ has $\mathcal{F}_n$-preenvelope $\mathrm{coker}(g) \to P^1$ with P^1 n-projective. Continuing this process, we can get $\mathrm{Hom}(-, \mathcal{F}_n)$ an exact complex $0 \to M \to P^0 \to P^1 \to \cdots$ with each P^i n-projective. Note that $\mathrm{Ext}^i(M, ({}_R R)^+) = 0$ for all $i \geq 1$ and $\mathrm{Ext}^0(M, ({}_R R)^+) \cong M^+$ since M has an exact right $\mathcal{F}_n$ resolution and $({}_R R)^+$ is injective. Accordingly, the complex $0 \to M \to P^0 \to P^1 \to \cdots$ is exact as $({}_R R)^+$ is a cogenerator. On the other hand, there is $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i n-projective since $\mathrm{Ext}^i(M, F) = 0$ for all n-flat right R-modules F and all $i \geq 1$. Now we get a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of n-projective right R-modules with $M \cong \mathrm{coker}(P_0 \to P^0)$. Hence, M is strongly Gorenstein n-flat.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1). By the proof of (3) implies (1), we obtain a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact complex $\mathcal{E} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of n-projective right R-modules such that $M \cong \mathrm{coker}(P_0 \to P^0)$. Next we will show that $\mathrm{Hom}(\mathcal{E}, N)$ is exact for any right R-module N with $\mathrm{f_n d}(N) = n < \infty$. We proceed by induction on n. The case $n = 0$ is clear. Consider $n \geq 1$, then there is an exact sequence $0 \to K \to P \to N \to 0$ with P n-projective, which induces an exact sequence $0 \to \mathrm{Hom}(\mathcal{E}, K) \to \mathrm{Hom}(\mathcal{E}, P) \to \mathrm{Hom}(\mathcal{E}, N) \to 0$ of complexes. Note that $\mathrm{f_n d}(K) = n - 1$, and therefore $\mathrm{Hom}(\mathcal{E}, K)$ is exact by induction. Thus, $\mathrm{Hom}(\mathcal{E}, N)$ is exact. In particular, $\mathrm{Hom}(\mathcal{E}, ({}_R R)^+)$ is exact since $\mathrm{f_n d}(({}_R R)^+)) = n\text{-}\mathrm{FP}\text{-}\mathrm{id}({}_R R) < \infty$. Therefore, $\mathcal{E}$ is an exact sequence. Thus, M is strongly Gorenstein n-flat. □

Proposition 2. *Let R be a left n-coherent ring, and $0 \to A \to B \to C \to 0$ an exact sequence of right R-modules.*

(1) If A and C are strongly Gorenstein n-flat, then so is B.

(2) If B and C are strongly Gorenstein n-flat, then so is A.

(3) If A and B are strongly Gorenstein n-flat, then C is strongly Gorenstein n-flat if and only if $\mathrm{Ext}^1(C, F) = 0$ for all n-flat right R-modules F.

Thus, the class of strongly Gorenstein n-flat right R-modules is closed under direct summands.

Proof. If C is strongly Gorenstein n-flat, then $\mathrm{Ext}^1(C, F) = 0$ for all n-flat right R-modules by Lemma 1. But the fact that $\mathrm{Ext}^1(C, F) = 0$ for all n-flat right R-modules F implies that $0 \to A \to B \to C \to 0$ is $\mathrm{Hom}(-, \mathcal{F}_n)$ exact. Thus, by [7, Theorem 8.2.5(2)], we obtain the long exact sequence,

$$\begin{aligned}\cdots &\to \mathrm{Ext}_n(A, ({}_R R)^+) \to \mathrm{Ext}_{n-1}(C, ({}_R R)^+) \to \mathrm{Ext}_{n-1}(B, ({}_R R)^+) \\ &\to \mathrm{Ext}_{n-1}(A, ({}_R R)^+) \to \cdots \to \mathrm{Ext}_1(A, ({}_R R)^+) \to \mathrm{Ext}_0(C, ({}_R R)^+) \\ &\to \mathrm{Ext}_0(B, ({}_R R)^+) \to \mathrm{Ext}_0(A, ({}_R R)^+) \to 0.\end{aligned}$$

Thus (1), (2) and (3) follows from Theorem 3. The last statement holds by (1), (2), Remark 1(2) and [9, Proposition 1.4] □

Proposition 3. *The following statements are equivalent for an n-FC ring R.*

(1) $wD(R) \leq n$.

(2) Every Gorenstein n-flat right R-module is n-flat.

(3) Every strongly Gorenstein n-flat right R-module is n-projective.

(4) Every strongly Gorenstein n-flat right R-module is n-flat.

Proof. (1) $\Rightarrow$ (2) holds from [4, Theorem 13].

(2) $\Rightarrow$ (3). Let M be a strongly Gorenstein n-flat right R-module. Then M is Gorenstein n-flat by Proposition 1. Thus, M is n-flat by (2) and hence M is n-projective by Remark 1(1).

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) Assume that every strongly Gorenstein n-flat left R-module is n-flat. We prove that $w\mathrm{D}(R) \leq n$. Let M be any left R-module. Consider a projective resolution $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is projective and $K_n = \ker(P_{n-1} \to P_{n-2})$ is the n-th syzygy of M. By dimension shifting, it is enough to show that $\mathrm{fd}_R(K_n) \leq n$. Since R is an n-FC ring, there exists a short exact sequence $0 \to K_n \to P \to K_n \to 0$, where P is an n-projective left R-module and $P \to K_n \to 0$ is an n-projective precover of K_n. Using this short exact sequence, we construct the doubly infinite exact complex $\mathbf{P} = \cdots \to P \xrightarrow{d} P \xrightarrow{d} P \xrightarrow{d} P \to \cdots$, such that $K_n \cong \ker(P \xrightarrow{d} P)$. Because $\mathrm{Im}(d) = K_n$ and $K_n \cong \ker(d)$, the complex $\mathbf{P}$ is exact. Let $F \in \mathcal{F}_n$ be any left R-module. Since P is n-projective, we have $\mathrm{Ext}^1_R(P, F) = 0$. Applying the functor $\mathrm{Hom}_R(-, F)$ to the above short exact sequence yields the exact sequence

$$0 \to \mathrm{Hom}_R(K_n, F) \to \mathrm{Hom}_R(P, F) \to \mathrm{Hom}_R(K_n, F) \to \mathrm{Ext}^1_R(K_n, F) \to 0.$$

Moreover, the map $\mathrm{Hom}_R(P, F) \to \mathrm{Hom}_R(K_n, F)$ is surjective because $P \to K_n \to 0$ is an n-projective precover of K_n. Hence, $\mathrm{Ext}^1_R(K_n, F) = 0$. Therefore, $\mathrm{Hom}_R(-, F)$ preserves the exactness of the complex $\mathbf{P}$ for every $F \in \mathcal{F}_n$. It follows that K_n is a strongly Gorenstein n-flat left R-module. By hypothesis, K_n is n-flat and hence $\mathrm{fd}_R(K_n) \leq n$. Finally, by dimension shifting, $\mathrm{Tor}^R_{n+k}(M, N) \cong \mathrm{Tor}^R_k(K_n, N)$ for every finitely presented left R-module N and every $k \geq 1$. Since $\mathrm{fd}_R(K_n) \leq n$, we have $\mathrm{Tor}^R_k(K_n, N) = 0$ for all $k > n$. Hence, $\mathrm{Tor}^R_{n+k}(M, N) = 0$ for all $k \geq 1$. Therefore, $\mathrm{fd}_R(M) \leq n$. As M was arbitrary, we conclude that

$$wD(R) = \sup\{\mathrm{fd}_R(M) \mid M \text{ is a left } R\text{-module}\} \leq n.$$

□

Proposition 4. *The following statements are equivalent for a right coherent ring R.*

(1) R is an n-FC ring.

(2) Every finitely presented R-module is strongly Gorenstein n-flat.

Proof. (1) $\Rightarrow$ (2). Let M be a finitely presented right R-modules and F be a n-flat right R-module. Then F is n-FP-injective by [21, Proposition 3.4]. Therefore, $\mathrm{Ext}^i(M, F) = 0$ for all $i \geq 1$. Thus, M is strongly Gorenstein n-flat by Theorem 3.

(2) $\Rightarrow$ (1). For any finitely presented right R-module N, $\mathrm{Ext}^1(N, {}_R R) = 0$. Hence, R_R is n-FP-injective. Similarly, ${}_R R$ is n-FP-injective. In addition, every finitely presented R-module has a $\mathcal{F}_n$-preenvelope. Therefore, R is left and right n-coherent. □

4 Existence of the strongly Gorenstein n-flat dimension

The aim of this section is to investigate the strongly Gorenstein n-flat dimension and discuss the results related to precovers and preenvelopes. Now, we start with the following definition of the strongly Gorenstein flat dimension.

Definition 4 ([4]). *For a right R-module M, let $\mathrm{SGF}(M)$ denote the infimum of the set of n such that there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules, where each G_i is strongly Gorenstein flat and call $\mathrm{SGF}(M)$ the strongly Gorenstein flat dimension of M. If no such n exists, set $\mathrm{SGF}(M) = \infty$*

Motivated by the above definition, we now introduce a definition of the strongly Gorenstein flat n-dimension.

Definition 5. *For a right R-module M, let $\mathrm{SGf_nd}(M)$ denote the infimum of the set of n such that there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules, where each G_i is strongly Gorenstein n-flat and call $\mathrm{SGf_nd}(M)$ the strongly Gorenstein n-flat dimension of M. If no such n exists, set $\mathrm{SGf_nd}(M) = \infty$*

Example 2. (1) From Example 1, $I = (x)$ is a strongly Gorenstein n-flat module, and hence $\mathrm{SGf_nd}(I) = 0$.

(2) Let $R = k[x, y]$ where k is field, and consider the R-module $M = R/(x, y) \cong k$. Then the residue field k has projective dimension 2 because R is Noetherian ring of global dimension 2. A minimal free resolution of k is given by the Koszul complex

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{(x\ \ y)} R \longrightarrow k \longrightarrow 0.$$

Since every term in the resolution is a free R-module, each module is in particular flat and hence strongly Gorenstein flat. Therefore, the above complex provides a strongly Gorenstein n-flat resolution of k of length 2. It is also a strongly Goresntein n-flat resolution of k of length 2. This implies that $\mathrm{SGf_nd}(M) \leq 2$. Since $\mathrm{pd}_R(k) = 2$, strongly Gorenstein n-flat resolution of k of length < 2 is not possible. Therefore $\mathrm{SGf_nd}(M) = 2$.

Lemma 2. *Let R be a left n-coherent ring, and n be a fixed non-negative integer. Consider the following conditions for a right R-module M.*

(1) $\mathrm{SGf_nd}(M) \leq n$.

(2) For any exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ with each P_i n-projective, K_n is strongly Gorenstein n-flat.

(3) $\mathrm{Ext}^{n+i}(M, F) = 0$ *for any n-flat right R-module F and any $i \geq 1$.*

Then (1) and (2) are equivalent and imply (3). Moreover, (3) implies (2) if n-FP-$\mathrm{id}({}_RR) < \infty$.

Proof. (1) $\Rightarrow$ (2). By (1), there exists an exact sequence $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$ of right R-modules with each G_i strongly Gorenstein n-flat. By Proposition 2, K_n is strongly Gorenstein n-flat.

(2) $\Rightarrow$ (1). It is clear from Definition 5.

(2) $\Rightarrow$ (3). For any n-flat right R-module F and $i \geq 1$, we have $\mathrm{Ext}^{n+i}(M, F) \cong \mathrm{Ext}^{i}(K_n, F) = 0$ by (2) and Lemma 1.

(3) $\Rightarrow$ (2) follows from Theorem 3. □

The following Lemma follows from the standard dimension shifting isomorphism, found in basic level homological algebra [20, Proposition 2.4.1].

Lemma 3. *Consider an exact sequence* $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, *where all* G_i *are strongly Gorenstein n-flat. Then,*

$$\mathrm{Ext}^{i}_{R}(G_n, F) \cong \mathrm{Ext}^{n+i}_{R}(M, F)$$

for all integers $i > 0$ *and all modules* F *with* $\mathrm{f_nd}(M) < \infty$.

Theorem 4. *The following statements are equivalent for a two-sided n-coherent ring* R *and a fixed integer* $n \geq 1$.

(1) R *is an n-FC ring.*

(2) $\mathrm{SGf_nd}(M) \leq n$ *for any finitely presented* R*-module* M.

(3) The nth $\mathcal{F}_n$*-cosyzygy of every finitely presented* R*-module is strongly Gorenstein n-flat.*

Proof. (1) $\Rightarrow$ (2). Let M be a finitely presented R-module and F be a n-flat R-module. Then $n\text{-}\,\mathrm{FP}\text{-}\,\mathrm{id}(F) \leq n$ by (1) and thus $\mathrm{Ext}^{n+i}(M, F) = 0$ for all $i \geq 1$. Thus $\mathrm{SGf_nd}(M) \leq n$ by Lemma 2.

(2) $\Rightarrow$ (1). For any finitely presented right R-module M, $\mathrm{Ext}^{n+1}(M, R_R) = 0$ by (2) and Lemma 2. Thus, $n\text{-}\,\mathrm{FP}\text{-}\,\mathrm{id}(R_R) \leq n$. Similarly, it follows that n-FP-id$({}_RR) \leq n$.

(1) $\Rightarrow$ (3). Let M be a finitely presented R-module and $0 \to M \to P^0 \to P^1 \to \cdots$ be a right $\mathcal{F}_n$-resolution of M with each P^i finitely generated n-projective and L^n the nth $\mathcal{F}_n$-cosyzygy. Then the sequence is exact at P^k for $k \geq n-1$ by [7, Theorem 8.4.31]. Therefore, we get an exact sequence, $0 \to L^n \to P^n \to \cdots \to P^{2n-1} \to L^{2n} \to 0$. Note that L^n and L^{2n} are finitely presented. Let F be a n-flat R-module and $n\text{-}\,\mathrm{FP}\text{-}\,\mathrm{id}(F) \leq n$. So, $\mathrm{Ext}^{i}(L^n, F) \cong \mathrm{Ext}^{n+i}(L^{2n}, F) = 0$ for all $i \geq 1$. Thus, L^n is strongly Gorenstein n-flat by Theorem 3.

(3) $\Rightarrow$ (1). Let N be a finitely presented right R-module. Then N admits a right $\mathcal{F}_n$-resolution $\mathcal{E} : 0 \to N \to P^0 \to P^1 \to \cdots$. Note that the nth $\mathcal{F}_n$-cosyzygy L^n is strongly Gorenstein n-flat by (3). Therefore, there is a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $0 \to L^n \to F^1 \to F^2 \to \cdots$ with each F^i n-projective. Thus, $\mathcal{E}$ is exact at P^k for $k \geq n-1$. Hence, $n\text{-}\,\mathrm{FP}\text{-}\,\mathrm{id}({}_RR) \leq n$ by [7, Theorem 8.4.31]. Similarly, $n\text{-}\,\mathrm{FP}\text{-}\,\mathrm{id}(R_R) \leq n$. □

The next result gives a structural description of the strongly Gorenstein n-flat dimension of modules.

Theorem 5. *Let R be an n-coherent ring and M be an R-module with a finite strongly Gorenstein n-flat dimension, and let n be an integer. Then, the following conditions are equivalent:*

1. $\mathrm{SGf_n d}(M) \leq n$
2. $\mathrm{Ext}^i_R(M, L) = 0$ *for all* $i > n$ *and all* R*-modules* L *with* $\mathrm{f_n d}(L) < \infty$
3. $\mathrm{Ext}^i_R(M, F) = 0$ *for all* $i > n$ *and all* n*-flat* R*-modules* F
4. *For every exact sequence* $0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ *where* $G_0, \cdots, G_{n-1}$ *are strongly Gorenstein* n*-flat,* K_n *is also strongly Gorenstein* n*-flat.*

Proof. (1) $\Rightarrow$ (2). Assume that $\mathrm{SGf_n d}(M) \leq n$. By definition, we have $0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, where each G_i is strongly Gorenstein n-flat. By Lemma 3 and Lemma 1(1) we conclude that the equalities, $\mathrm{Ext}^i_R(M, L) \cong \mathrm{Ext}^{i-m}_R(G_n, L)$ whenever L has a finite n-flat dimension and $i > n$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4). Consider $\mathrm{SGf_n d}(M) = m < \infty$ and suppose $n < m$. Consider an exact sequence $0 \to G_m \to \cdots \to G_0 \to M \to 0$, where all G_i are a strongly Gorenstein n-flat module. Set $K_{m-1} = \mathrm{coker}(G_m \to G_{m-1})$. Thus, by the hypothesis and Lemma 3, from the exact sequence $0 \to K_{m-1} \to G_{m-2} \to \cdots \to G_0 \to M \to 0$ we get $\mathrm{Ext}^1_R(K_{m-1}, F) = \mathrm{Ext}^m_R(M, F) = 0$ for all n-flat module F. Hence, by applying Proposition 2 in the short exact sequence $0 \to G_m \to G_{m-1} \to K_{m-1} \to 0$, we conclude that K_{m-1} is a strongly Gorenstein n-flat module. Then $\mathrm{SGf_n d}(M) \leq m - 1$, which is impossible. This contradiction completes the proof.

(4) $\Rightarrow$ (1) is obvious. □

Let $\mathcal{SGF}_n$ be the class of all strongly Gorenstein n-flat R-modules, and $rSGf_nd(R)$ stands for right strongly Gorenstein n-flat dimension of a ring. We now discuss some results related to precovers and preenvelopes by strongly Gorenstein n-flat modules.

Theorem 6. *Let R be a left n-coherent ring and n a fixed non-negative integer such that $rSGf_nd(N) \leq n$ for any finitely presented right R-module N. Then every finitely presented right R-module has a $\mathcal{SGF}_n$-preenvelope.*

Proof. Let M be a finitely presented right R-module. Then there is $\mathrm{Hom}(-, \mathcal{F}_n)$ an exact complex $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L^n \to 0$ with each F_i finitely generated n-projective, since R is left n-coherent. Note that there exists an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to L^n \to 0$ with each P_i n-projective. Thus, K_n is strongly Gorenstein n-flat by Lemma 2, since $\mathrm{SGf_n d}(L^n) \leq n$. For any strongly Gorenstein n-flat right R-module G and any homomorphism $f : M \to G$, there is a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $0 \to G \to H^0 \to H^1 \to \cdots \to H^{n-1} \to Q^n \to 0$ with each H^i n-projective and we obtain the following commutative diagram

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & \cdots & \longrightarrow & F^{n-1} & \longrightarrow & L^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & G & \longrightarrow & H^0 & \longrightarrow & H^1 & \longrightarrow & \cdots & \longrightarrow & H^{n-1} & \longrightarrow & Q^n & \longrightarrow & 0
\end{array}$$

But we also have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \longrightarrow & F^0 & \longrightarrow & F^1 & \longrightarrow & \cdots & \longrightarrow & F^{n-1} & \longrightarrow & L^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_n} & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & L^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_n} & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & G & \longrightarrow & H^0 & \longrightarrow & H^1 & \longrightarrow & \cdots & \longrightarrow & H^{n-1} & \longrightarrow & Q^n & \longrightarrow & 0
\end{array}$$

Then the usual homotopy argument shows that $f - g_n h_n$ factors through F^0. Hence, $f : M \to G$ factors through the strongly Gorenstein n-flat right R-module $K_n \oplus F^0$. Thus $M \to K_n \oplus F^0$ is an $\mathcal{SGF}_n$-preenvelope. □

Theorem 7. *Let R be an n-coherent ring. Then every finitely presented right R-module M has a $\mathcal{SGF}_n$-preenvelope $M \to P$ with P finitely presented.*

Proof. This result follows from Theorem 4 and Theorem 6. □

Theorem 8. *Let R be a left n-coherent ring and n be a fixed nonnegative integer with $rSGf_nd(R) \leq n$. Then every right R-module M has a $\mathcal{SGF}_n$-precover $\alpha : N \to M$ such that if $0 \to K \to N \xrightarrow{\alpha} M \to 0$ is exact, then $\mathrm{p_n d}(K) \leq n-1$ whenever $n \geq 1$. Moreover, if $\mathrm{p_n d}(M) < \infty$, then N is n-projective.*

Proof. By hypothesis, there is an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such that each P_i is n-projective and K_n is strongly Gorenstein n-flat by Lemma 2 since $\mathrm{SGf_n d}(M) \leq n$. Then there exists a $\mathrm{Hom}(-, \mathcal{F}_n)$ exact sequence $0 \to K_n \to P^0 \to \cdots \to P^{n-2} \to P^{n-1} \to L^n \to 0$ with each P^i n-projective. Note that L^n is strongly Gorenstein n-flat. Therefore, the following diagram commutes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & K_n & \longrightarrow & P^0 & \longrightarrow & \cdots & \longrightarrow & P^{n-2} & \longrightarrow & P^{n-1} & \longrightarrow & L^n & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h_0} & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h_n} & & \\
0 & \longrightarrow & K_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

By [7], we get an exact sequence

$$0 \to K_n \to P^0 \oplus K_n \to P^1 \oplus P_{n-1} \to \cdots \to P^{n-1} \oplus P_1 \to L^n \oplus P_0 \to M \to 0$$

which gives the exactness of the sequence

$$0 \to P^0 \to P^1 \oplus P_{n-1} \to \cdots \to P^{n-1} \oplus P_1 \to L^n \oplus P_0 \to M \to 0.$$

Let $N = L^n \oplus P_0$. Consider the short exact sequence $0 \to K \to N \to M \to 0$. It is clear that $\mathrm{p_n d}(K) \leq n-1$ and N is strongly Gorenstein n-flat. Thus, $\mathrm{Ext}^1(H, K) = 0$ for any strongly Gorenstein n-flat right R-module H by Lemma 1(1). Thus, $N \to M$ is an $\mathcal{SGF}_n$-precover. If $\mathrm{p_n d}(M) < \infty$, then $\mathrm{p_n d}(N) < \infty$ since $\mathrm{p_n d}(K) < \infty$. It follows that N is n-projective by Lemma 1(2). □

Now, we are ready to prove the following theorem, for any R-module M with finite n-flat dimension, the strongly Gorenstein n-flat dimension of M coincides with the n-projective dimension.

Theorem 9. *Let R be an n-coherent ring. If $M \in \operatorname{Mod} R$ with $\mathrm{f_n d}(M) < \infty$, then $\mathrm{SGf_n d}(M) = \mathrm{p_n d}(M)$*

Proof. It is enough to show that $\mathrm{p_n d}(M) \leq \mathrm{SGf_n d}(M)$. We may assume that $\mathrm{SGf_n d}(M) < \infty$. If M is strongly Gorenstein n-flat, then there exists an exact sequence $0 \to M \to P \to M' \to 0$ in $\operatorname{Mod} R$, where $M^{'}$ is strongly Gorenstein n-flat and P is n-projective. Since $\mathrm{f_n d}(M) < \infty$, $\operatorname{Ext}^1_R(M^{'}, M) = 0$ by Theorem 3. This implies that M is a direct summand of P, and therefore M is n-projective. Now, let $\mathrm{SGf_n d}(M) = n > 0$. By Theorem 8, there exists an exact sequence $0 \to K \to G \to M \to 0$ in $\operatorname{Mod} R$, where G is strongly Gorenstein n-flat and $\mathrm{p_n d}(K) = n - 1$. Since G is strongly Gorenstein n-flat, there exists an exact sequence $0 \to G \to P \to G^{'} \to 0$ in $\operatorname{Mod} R$, where P is n-projective and $G^{'}$ is strongly Gorenstein n-flat. Now consider the following push-out diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & W & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since P is n-projective and $\mathrm{p_n d}\, K = n-1$, we have $\mathrm{p_n d}\, W \leq n$ by the exactness of the middle row in the above diagram. Note that $G^{'}$ is strongly Gorenstein n-flat, then $\operatorname{Ext}^1(G^{'}, M) = 0$ by Theorem 3, and hence $W \cong M \oplus G^{'}$. This implies $\mathrm{p_n d}(M) \leq \mathrm{p_n d}(W) \leq n$. □

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