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State-dependent memory in fractional integro-differential equations

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Abstract. This paper studies a class of nonlinear fractional integro-differential equations in which the memory term depends on the state of the unknown function, subject to nonlocal multi-point boundary conditions in the space $C([0,1],E)$, where E is an infinite-dimensional Banach space. The problem under consideration features a nonlinear integral term governed by the unknown function, together with a generalized nonlocal initial condition. Existence of solutions is derived through a combination of the measure of noncompactness and Darbo's fixed point theorem; under additional Lipschitz-type conditions, uniqueness and Ulam-Hyers stability of the solution are also established. The theory is illustrated through three examples, for one of which numerical computations are also presented.

Keywords: Fractional integro-differential equations; nonlocal conditions; Kuratowski measure of noncompactness; Darbo's fixed point theorem; Ulam-Hyers stability
AMS Subject Classification 2010: 34A08, 47H10.

1 Introduction

Over the past few decades, fractional calculus has emerged as a powerful tool for modeling phenomena whose evolution depends not only on the present state of the system but also on its entire past history. This capacity to encode memory effects arises naturally in viscoelasticity, control theory, population biology, and several branches of engineering. Readers seeking a detailed treatment of fractional calculus and its applications may consult [15, 20–22, 27].

To fix ideas, recall that for $0 < \alpha < 1$ the Caputo derivative of a sufficiently smooth function ω is

$$D^{\alpha}\omega(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\,\omega'(s)\,\mathrm{d}s,$$

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that is, a weighted integral of the whole past of ω' over $[0,t]$ with the weight $\dfrac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}$. This weight is largest for instants s close to t and decays slowly (not exponentially) as s moves back to 0, which is precisely the memory effect; as $\alpha \longrightarrow 1^-$ the weight concentrates at $s=t$ and $D^{\alpha}\omega(t) \to \omega'(t)$ (see Figure 1).

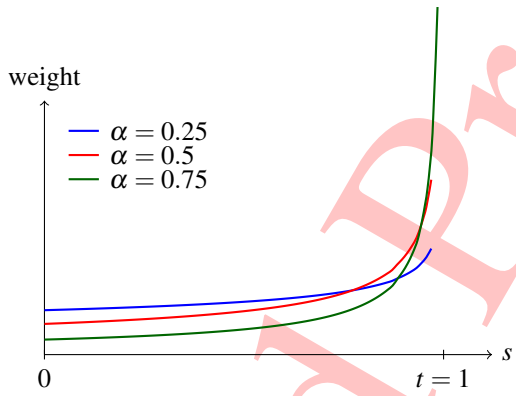


Figure 1: The memory weight $\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}$ of the Caputo derivative at $t=1$ for three values of α

The study of differential equations subject to nonlocal conditions has grown considerably, motivated by physical situations in which the initial state cannot be reduced to a single point value but instead depends on the full history of the system. Byszewski [11] was among the first to establish existence and uniqueness results for semilinear nonlocal Cauchy problems; this line of research was later extended by several authors, notably in [2, 8]. Fractional problems with an integral boundary condition of the type $u(0) = \lambda \int_0^1 g(s)u(s)\,\mathrm{d}s + d$ were studied in [25], where existence and uniqueness of positive solutions are obtained for a generalized Caputo derivative by means of upper and lower solutions and the Schauder and Banach fixed point theorems.

Fractional integro-differential equations involving memory terms have likewise attracted growing interest, owing to their ability to capture processes with after-effects. Integral terms whose kernel depends jointly on present and past states yield models that are more faithful to physical reality, which has motivated a substantial body of work on existence and qualitative properties of solutions [1].

Among the tools available for this purpose, Darbo's fixed point theorem [12] occupies a central place, as it generalizes the classical Schauder theorem through the notion of measure of noncompactness [5, 6]. This approach has proved effective across a wide range of nonlinear problems.

The use of measures of noncompactness in the analysis of fractional integro-differential equations has advanced considerably in recent years. Several authors have obtained existence results for fractional integral and differential equations by combining Darbo or Mönch-type theorems with generalized measures of noncompactness. For example, the authors of [14, 23] established solvability results for fractional functional equations, while [7] investigated Hilfer-type fractional differential equations with mixed fractional integral boundary conditions through this same approach. In addition, [13] established new fixed point theorems via measure of noncompactness and applied them to a fractional integral equation, while [4] studied (k,ψ)-Riemann-Liouville fractional differential equations with nonlocal conditions on

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an unbounded domain. Controllability aspects of fractional systems using this framework were examined in [17], and more recent contributions, such as [16], have addressed hybrid fractional differential equations. Impulsive differential equations and inclusions are treated in [9].

Building on these contributions, the present paper addresses a class of nonlinear fractional integro-differential equations featuring state-dependent memory effects. By state-dependent memory we mean a memory term of the form $\int_0^t \eta(t,s,\omega(s))\,\mathrm{d}s$ in which the integrand depends nonlinearly on the value $\omega(s)$ taken by the unknown function at past instants s, so that the contribution of the past to the dynamics is determined by the state itself. This differs from the classical memory term $\int_0^t k(t,s)\,\omega(s)\,\mathrm{d}s$ (or $\int_0^t k(t,s)\varphi(s)\,\mathrm{d}s$ with φ prescribed), in which the past acts through fixed linear weights. Strictly speaking, $\int_0^t \eta(t,s,\omega(s))\,\mathrm{d}s$ is a nonlinear Volterra integral operator of Urysohn type; what is new here is its presence inside the nonlinearity of a fractional equation, coupled with a nonlocal initial condition that itself involves an integral term, in an infinite-dimensional Banach space, where compactness fails (see Remark 1).

To handle this problem, we rely on the Kuratowski measure of noncompactness [5] together with Darbo's fixed point theorem [12], which jointly allow us to bypass the compactness obstruction described in Remark 1. We further analyze the Ulam-Hyers stability [10,19,24,26] of the resulting solution, which guarantees robustness with respect to small perturbations.

2 Preliminaries

In this section, we introduce some basic definitions and results that will be used throughout the paper. Let E be a Banach space of infinite dimension, with norm $\|\cdot\|$, and let $\chi = C([0,1],E)$ be the Banach space of continuous functions on $[0,1]$ equipped with the norm

$$\|\omega\| = \sup_{t\in[0,1]} \|\omega(t)\|.$$

Integrals of E-valued functions are understood in the Bochner sense.

Definition 1. *([20, 22]) Let $\omega \in L^1([0,1],E)$. The Riemann-Liouville fractional integral of order $\alpha > 0$ associated with ω is the map defined for $t \in [0,1]$ by*

$$I^\alpha \omega(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\omega(s)\,\mathrm{d}s.$$

Definition 2. *([15, 20]) For $0 < \alpha < 1$ and $\omega \in \chi$, put $v = I^{1-\alpha}(\omega - \omega(0))$. The Caputo fractional derivative of order α of ω is*

$$D^\alpha \omega(t) = v'(t), \qquad t \in [0,1],$$

whenever v is continuously differentiable on $[0,1]$. If $\omega(t) = \omega(0) + \int_0^t \omega'(s)\,\mathrm{d}s$ with $\omega' \in L^1([0,1],E)$ (in particular if ω is absolutely continuous), this reduces to the usual formula

$$D^\alpha \omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha}\,\omega'(s)\,\mathrm{d}s, \qquad t \in [0,1].$$

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Lemma 1. *([15, 20]) Let $0<\alpha<1$, $\omega_0 \in E$, and $f \in C([0,1],E)$. A function $\omega \in \chi$ is a solution of the problem*

$$\begin{cases} D^{\alpha}\omega(t) = f(t), & t \in [0,1] \\ \omega(0) = \omega_0, \end{cases} \tag{1}$$

if and only if

$$\omega(t) = \omega_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s.$$

Proof. ($\Leftarrow$) Put $u = \omega - \omega_0 = I^{\alpha} f$, so $\omega(0) = \omega_0$.
By the semigroup property, $I^{1-\alpha}u = I^{1-\alpha} \circ I^{\alpha} f = I^1 f$, which is continuously differentiable with derivative f. Hence $D^{\alpha}\omega = f$.

($\Rightarrow$) Let $u = \omega - \omega_0 \in \chi$, so $u(0) = 0$, and $v = I^{1-\alpha}u$. By assumption $v \in C^1$ and $v' = f$. Since u is continuous with $u(0) = 0$, we have $\|v(t)\| \leq \dfrac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sup_{s\in[0,t]} \|u(s)\| \to 0$ as $t \to 0$, so $v(0) = 0$ and $v = I^1 f$. Applying I^{α} gives $I^1 u = I^{\alpha} \circ I^{1-\alpha} u = I^{\alpha} \circ I^1 f = I^1 (I^{\alpha} f)$. Both u and $I^{\alpha} f$ are continuous, so differentiating yields $u = I^{\alpha} f$ [20]. □

Definition 3. *([5]) Let X be a Banach space ($X = \chi$ or $X = E$) and let $F \subset X$ be bounded. The Kuratowski measure of noncompactness of F, denoted $\mathscr{M}(F)$, is given by*

$$\mathscr{M}(F) = \inf\{d > 0 : \ F \text{ admits a finite cover by sets of diameter at most } d\}.$$

Let $\Omega \subset \chi$. An operator $\mathscr{A} : \Omega \longrightarrow \chi$ is called a δ-contraction if it is continuous, maps bounded sets into bounded sets, and there exists $\delta \in [0,1)$ such that

$$\mathscr{M}(\mathscr{A}(F)) \leq \delta \mathscr{M}(F) \quad \text{for every bounded } F \subset \Omega.$$

Theorem 1. *([5, 12]) Let $\Omega \subset \chi$ be nonempty, closed, bounded and convex, and let $\mathscr{A} : \Omega \to \Omega$ be a δ-contraction for some $\delta \in [0,1)$. Then $\mathscr{A}$ admits at least one fixed point in Ω.*

Proof. This is Darbo's fixed point theorem [12] (see also [5]). □

Let $F, F_1, F_2 \subset X$ be bounded. We shall also use the following standard properties of $\mathscr{M}$ [3, 5, 18]:
(**i**) $\mathscr{M}(F_1 + F_2) \leq \mathscr{M}(F_1) + \mathscr{M}(F_2)$, $\mathscr{M}(\lambda F) = |\lambda| \mathscr{M}(F)$ for $\lambda \in \mathbb{R}$, and $\mathscr{M}(F + x) = \mathscr{M}(F)$.
(**ii**) If T is a bounded linear operator, then $\mathscr{M}(TF) \leq \|T\| \mathscr{M}(F)$; more generally $\mathscr{M}(f(F)) \leq \ell \mathscr{M}(F)$ if f is ℓ-Lipschitz. Moreover, with the norm $\|(x,y)\| = \|x\| + \|y\|$ on $E \times E$, $\mathscr{M}(F_1 \times F_2) \leq \mathscr{M}(F_1) + \mathscr{M}(F_2)$. For $F \subset \chi$ and $t \in [0,1]$ we write $F(t) = \{\omega(t) : \omega \in F\}$; then $\mathscr{M}(F(t)) \leq \mathscr{M}(F)$.
(**iii**) Let $W \subset L^1([a,b],E)$ be integrably bounded, and suppose $\mathscr{M}(W(s)) \leq \vartheta(s)$ for a.e. s, where $W(s) = \{w(s) : w \in W\}$ and $\vartheta \in L^1([a,b],\mathbb{R}_+)$. Then

$$\mathscr{M}\left(\left\{\int_a^b w(s)\,\mathrm{d}s : \ w \in W\right\}\right) \leq 2\int_a^b \vartheta(s)\,\mathrm{d}s.$$

(**iv**) If $F \subset \chi$ is bounded and equicontinuous, then $\mathscr{M}(F) = \sup_{t\in[0,1]} \mathscr{M}(F(t))$.

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3 Main result

Consider the boundary value problem:

$$\begin{cases} D^{\alpha}\omega(t) = h(t),\ t \in [0,1],\ 0 < \alpha < 1, \\ \omega(0) = \omega_0 + \sum_{i=1}^{n} b_i \omega(t_i) + \int_0^1 g(s, \omega(s))\,\mathrm{d}s, \end{cases} \tag{2}$$

where $\omega_0 \in E$, $b_i \in \mathbb{R}$, $t_i \in [0,1]$, $i = 1,...,n$, D^{α} is the Caputo derivative of order α, with$0 < \alpha < 1$ and $h : [0,1] \longrightarrow E$, $g : [0,1] \times E \longrightarrow E$ are continuous functions.

Lemma 2. *Let $h \in C([0,1], E)$. A function ω solves problem (2) if and only if*

$$\omega(t) = \omega_0 + \sum_{i=1}^{n} b_i \omega(t_i) + \int_0^1 g(s, \omega(s))\,\mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s, \quad t \in [0,1].$$

Proof. If ω solves (2), Lemma 1 (with initial value $\omega(0)$) gives $\omega(t) = \omega(0) + I^{\alpha}h(t)$; inserting the nonlocal condition for $\omega(0)$ gives the formula.
Conversely, taking $t = 0$ in the formula and using $I^{\alpha}h(0) = 0$ gives the nonlocal condition;
then $\omega(t) = \omega(0) + I^{\alpha}h(t)$ and Lemma 1 gives $D^{\alpha}\omega = h$. □

Now we turn to the following problem:

$$\begin{cases} D^{\alpha}\omega(t) = O\omega(t) + h\left(t, \omega(t), \int_0^t \eta(t,s,\omega(s))\,\mathrm{d}s\right),\ t \in [0,1] \\ \omega(0) = \omega_0 + \sum_{i=1}^{n} b_i \omega(t_i) + \int_0^1 g(s, \omega(s))\ \mathrm{d}s. \end{cases} \tag{3}$$

Throughout, D^{α} stands for the Caputo fractional derivative of order $0 < \alpha < 1$ (Definition 2).

Here $O \in \mathscr{L}(\chi)$, where $\mathscr{L}(\chi)$ denotes the space of bounded linear operators on χ equipped with the operator norm $\|O\|_{\mathscr{L}(\chi)}$, and $K := \|O\|_{\mathscr{L}(\chi)}$, so that $\|O\omega\| \leq K\|\omega\|$, for all $\omega \in \chi$. The maps $h : [0,1] \times E \times E \longrightarrow E$, $g : [0,1] \times E \longrightarrow E$, and $\eta : [0,1] \times [0,1] \times E \longrightarrow E$ are continuous functions. A solution of (3) is a function $\omega \in \chi$ for which $D^{\alpha}\omega$ exists in the sense of Definition 2 and (3) holds.

Remark 1. *Since $\dim E = \infty$, the closed unit ball of E is not relatively compact. Consequently, bounded subsets of E may fail to be relatively compact. The operator O is assumed only to be bounded and not necessarily compact (for instance $O = \kappa\mathrm{Id}$ with $\kappa \neq 0$). In general neither $\mathscr{O}_1$ nor $\mathscr{O}_2$ below is compact. Indeed, the Arzelà-Ascoli theorem requires, besides equicontinuity, that the sets $\{\omega(t) : \omega \in F, t \in [0,1]\}$ be relatively compact in E; this may fail for bounded sets here. Therefore, Schauder's theorem is not available, which motivates the use of the Kuratowski measure of noncompactness and Darbo's fixed point theorem together with suitable assumptions on the measure of noncompactness of the nonlinear terms. We note that for $E = \mathbb{R}$ (or $\dim E < \infty$) the situation is different; I^{α} maps bounded sets into bounded equicontinuous sets, so $\mathscr{A}$ is compact by the Arzelà-Ascoli theorem and existence follows directly from Schauder's theorem; the measure of noncompactness is genuinely needed only in the infinite-dimensional setting considered here. For the same reason, the inequalities (7) below concern bounded subsets B, D of E; they would be trivially satisfied for $E = \mathbb{R}$, where every bounded set has a measure of noncompactness*

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of zero, but they are substantive when $\dim E = \infty$. *They are transferred to families of functions of* χ *in the proof of Theorem 2 through the properties* (**ii**)-(**iv**) *of* $\mathscr{M}$*; pointwise bounds on* $\mathscr{M}(F(s))$*, an integral estimate* (**iii**)*, and the equicontinuity of* $\mathscr{O}_2 F$*, which allows us to pass from pointwise estimates to* $\mathscr{M}(\mathscr{O}_2 F)$ (**iv**).

We establish below the existence of a solution to problem (3).

Theorem 2. *Suppose there exist nonnegative functions* $\rho \in C([0,1],\mathbb{R}_+)$, $\mu \in L^1([0,1],\mathbb{R}_+)$ *such that, for all* $t, s \in [0,1]$, *and for all* $(x,y) \in E \times E$,

$$\|h(t,x,y)\| \leq \rho(t)(\|x\| + \|y\|), \tag{4}$$

$$\|\eta(t,s,y)\| \leq c\|y\|, \quad c > 0, \tag{5}$$

and

$$\|g(t,x)\| \leq \mu(t)\|x\|. \tag{6}$$

Moreover, for all $t, s \in [0,1]$, *and for all bounded sets* $B, D \subset E$,

$$\begin{aligned} \mathscr{M}(h(t,B,D)) &\leq \rho(t)(\mathscr{M}(B) + \mathscr{M}(D)), \\ \mathscr{M}(\eta(t,s,B)) &\leq c\,\mathscr{M}(B), \\ \mathscr{M}(g(t,B)) &\leq \mu(t)\,\mathscr{M}(B). \end{aligned} \tag{7}$$

If

$$\sum_{i=1}^{n} |b_i| + 2\left(\mu_0 + \frac{K + \rho_0(1+2c)}{\Gamma(\alpha+1)}\right) < 1, \tag{8}$$

where $\mathscr{M}$ *is as in Definition 3,* $\|O\omega\| \leq K\|\omega\|$, $\rho_0 = \sup_{s\in[0,1]} \rho(s)$, *and* $\mu_0 = \int_0^1 \mu(t)\,\mathrm{d}t$, *then problem* (3) *admits at least one solution in* χ.

Proof. **Step 1:** Fix $R > 0$ large enough, and set $\overline{B}_R = \{\omega \in \chi : \|\omega\| \leq R\}$. For $\omega \in \chi$ the function $s \longmapsto O\omega(s) + h\left(s, \omega(s), \int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi\right)$ belongs to $C([0,1],E)$. By Lemma 2, $\omega \in \chi$ solves problem (3) if and only if $\omega = \mathscr{A}\omega$, where $\mathscr{A} := \mathscr{O}_1 + \mathscr{O}_2$, with

$$\mathscr{O}_1\omega(t) = \omega_0 + \sum_{i=1}^{n} b_i\omega(t_i) + \int_0^1 g(s,\omega(s))\,\mathrm{d}s, \tag{9}$$

$$\mathscr{O}_2\omega(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\mathscr{O}\omega(s) + h\left(s,\omega(s),\int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi\right)\right)\mathrm{d}s. \tag{10}$$

In particular, a fixed point of $\mathscr{A}$ is a solution in the sense of Definition 2.

Step 2: $\mathscr{A}(\overline{B}_R) \subset \overline{B}_R$. Let $\omega \in \overline{B}_R$, $t \in [0,1]$. Using (9), (6) and $\mu_0 = \int_0^1 \mu(t)\,\mathrm{d}t$, we have

$$\|\mathscr{O}_1\omega(t)\| \leq \|\omega_0\| + \sum_{i=1}^{n} |b_i|\,\|\omega(t_i)\| + \int_0^1 \|g(s,\omega(s))\|\,\mathrm{d}s$$

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$$\leq \|\omega_0\| + \Big(\sum_{i=1}^{n} |b_i| + \mu_0\Big)\|\omega\|.$$

Using (10), the estimate $\|O\omega(s)\| \leq \|O\omega\| \leq K\|\omega\|$ for every $s \in [0,1]$, and (4), (5),

$$\|\mathscr{O}_2\omega(t)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(K\|\omega\| + \rho(s)\Big(\|\omega\| + \int_0^s \|\eta(s,\xi,\omega(\xi))\|\,\mathrm{d}\xi\Big)\Big)\,\mathrm{d}s$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(K\|\omega\| + \rho(s)\big(\|\omega\| + c\|\omega\|s\big)\Big)\,\mathrm{d}s.$$

Since $0 \leq s \leq 1$,

$$\|\mathscr{O}_2\omega(t)\| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(K\|\omega\| + \rho(s)(1+c)\|\omega\|\big)\,\mathrm{d}s$$
$$\leq \frac{K\|\omega\|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s + \frac{(1+c)\|\omega\|}{\Gamma(\alpha)}\rho_0\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s$$
$$\leq \frac{K+(1+c)\rho_0}{\Gamma(\alpha+1)}\|\omega\|.$$

It follows that

$$\|\mathscr{A}\omega\| \leq \|\omega_0\| + L\|\omega\| \leq \|\omega_0\| + LR \leq R,$$

where

$$L := \sum_{i=1}^{n} |b_i| + \mu_0 + \frac{K + \rho_0(1+c)}{\Gamma(\alpha+1)}, \tag{11}$$

provided $R \geq \dfrac{\|\omega_0\|}{1-L}$ and $L < 1$. Set

$$\delta := \sum_{i=1}^{n} |b_i| + 2\left(\mu_0 + \frac{K + \rho_0(1+2c)}{\Gamma(\alpha+1)}\right). \tag{12}$$

Comparing with (11), we get

$$\delta - L = \mu_0 + \frac{K + \rho_0(1+3c)}{\Gamma(\alpha+1)} \geq 0.$$

Since μ_0, K, ρ_0, $c \geq 0$ and $\Gamma(\alpha+1) > 0$, we have $L \leq \delta < 1$ by (8). If $\omega_0 = 0$, any $R > 0$ satisfies the required inequality, since $LR \leq R$ whenever $L < 1$. Consequently, $\mathscr{A}(\overline{B}_R) \subset \overline{B}_R$.

Step 3: Continuity. Let $\omega_k \to \omega$ in $\overline{B}_R$. The set $\{(t,\omega(t)) : t \in [0,1]\}$ is compact in $[0,1] \times E$, and g, η, h are continuous; hence, for every $\varepsilon > 0$ there is $r > 0$ such that $\|g(t,y) - g(t,\omega(t))\| < \varepsilon$ whenever $\|y - \omega(t)\| < r$, for all t (and similarly for η and h on the corresponding compact sets); thus, $g(\cdot,\omega_k(\cdot)) \to g(\cdot,\omega(\cdot))$ uniformly, $\int_0^s \eta(s,\xi,\omega_k(\xi))\,\mathrm{d}\xi \to \int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi$ uniformly in s, so that $h(\cdot,\omega_k(\cdot),\cdot) \to$

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$h(\cdot,\omega(\cdot),\cdot)$ uniformly. Since O is bounded and I^α is linear and continuous from $C([0,1],E)$ into itself, $\mathscr{O}_1\omega_k \to \mathscr{O}_1\omega$ and $\mathscr{O}_2\omega_k \to \mathscr{O}_2\omega$ in χ. Thus, $\mathscr{A}$ is continuous on $\overline{B}_R$.

Step 4: $\mathscr{A}$ is a δ-contraction.

It remains to show that there exists $\delta \in [0,1)$ such that $\mathscr{M}(\mathscr{A}F) \le \delta\,\mathscr{M}(F)$, for every $F \subset \overline{B}_R$. Since $\mathscr{A}\omega = \mathscr{O}_1\omega + \mathscr{O}_2\omega$, the subadditivity of $\mathscr{M}$ gives

$$\mathscr{M}(\mathscr{A}F) \le \mathscr{M}(\mathscr{O}_1F) + \mathscr{M}(\mathscr{O}_2F).$$

Estimate of $\mathscr{M}(\mathscr{O}_1F)$. Each element of $\mathscr{O}_1F$ is a constant function, so $\mathscr{M}(\mathscr{O}_1F)$ equals the measure of noncompactness in E of

$$S = \left\{ \omega_0 + \sum_{i=1}^{n} b_i\,\omega(t_i) + \int_0^1 g(s,\omega(s))\,\mathrm{d}s : \ \omega \in F \right\}.$$

By (**i**) and (**ii**), $\mathscr{M}\left(\left\{\sum_i b_i\omega(t_i)\right\}\right) \le \sum_i |b_i|\,\mathscr{M}(F)$. For the integral term, $W = \{g(\cdot,\omega(\cdot)) : \omega \in F\}$ is integrably bounded by μR, and by (7),

$$\mathscr{M}(W(s)) \le \mathscr{M}(g(s,F(s))) \le \mu(s)\mathscr{M}(F);$$

by (**iii**),

$$\mathscr{M}\left(\left\{\int_0^1 g(s,\omega(s))\,\mathrm{d}s\right\}\right) \le 2\int_0^1 \mu(s)\,\mathscr{M}(F)\,\mathrm{d}s = 2\mu_0\,\mathscr{M}(F).$$

Hence

$$\mathscr{M}(\mathscr{O}_1F) \le \left(\sum_{i=1}^{n} |b_i| + 2\mu_0\right)\mathscr{M}(F).$$

Estimate of $\mathscr{M}(\mathscr{O}_2F)$. Write $\varphi_\omega(s) = O\omega(s) + h\left(s,\omega(s),\int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi\right)$. For $\omega \in F$, $\|\varphi_\omega(s)\| \le C_R := (K + \rho_0(1+c))R$. Hence, for $0 \le t_1 < t_2 \le 1$,

$$\begin{aligned}\|\mathscr{O}_2\omega(t_2) - \mathscr{O}_2\omega(t_1)\| &\le \frac{C_R}{\Gamma(\alpha+1)}\left(t_1^\alpha - t_2^\alpha + 2(t_2-t_1)^\alpha\right)\\ &\le \frac{2C_R}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha,\end{aligned}$$

so $\mathscr{O}_2F$ is bounded and equicontinuous, and by (**iv**)

$$\mathscr{M}(\mathscr{O}_2F) = \sup_{t\in[0,1]} \mathscr{M}(\mathscr{O}_2F(t)).$$

Let $W(s) = \{\varphi_\omega(s) : \omega \in F\}$ and $D(s) = \left\{\int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi : \omega \in F\right\}$. Since $\{O\omega(s) : \omega \in F\} = (OF)(s)$, (**ii**) gives

$$\mathscr{M}\{O\omega(s) : \omega \in F\} \le \mathscr{M}(OF) \le K\,\mathscr{M}(F).$$

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By (**ii**), (**iii**) and (7), $\mathscr{M}(D(s)) \le 2\int_0^s c\,\mathscr{M}(F)\,\mathrm{d}\xi \le 2c\,\mathscr{M}(F)$. Using (7) and $\mathscr{M}(F(s)) \le \mathscr{M}(F)$,

$$\begin{aligned}\mathscr{M}(W(s)) &\le K\,\mathscr{M}(F) + \rho(s)\big(\mathscr{M}(F) + 2c\,\mathscr{M}(F)\big)\\ &\le \big(K + \rho_0(1+2c)\big)\mathscr{M}(F).\end{aligned}$$

Applying (**iii**) to the family $\left\{\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\varphi_\omega(s)\right\}$ gives

$$\begin{aligned}\mathscr{M}(\mathscr{O}_2F(t)) &\le \frac{2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big(K+\rho_0(1+2c)\big)\mathscr{M}(F)\,\mathrm{d}s\\ &\le 2\,\frac{K+\rho_0(1+2c)}{\Gamma(\alpha+1)}\,\mathscr{M}(F).\end{aligned}$$

Adding the two estimates,

$$\mathscr{M}(\mathscr{A}F) \le \delta\,\mathscr{M}(F)$$

with δ given by (12). By (8), $\delta < 1$; clearly $\delta \ge 0$ (the case $\delta = 0$ occurs only if all the constants vanish). Recall that $L \le \delta < 1$ by Step 2. Hence, $\mathscr{A}$ is a δ-contraction. By Theorem 1 with $\Omega = \overline{B}_R$, $\mathscr{A}$ has a fixed point in $\overline{B}_R$, and by Step 1 problem (3) admits at least one solution in χ. □

We now investigate the stability of problem (3) in χ.

Definition 4. *([10, 19, 24, 26]) Let $\varepsilon > 0$. A function $\widetilde{\omega} \in \chi$ is called an ε-solution of problem (3) if $D^\alpha\widetilde{\omega}$ exists and belongs to χ, and*

$$\left\| D^\alpha\widetilde{\omega}(t) - O\widetilde{\omega}(t) - h\left(t,\widetilde{\omega}(t),\int_0^t \eta\,(t,s,\widetilde{\omega}(s))\,\mathrm{d}s\right)\right\| \le \varepsilon,\quad t\in[0,1],$$

$$\left\| \widetilde{\omega}(0) - \omega_0 - \sum_{i=1}^n b_i\,\widetilde{\omega}(t_i) - \int_0^1 g(s,\widetilde{\omega}(s))\,\mathrm{d}s\right\| \le \varepsilon.$$

The solutions of problem (3) are said to be Ulam-Hyers stable if there exists $\sigma > 0$ such that, for every $\varepsilon > 0$ and every ε-solution $\widetilde{\omega}$ of (3), there exists a solution ω of (3) with

$$\|\omega - \widetilde{\omega}\| \le \sigma\varepsilon,$$

where σ is independent of $\widetilde{\omega}$.

Theorem 3. *Assume that ρ, μ and c are as in Theorem 2, that (8) holds, and that, for all $t, s \in [0,1]$, and all $(x_1,y_1),(x_2,y_2) \in E\times E$,*

$$\begin{aligned}\|h(t,x_1,y_1) - h(t,x_2,y_2)\| &\le \rho(t)\,(\|x_1-x_2\| + \|y_1-y_2\|),\\ \|\eta(t,s,y_1) - \eta(t,s,y_2)\| &\le c\,\|y_1-y_2\|,\\ \|g(t,x_1) - g(t,x_2)\| &\le \mu(t)\,\|x_1-x_2\|.\end{aligned}$$

Then problem (3) has a unique solution ω in χ, and it is Ulam-Hyers stable in χ.

Corrected Proof

Proof. For $\omega_1, \omega_2 \in \chi$, the same computations as in Step 2 of the proof of Theorem 2, applied to differences (and using that O is linear), give $\|\mathscr{A}\omega_1 - \mathscr{A}\omega_2\| \le L\|\omega_1 - \omega_2\|$, with L given by (11). Since $L \le \delta < 1$ by (8), $\mathscr{A}$ is a contraction on χ, and by Banach's fixed point theorem and Step 1 of Theorem 2, problem (3) has a unique solution $\omega \in \chi$.

Let $\widetilde{\omega}$ be an ε-solution of (3). Put

$$\tau(t) = D^{\alpha}\widetilde{\omega}(t) - O\widetilde{\omega}(t) - h\Big(t, \widetilde{\omega}(t), \int_0^t \eta(t,s,\widetilde{\omega}(s))\,\mathrm{d}s\Big),$$

$$\theta = \widetilde{\omega}(0) - \omega_0 - \sum_{i=1}^{n} b_i \widetilde{\omega}(t_i) - \int_0^1 g(s, \widetilde{\omega}(s))\,\mathrm{d}s.$$

Then $\tau \in \chi$, $\|\tau(t)\| \le \varepsilon$ for $t \in [0,1]$, and $\|\theta\| \le \varepsilon$. Since τ is continuous, Lemma 1 applies to $\widetilde{\omega}$

$$\widetilde{\omega}(t) = \widetilde{\omega}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[O\widetilde{\omega}(s) + h\left(s, \widetilde{\omega}(s), \int_0^s \eta(s,\xi,\widetilde{\omega}(\xi))\,\mathrm{d}\xi\right) + \tau(s)\right]\mathrm{d}s,$$

while $\omega(t) = \omega(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[O\omega(s) + h\left(s, \omega(s), \int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi\right)\right]\mathrm{d}s.$

Subtracting the second identity from the first and using the hypotheses on h, η, g, μ, ρ, we obtain

$$\begin{aligned}\|\omega(t) - \widetilde{\omega}(t)\| \le \varepsilon + \left(\sum_{i=1}^{n}|b_i| + \mu_0\right)\|\omega - \widetilde{\omega}\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\|O(\omega-\widetilde{\omega})(s)\| \\ + \left\|h\left(s, \omega(s), \int_0^s \eta(s,\xi,\omega(\xi))\,\mathrm{d}\xi\right) - h\left(s, \widetilde{\omega}(s), \int_0^s \eta(s,\xi,\widetilde{\omega}(\xi))\,\mathrm{d}\xi\right)\right\| + \|\tau(s)\|\Big]\,\mathrm{d}s,\end{aligned}$$

where $\omega(0) - \widetilde{\omega}(0) = \sum_{i=1}^{n} b_i(\omega - \widetilde{\omega})(t_i) + \int_0^1 [g(s,\omega(s)) - g(s,\widetilde{\omega}(s))]\,\mathrm{d}s - \theta$.

Since $\|\tau(s)\| \le \varepsilon$ for all $s \in [0,1]$,

$$\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\tau(s)\|\,\mathrm{d}s \le \frac{\varepsilon}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mathrm{d}s = \frac{\varepsilon t^{\alpha}}{\Gamma(\alpha+1)} \le \frac{\varepsilon}{\Gamma(\alpha+1)},$$

so

$$\|\omega(t) - \widetilde{\omega}(t)\| \le \left(1 + \frac{1}{\Gamma(\alpha+1)}\right)\varepsilon + \left(\sum_{i=1}^{n}|b_i| + \mu_0 + \frac{K + (1+c)\rho_0}{\Gamma(\alpha+1)}\right)\|\omega - \widetilde{\omega}\|.$$

Taking the supremum over $t \in [0,1]$ yields

$$\|\omega - \widetilde{\omega}\| \le \left(1 + \frac{1}{\Gamma(\alpha+1)}\right)\varepsilon + L\|\omega - \widetilde{\omega}\|.$$

Then, since $1 - L > 0$ (here the condition $L < 1$ is used),

$$\|\omega - \widetilde{\omega}\| \le \frac{1 + \Gamma(\alpha+1)}{(1-L)\Gamma(\alpha+1)}\varepsilon = \sigma\varepsilon,$$

Corrected Proof

with $\sigma := \dfrac{1+\Gamma(\alpha+1)}{(1-L)\,\Gamma(\alpha+1)} > 0$. Therefore, the solution of (3) is Ulam-Hyers stable in χ. □

The following examples illustrate the existence and stability results established above by verifying the hypotheses analytically. Example 1 is further complemented by a numerical computation. In each example, $E = \ell^2$, the Hilbert space of real sequences $x = (x_i)_{i\geq 1}$ with $\sum_i x_i^2 < \infty$, endowed with the norm $\|x\| = (\sum_i x_i^2)^{\frac{1}{2}}$; we write $e_1 = (1,0,0,\dots)$ and $\pi_1 x = x_1$ for the first coordinate of x. The functions sin and tanh act coordinatewise; both are 1-Lipschitz and bounded in modulus by $|\cdot|$, so they map ℓ^2 into itself with $\|\sin x - \sin y\| \leq \|x-y\|$ and $\|\tanh x - \tanh y\| \leq \|x-y\|$. The operator O is a multiplication by a scalar function, hence it is linear and bounded on χ but not compact (since $\dim E = \infty$). In Examples 1 and 3 the nonlinearities are Lipschitz, so by (**ii**) the measure of noncompactness conditions (7) follow from the Lipschitz-type conditions of Theorem 3 with the same constants; in Example 2 the function h is not Lipschitz and (7) is verified directly. Moreover, $\omega_0 \neq 0$, so $\omega \equiv 0$ is not a solution.

Example 1. Consider $\chi = C([0,1],\ell^2)$ and the following fractional nonlocal problem:

$$\begin{cases} D^{\frac{1}{4}}\omega(t) = \dfrac{1}{10}\omega(t) + h\left(t, \omega(t), \displaystyle\int_0^t \eta\,(t,s,\omega(s))\,\mathrm{d}s\right), \ t \in [0,1] \\ \omega\,(0) = \omega_0 + \dfrac{1}{5}\omega\left(\dfrac{1}{3}\right) + \dfrac{1}{6}\omega\left(\dfrac{2}{3}\right) + \dfrac{1}{8}\displaystyle\int_0^1 \omega(s)\mathrm{d}s, \end{cases} \tag{13}$$

where

$$O\omega(t) = \frac{1}{10}\,\omega(t), \quad h(t,x,y) = \frac{1}{24}(x+y), \quad g(t,x) = \frac{1}{8}x, \quad \eta\,(t,s,x) = \frac{1}{8}x, \quad \omega_0 = e_1.$$

The hypotheses of Theorem 2 are verified as follows:

$$\|h(t,x,y)\| \leq \frac{1}{24}(\|x\| + \|y\|), \qquad \|\eta\,(t,s,x)\| \leq \frac{1}{8}\|x\|, \qquad \|g(t,x)\| \leq \frac{1}{8}\|x\|,$$

then $K = \frac{1}{10}$, $c = \frac{1}{8}$, $\rho(t) = \frac{1}{24} \Rightarrow \rho_0 = \frac{1}{24}$, $\mu(t) = \frac{1}{8} \Rightarrow \mu_0 = \frac{1}{8}$, and $b_1 = \frac{1}{5}$, $b_2 = \frac{1}{6} \Rightarrow \sum_{i=1}^{n} |b_i| = \frac{11}{30} < 1$. So, with $\Gamma(\frac{5}{4}) \simeq 0.9064$,

$$L = \sum_{i=1}^{n} |b_i| + \mu_0 + \frac{K + (1+c)\rho_0}{\Gamma(\alpha+1)} \simeq 0.6537, \qquad \delta = \sum_{i=1}^{n} |b_i| + 2\mu_0 + \frac{2\big(K + \rho_0(1+2c)\big)}{\Gamma(\alpha+1)} \simeq 0.9522 < 1.$$

Since h, η, g are linear, they satisfy the Lipschitz-type conditions required in Theorem 3, with the same constants as above. Since $\delta < 1$, all the assumptions of Theorems 2 and 3 are satisfied; problem (13) therefore admits a unique solution, which lies in $\overline{B}_R$ for any $R \geq \|\omega_0\|/(1-L) \simeq 2.89$ and is Ulam-Hyers stable in χ with $\sigma \simeq 6.07$.

Numerical computation.

Since all the data of problem (13) act coordinatewise and $\omega_0 = e_1$, the solution has the form $\omega(t) = u(t)e_1$, where u solves the corresponding scalar problem. We used the uniform grid $t_j = jh$, $j = 0,\dots,N$, with $N = 600$ and $h = \dfrac{1}{N}$. The points $t = \dfrac{1}{3}$ and $t = \dfrac{2}{3}$ are grid nodes ($j = 200$ and $j = 400$), so no interpolation is needed in the nonlocal condition. The computations were performed in Python 3 with NumPy.

Corrected Proof

Quadrature. The fractional integral is approximated by the product trapezoidal rule (see [15]): f is replaced by its piecewise linear interpolant, which is then integrated exactly against the kernel,

$$I^{\alpha} f(t_n) \approx \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n} f(t_j),$$

with $a_{0,n} = (n-1)^{\alpha+1} - (n-\alpha-1)\, n^{\alpha}$, $a_{j,n} = (n-j+1)^{\alpha+1} + (n-j-1)^{\alpha+1} - 2(n-j)^{\alpha+1}$ for $1 \le j \le n-1$, and $a_{n,n} = 1$ (here $\alpha = \dfrac{1}{4}$). The integrals $\int_0^{t_n} \eta(t_n, \xi, \omega(\xi))\, \mathrm{d}\xi$ and $\int_0^1 \omega(s)\, \mathrm{d}s$ are approximated by the composite trapezoidal rule on the same grid.

Picard iteration and nonlocal condition. We computed u by the Picard iteration

$$\begin{cases} \omega^{[i+1]} = \mathscr{A}\omega^{[i]}, \quad i \ge 0, \\ \omega^{[0]} \equiv 0. \end{cases}$$

The nonlocal condition is not solved as a separate system; at each iteration the constant $\omega_0 + \frac{1}{5}\omega^{[i]}(\frac{1}{3}) + \frac{1}{6}\omega^{[i]}(\frac{2}{3}) + \frac{1}{8}\int_0^1 \omega^{[i]}(s)\, \mathrm{d}s$ is computed from the previous iterate and added to the discrete fractional integral, exactly as in the definition of the operator $\mathscr{A}$. The iteration is stopped when $\max_j \|\omega^{[i+1]}(t_j) - \omega^{[i]}(t_j)\| < 10^{-12}$, which is reached after 59 iterations.

Reported quantities. Let $e_i = \max_j \|\omega^{[i+1]}(t_j) - \omega^{[i]}(t_j)\|$ and $q_i = \dfrac{e_i}{e_{i-1}}$. We find $q_1 \simeq 0.653$, $q_2 \simeq 0.626$, $q_3 \simeq 0.621$, and $q_i \simeq 0.62$ for larger i; all ratios are $\le L \simeq 0.6537$, as predicted by Theorem 3. The norm of the last iterate is $\|\omega\| = \max_j |u(t_j)| \simeq 2.73$; it is attained at $t = 1$ ($u(0) \simeq 2.30$, $u(1) \simeq 2.73$), and $2.73 \le R \simeq 2.89$ (Figure 2).

Stability test. For a given $\varepsilon > 0$, the ε-solution $\widetilde{\omega} = \tilde{u} e_1$ is the solution of the perturbed problem

$$\begin{cases} D^{\frac{1}{4}} \widetilde{\omega} = \mathscr{O}\widetilde{\omega} + h\Big(t, \widetilde{\omega}, \int_0^t \eta(t, s, \widetilde{\omega}(s))\, \mathrm{d}s\Big) + \tau(t), \\ \widetilde{\omega}(0) = \omega_0 + \frac{1}{5}\widetilde{\omega}(\frac{1}{3}) + \frac{1}{6}\widetilde{\omega}(\frac{2}{3}) + \frac{1}{8}\int_0^1 \widetilde{\omega}(s)\, \mathrm{d}s + \theta, \end{cases}$$

with $\tau(s) = \varepsilon \sin(5s)\, e_1$ and $\theta = \frac{\varepsilon}{2} e_1$. Since $\|\tau(s)\| \le \varepsilon$ and $\|\theta\| \le \varepsilon$, $\widetilde{\omega}$ is an ε-solution in the sense of Definition 4. It is computed with the same grid, quadrature, initial iterate, and stopping criterion as above (τ is added to the integrand of I^{α} and θ to the constant term), and $\|\omega - \widetilde{\omega}\| = \max_j |u(t_j) - \tilde{u}(t_j)|$. For $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ we obtained $\|\omega - \widetilde{\omega}\| \simeq 2.81\, \varepsilon$, which is consistent with the bound $\sigma\varepsilon$, $\sigma \simeq 6.07$. Since the problem is linear, this ratio does not depend on ε.

Example 2. Consider $\chi = C([0,1], \ell^2)$ and problem (3) with the following parameters:

$$\begin{cases} D^{\frac{1}{8}} \omega(t) = O\omega(t) + h\left(t, \omega(t), \int_0^t \eta\,(t, s, \omega(s))\, \mathrm{d}s\right), \ t \in [0,1] \\ \omega\,(0) = \omega_0 + \dfrac{1}{6}\omega\left(\dfrac{1}{8}\right) + \dfrac{1}{7}\omega\left(\dfrac{7}{8}\right) + \int_0^1 g\,(s, \omega(s))\, \mathrm{d}s, \end{cases} \tag{14}$$

Corrected Proof

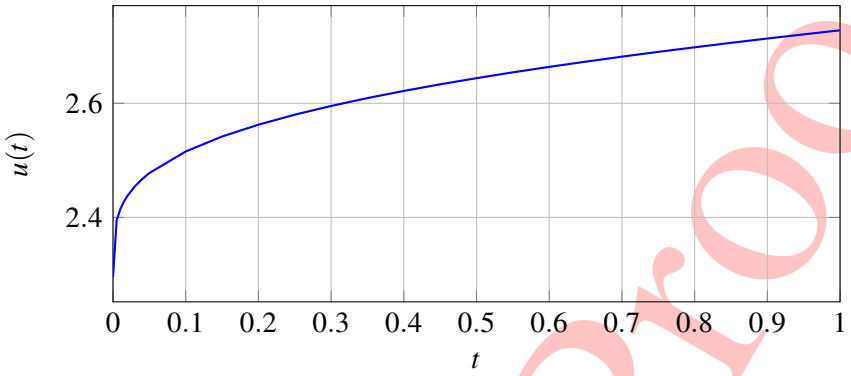


Figure 2: Numerical solution of problem (13):

$$\text{first coordinate } u \text{ of } \omega = u e_1, \alpha = \frac{1}{4}$$

with $\alpha = \frac{1}{8}$, $O\omega(t) = -\frac{1}{8}\,\omega(t)$, $g(t,x) = \frac{t}{10}x$, $\eta(t,s,x) = \dfrac{t+s}{8}x$, $\omega_0 = (2^{-k})_{k\geq 1}$ (so $\|\omega_0\| = \dfrac{\sqrt{3}}{3}$), and

$$h(t,x,y) = \frac{1}{24}(x+\tanh y) + \frac{1}{24}\,\theta(\pi_1 x)\,e_1,$$
$$\theta(r) = |r|\sin\frac{1}{|r|}\ (\theta(0)=0).$$

The function θ is continuous, $|\theta(r)| \leq |r|$, but it is not Lipschitz at 0 (its derivative is unbounded near 0); hence h is continuous but not Lipschitz. The hypotheses of Theorem 2 are verified as follows. Since $|\pi_1 x| \leq \|x\|$,

$$\|h(t,x,y)\| \leq \frac{1}{24}(2\|x\| + \|y\|) \leq \frac{1}{12}(\|x\| + \|y\|),$$
$$\|\eta(t,s,x)\| \leq \frac{t+s}{8}\|x\| \leq \frac{1}{4}\|x\|,$$
$$\|g(t,x)\| \leq \frac{t}{10}\|x\|.$$

Then $K = \frac{1}{8}$, $c = \frac{1}{4}$, $\rho(t) = \frac{1}{12} \Rightarrow \rho_0 = \frac{1}{12}$, $\mu(t) = \frac{t}{10} \Rightarrow \mu_0 = \frac{1}{20}$, and $b_1 = \frac{1}{6}$, $b_2 = \frac{1}{7} \Rightarrow \sum_{i=1}^{n} |b_i| = \frac{13}{42}$.

For (7), $\theta(\pi_1 B)e_1$ is a bounded subset of a one-dimensional space, hence relatively compact, so $\mathscr{M}(h(t,B,D)) \leq \frac{1}{24}(\mathscr{M}(B) + \mathscr{M}(D)) \leq \rho(t)(\mathscr{M}(B) + \mathscr{M}(D))$; since η and g are linear, the other two inequalities hold with $c = \frac{1}{4}$ and $\mu(t) = \frac{t}{10}$. With $\Gamma(\frac{9}{8}) \simeq 0.9417$,

$$L \simeq 0.6029, \qquad \delta \simeq 0.9405 < 1.$$

Consequently, problem (14) possesses at least one solution, which lies in $\overline{B}_R$ for $R \geq \|\omega_0\|/(1-L) \simeq 1.45$. Theorem 3 does not apply here (h is not Lipschitz), so we make no claim on uniqueness or stability; neither Banach's theorem nor Schauder's theorem is directly available (the operator O and the identity term of h are not compact and $\dim E = \infty$), so existence rests on Darbo's theorem.

Corrected Proof

Example 3. Consider $\chi = C([0,1],\ell^2)$, and problem (3) with the following parameters:

$$\begin{cases} D^{\frac{1}{3}}\omega(t) = O\omega(t) + h\left(t,\omega(t),\int_0^t \eta\,(t,s,\omega(s))\,\mathrm{d}s\right),\ t\in[0,1] \\ \omega\,(0) = \omega_0 + \frac{1}{5}\omega\left(\frac{1}{4}\right) + \frac{1}{9}\omega\left(\frac{3}{4}\right) + \int_0^1 g\,(s,\omega(s))\,\mathrm{d}s, \end{cases} \tag{15}$$

with

$$\alpha = \frac{1}{3}, \qquad O\omega(t) = \frac{1}{12}\cos(t)\,\omega(t), \qquad h(t,x,y) = \frac{1}{7}(\sin x + \tanh y),$$
$$g(t,x) = \frac{t^2}{15}\tanh x, \qquad \eta(t,s,x) = \frac{\exp(-t-s)}{10}\sin x, \qquad \omega_0 = e_1.$$

The hypotheses of Theorem 2 are verified as follows:

$$\|h(t,x,y)\| \le \frac{1}{7}(\|\sin x\| + \|\tanh y\|) \le \frac{1}{7}(\|x\| + \|y\|),$$
$$\|\eta\,(t,s,x)\| \le \frac{e^{-t-s}}{10}\|\sin x\| \le \frac{1}{10}\|x\|,$$

and

$$\|g(t,x)\| \le \frac{t^2}{15}\|\tanh x\| \le \frac{t^2}{15}\|x\|.$$

Then $K = \frac{1}{12}$, $c = \frac{1}{10}$, $\rho(t) = \frac{1}{7} \Rightarrow \rho_0 = \frac{1}{7}$, $\mu(t) = \frac{t^2}{15} \Rightarrow \mu_0 = \frac{1}{45}$, and $b_1 = \frac{1}{5}$, $b_2 = \frac{1}{9} \Rightarrow \sum_{i=1}^{n} |b_i| = \frac{14}{45} < 1$. So, with $\Gamma(\frac{4}{3}) \simeq 0.8930$,

$$L \simeq 0.6026, \qquad \delta \simeq 0.9261 < 1.$$

Since sin and tanh are both globally Lipschitz with constant 1, and since $e^{-t-s} \le 1$ is bounded, it follows that h, η, and g each satisfy the required Lipschitz-type conditions, with the same constants as above. Since $\delta < 1$, the assumptions of Theorems 2 and 3 hold; consequently, problem (15) has a unique solution, which lies in $\overline{B}_R$ for $R \ge \|\omega_0\|/(1-L) \simeq 2.52$ and is Ulam-Hyers stable in χ with $\sigma \simeq 5.33$.

4 Conclusion

This paper has established existence and Ulam-Hyers stability results for a class of nonlinear fractional integro-differential equations involving a state-dependent memory term and a nonlocal multi-point boundary condition in $C([0,1],E)$, with E an infinite-dimensional Banach space. The approach combines the Kuratowski measure of noncompactness with Darbo's fixed point theorem, circumventing the lack of compactness in infinite dimensions. Three examples show the applicability of the theoretical results. Future work may explore extensions to Hilfer-type fractional derivatives, problems on unbounded intervals, or weaker forms of dependence of the memory kernel on the state.

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Corrected Proof

Conflict of interest

The author declares no conflict of interest.

AI-use disclosure

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References

[1] S. Abbas, M. Benchohra, G.M. N'Guérékata, *Topics in fractional differential equations, Developments in Mathematics*, **27**, Springer, New York, 2012.

[2] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions,* Acta Appl. Math., **109** (2010) 973–1033.

[3] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B. N. Sadovskii, *Measures of noncompactness and condensing operators*, Birkhäuser, Basel, 1992.

[4] A. Aphithana, W. Sudsutad, J. Kongson, C. Thaiprayoon, *Measure of noncompactness for nonlocal boundary value problems via (k,ψ)-Riemann-Liouville derivative on unbounded domain*, AIMS Math. **8(9)** (2023) 10018–10047.

[5] J. Banaś, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, **60**, Marcel Dekker, New York, 1980.

[6] J. Banaś, M. Mursaleen, *Measures of non-compactness. In sequence spaces and measures of noncompactness with applications to differential and integral equations,* New Delhi: Springer India, (2014) 147–184.

[7] M. Benbachir, A. Boutiara, *Measure of noncompactness for nonlinear Hilfer fractional differential equation with mixed fractional integral boundary conditions in Banach space,* Innov. Appl. Math. Comput. Sci., **2** (2022) 27–42.

[8] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order and nonlocal conditions*, Nonlinear Anal., **71** (2009) 2391–2396.

[9] M. Benchohra, J. Henderson, S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporary Mathematics and Its Applications, **2**, Hindawi Publishing Corporation, New York, 2006.

[10] M. Benchohra, J.E. Lazreg, *Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative,* Stud. Univ. Babes-Bolyai Math. **62(1)** (2017) 27–38.

[11] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991) 494–505.

Corrected Proof

[12] G. Darbo, *Punti uniti in trasformazioni a codominio non compatto,* Rend. Sem. Mat. Univ. Padova, **24** (1955) 84–92.

[13] S. Deb, H. Jafari, A. Das, V. Parvaneh, *New fixed point theorems via measure of noncompactness and its applications on fractional integral equation involving an operator with iterative relations*, J. Inequal. Appl. **2023 (106)** (2023).

[14] A. Deep, B. Hazarika, *An existence result for Hadamard type two-dimensional fractional functional integral equations via measure of noncompactness*, Chaos, Solitons & Fractals, **147** (2021).

[15] K. Diethelm, *The analysis of fractional differential equations*, Lecture Notes in Mathematics, **2004**, Springer, Berlin, 2010.

[16] H. Djourdem, *A study of hybrid fractional differential equations via measure of noncompactness*, *Filomat*, **39(26)** (2025) 9075–9085.

[17] I. Haque, J. Ali, J.J. Nieto, *Controllability of ψ-Hilfer fractional differential equations with infinite delay via measure of noncompactness*, Nonlinear Anal. Model. Control, **29(2)** (2024) 379–399.

[18] H.P. Heinz, *On the behavior of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983) 1351–1371.

[19] R.W. Ibrahim, *Generalized Ulam-Hyers stability for fractional differential equations*, Int. J. Math. **23(5)** (2012) 1250056.

[20] A.A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, **204**, Elsevier, Amsterdam 2006, pp. 523.

[21] K.S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[22] I. Podlubny, *Fractional differential equations*, Mathematics in Science and Engineering, **198**, Academic Press, San Diego, 1999.

[23] M. Rabbani, A. Das, B. Hazarika, R. Arab, *Measure of noncompactness of a new space of tempered sequences and its application on fractional differential equations*, Chaos, Solit. Fractals **140** (2020).

[24] I.A. Rus, *Ulam stability of ordinary differential equations*, Stud. Univ. Babes-Bolyai Math. **54(4)** (2009) 125–133.

[25] H.A. Wahash, S.K. Panchal, M.S. Abdo, *Positive solutions for generalized Caputo fractional differential equations with integral boundary conditions*, J. Math. Model. **8(4)** (2020) 393–414.

[26] J. Wang, M. Feckan, Y. Zhou, *Ulam's type stability of impulsive ordinary differential equations*, J. Math. Anal. Appl. **395** (2012) 258–264.

[27] Y. Zhou, *Basic theory of fractional differential equations*, World Scientific, Singapore, 2014.

Corrected Proof