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Cohomology and extensions of Jacobi-Jordan superalgebras

Espérance Niyonkuru[†*], Sylvain Attan[‡], Donatien Gaparayi[§]

[†] *Centre de Recherche en Mathématiques et Physique (CRMP), Ecole Doctorale, Université du Burundi, BP 2700 Bujumbura, Burundi*
[‡] *Institut National des classes Préparatoires aux Etudes d'Ingénieur (INSPEI), BP 2282 Goho, Abomey, Bénin*
[§] *Section de Mathématiques, Ecole Normale Supérieure du Burundi, BP 6983 Bujumbura, Burundi*
Emails: esperanceni59@gmail.com, syltane2010@yahoo.fr, gapadona@yahoo.fr

Abstract. Let $S = S_0 \oplus S_1$ be a Jacobi-Jordan (J-J, for short) superalgebra and (V, π) be a representation on S. We prove necessary and sufficient conditions for an (anti-)supercommutator of S to be an (anti-)superderivation. Using two distinct differential morphisms, we construct on S a complete cohomology. Then we show the low degree cohomology of S with coefficients in V and interpret them. In particular, we consider the case $V \equiv S$ endowed with the adjoint representation. Finally, we describe the relationships between second cohomology supergroup with extensions.

Keywords: Jacobi-Jordan superalgebra, Representation, (Anti-)superderivation, Cohomology, Extension.
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1 Introduction

The cohomology of superalgebras is an important tool in Mathematics for applications in Physics as well as in anomaly detection for human body [9]. The cohomology of Lie superalgebras has been extensively studied in the literature, notably by Leites in 1976 [7] and more recently by Alvarez-Gómez in 2020 [2]. However, to the best of our knowledge, no cohomology theory for J-J superalgebras, that is, $\mathbb{Z}_2$-graded J-J algebras, has yet been developed. As it was explained in [5], JJ algebras are fundamentally different from Lie algebras. In 2015, the authors of [1] showed that J-J algebras possess a rich and highly interesting theory that deserves further development.

*Corresponding author
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In the present paper, we extend the zigzag cohomology theory developed in [3] for J-J algebras to J-J superalgebras. Since the $\mathbb{Z}_2$-graded structure of a J-J superalgebra $S = S_0 \oplus S_1$ consists of an even part S_0 and an odd part S_1, the techniques used in the classical case, i.e. where $S_1 \equiv 0$, cannot be directly adapted to obtain analogous results. Then we construct the cohomology theory of J-J superalgebras satisfying the supersymmetry and the super Jacobi identity, based on two differential operators δ^n and ρ^{n-1}. We show that, for all $n > 0$, $\delta^n \circ \rho^{n-1} \equiv 0$ and, we interpret the low degree cohomology superspaces.

The rest of this paper is organized as follows. In Section 2, we give some basic properties and notations for J-J superalgebras. Section 3 is devoted to the definition of representations of J-J superalgebras. Next, we establish necessary and sufficient conditions under which an (anti-)supercommutator of J-J superalgebra is an (anti-)superderivation. In Section 4, we develop a cohomology theory for J-J superalgebras and prove the main results concerning the associated differential operators. We then compute the cohomology groups in degrees 0 and 1 and the relationship between the second cohomology and abelian extensions.

Throughout the article, $\mathbb{K}$ is an algebraically closed field of characteristic 0.

2 Some basic notions on J-J superalgebras

Note that the notion of J-J superalgebras has been studied in [4] under the name mock-Lie superalgebras. The authors studied the double extensions and generalized double extensions of mock-Lie superalgebras and their isometries [4].

2.1 Notations and definition on J-J superalgebras

A $\mathbb{Z}_2$-graded vector space (or vector superspace) is a direct sum of $\mathbb{K}$-vector spaces $V = V_0 \oplus V_1$. Elements in V_0 (resp. V_1) are called even (resp. odd). Non-zero elements of $V_0 \cup V_1$ are called homogeneous and the degree of an homogeneous element $v \in V_i$ is $|v| = i$. In the sequel, we denote by $\mathcal{H}(V)$ the space of all homogeneous elements of V.

Definition 1. *A J-J superalgebra is a vector superspace $S = S_0 \oplus S_1$ together with a bilinear map $\bullet : S \times S \longrightarrow S$, satisfying:*

(i) $S_i \bullet S_j \subset S_{i+j}$ for $i, j \in \mathbb{Z}_2$, ($\mathbb{Z}_2$-grading),

(ii)

$$x \bullet y = (-1)^{|x||y|} y \bullet x, \ \textit{(supersymmetry)}, \tag{1}$$

(iii)

$$\circlearrowleft_{(x,y,z)} (-1)^{|x||z|} x \bullet (y \bullet z) = 0, (\textit{super Jacobi identity}), \tag{2}$$

for all $x, y, z \in \mathcal{H}(S)$ where

$$\circlearrowleft_{(x,y,z)} (-1)^{|x||z|} x \bullet (y \bullet z) := (-1)^{|x||z|} x \bullet (y \bullet z) + (-1)^{|x||y|} y \bullet (z \bullet x) + (-1)^{|y||z|} z \bullet (x \bullet y)$$

is called the super Jacobian of S.

Example 1. Let $J := J_0 \oplus J_1$ be a $\mathbb{Z}_2$-graded vector space where J_0 is spanned by e_1 and J_1 is spanned by e_2. Then $(J, \bullet)$ is a J-J superalgebra where non zero values are: $e_2 \bullet e_2 = e_1$.2

Example 2. Let $S := S_0 \oplus S_1$ be a $\mathbb{Z}_2$-graded vector space where S_0 is spanned by (e_1, e_2) and S_1 is spanned by (e_3, e_4). Then $(S, \bullet)$ is a J-J superalgebra where non zero values are: $e_1 \bullet e_1 = e_2 = e_3 \bullet e_3 = e_4 \bullet e_4$.

2.2 Some J-J superalgebras properties

Observe the super Jacobi identity (2) is equivalent to the super identity

$$(x \bullet y) \bullet z + \big(-1\big)^{|x|(|y|+|z|)}(y \bullet z) \bullet x + \big(-1\big)^{|z|(|x|+|y|)}(z \bullet x) \bullet y = 0 \tag{3}$$

for all $x, y, z \in \mathcal{H}(S)$.

Remark 1. *A J-J superalgebra of the form $S = S_0 \oplus 0$ is a J-J algebra. That is, the even part S_0 of S is a J-J algebra.*

Definition 2. *Let $(S, \cdot)$ be a $\mathbb{K}$-superalgebra. The super anti-associator of $(S, \cdot)$ is the even trilinear mapping $Aasso : S \times S \times S \longrightarrow S$ given by*

$$Aasso(x; y; z) := x \cdot (y \cdot z) + (x \cdot y) \cdot z, \tag{4}$$

for every $x, y, z \in \mathcal{H}(S)$. The superalgebra $(S, \cdot)$ is said to be an anti-associative superalgebra if $Aasso(x, y, z) = 0$ for any $x, y, z \in \mathcal{H}(S)$.

Remark 2. *Let $(S, \cdot)$ be anti-associative superalgebra and $*$ be the superproduct in S defined by*

$$x * y = x \cdot y + \big(-1\big)^{|x||y|} y \cdot x, \quad \forall x, y \in \mathcal{H}(S).$$

Then $(S, *)$ is a J-J superalgebra.

3 Representations and (anti-)superderivations of J-J superalgebras

Definition 3. *Let $V = V_0 \oplus V_1$ and $W = W_0 \oplus W_1$ be two superspaces.*

1. *A linear map f from V to W such that $f(V_i) \subset W_i$ for all $i \in \mathbb{Z}_2$ is called homomorphism of superspaces (or even linear map). An isomorphism of superspaces is a bijective homomorphism $f : V_0 \oplus V_1 \longrightarrow W_0 \oplus W_1$.*
2. *A linear mapping f from V to W such that $f(V_i) \subset W_{i+r}$ for all $i \in \mathbb{Z}_2$ is said homogeneous of degree r. In this case, r is denoted by $|f|$.*

Definition 4 ([8])**.** *Let $(S, \bullet)$ be a J-J superalgebra and V a vector superspace. An even representation of $(S, \bullet)$ in V is an even morphism $\pi : S \longrightarrow End(V)$ satisfying*

$$\pi(x \bullet y) := -\pi(x)\pi(y) - (-1)^{|x||y|}\pi(y)\pi(x), \quad \forall x, y \in \mathcal{H}(S). \tag{5}$$

Example 3. The mapping $ad : S \longrightarrow End(S)$ defined by $ad_x(y) := x \bullet y, \forall x, y \in \mathcal{H}(S)$ is an even representation of S called adjoint representation.

Definition 5. *Let $(S, \bullet)$ be a J-J superalgebra and (V, π) be a representation on it. A linear map $d : S \to S$ is said to be*

1. *a superderivation of degree $|d|$ of $(S, \bullet)$ with coefficients in V if*

$$d(x \bullet y) = (-1)^{|d||x|}\pi(x)d(y) + (-1)^{(|x|+|d|)|y|}\pi(y)d(x), \quad \forall x, y \in \mathcal{H}(S). \tag{6}$$

2. *an anti-superderivation of degree $|d|$ of $(S, \bullet)$ with coefficients in V if*

$$d(x \bullet y) = -(-1)^{|d||x|}\pi(x)d(y) - (-1)^{(|x|+|d|)|y|}\pi(y)d(x), \quad \forall x, y \in \mathcal{H}(S). \tag{7}$$

Remark 3. *Observe that a superderivation (respectively an anti-superderivation) of degree $|d|$ of S is a superderivation (respectively an anti-superderivation) of degree $|d|$ of S with coefficients in the adjoint representation S. In this case (6) and (7) then read respectively*

$$d(x \bullet y) = d(x) \bullet y + (-1)^{|d||x|}x \bullet d(y), \quad \forall x, y \in \mathcal{H}(S),$$
$$d(x \bullet y) = -d(x) \bullet y - (-1)^{|d||x|}x \bullet d(y), \quad \forall x, y \in \mathcal{H}(S).$$

We denote by $SDer(S)$ (respectively, $SDer(S, V)$) the space of all superderivations of S (respectively of S with coefficients in V) and $SADer(S)$ (respectively, $SADer(S, V)$) the one of all anti-superderivations of S (respectively of S with coefficients in V).

Proposition 1. *Let $(S, \bullet)$ be a J-J superalgebra and (V, π) be a representation on it. Define for any $v \in \mathcal{H}(V)$, the map $d_v : S \to V$ by*

$$d_v(x) := (-1)^{|v||x|}\pi(x)v, \forall x \in \mathcal{H}(S),$$

then, d_v is an anti-superderivation of degree $|v|$ of S with coefficients in V, called an inner anti-superderivation of degree $|v|$ of S with coefficients in V.

Proof. Let $v \in \mathcal{H}(V)$ and $x, y \in \mathcal{H}(S)$, then we have

$$\begin{aligned} d_v(x \bullet y) &:= (-1)^{|v|(|x|+|y|)}\pi(x \bullet y)v \\ &= -(-1)^{|v|(|x|+|y|)}\pi(x)(\pi(y)v) - (-1)^{|v|(|x|+|y|)+|x||y|}\pi(y)(\pi(x)v) \text{ (by (5))} \\ &= -(-1)^{|v||x|}\pi(x)d_v(y) - (-1)^{(|x|+|v|)|y|}\pi(y)d_v(x). \end{aligned}$$

Hence the result follows. □

Corollary 1. *Let S be a J-J superalgebra. For all $x \in \mathcal{H}(S)$, the linear map $l_x : S \longrightarrow S$ defined by*

$$l_x(y) := x \bullet y, \ \forall y \in \mathcal{H}(S) \tag{8}$$

is an anti-superderivation of degree $|x|$ of S. Such anti-superderivation is called an inner anti-superderivation of degree $|x|$ of S.

Let denote by $InnSADer(S)$ (respectively, $InnSADer(S,V)$) the space of all inner anti-superderivations of S (respectively of S with coefficients in V). Let denote by $SDer(S)$ the set of all superderivations of a J-J superalgebra $(S,\bullet)$. Given even maps $d_1, d_2 \in SADer(S) \cup SDer(S)$, define their supercommutator $[d_1,d_2]$ and their (anti-)supercommutator $\{d_1,d_2\}$ by

$$[d_1,d_2] := d_1d_2 - (-1)^{|d_1||d_2|}d_2d_1,$$
$$\{d_1;d_2\} := d_1d_2 + (-1)^{|d_1||d_2|}d_2d_1.$$

Lemma 1. *Let $(S,\bullet)$ be a J-J superalgebra.*

(i) For any $d_1,d_2 \in SDer(S)$, we have $[d_1,d_2] \in SDer(S)$.

(ii) For any $d_1,d_2 \in SADer(S)$, we have $[d_1,d_2] \in SDer(S)$.

We know (from [6], p. 16) that if $V = V_0 \oplus V_1$ is a vector superspace then, the set of the linear maps of V into itself which are homogeneous of degree $r \in \mathbb{Z}_2$ is denoted by $End_r(V) = \{f \in End(V), f(V_i) \subseteq V_{r+i}\}$, we obtain an associative superalgebra $End(V) = End_0(V) \oplus End_1(V)$. Hence, it is obvious to prove the following:

Proposition 2. *Let $(S,\bullet)$ be a J-J superalgebra. Then the space $SDer(S)$ of superderivations of S, endowed with the supercommutator $[d_1,d_2] := d_1d_2 - (-1)^{|d_1||d_2|}d_2d_1$, is a Lie superalgebra.*

Recall that a Lie superalgebra is a vector superspace $L = L_0 \oplus L_1$ together with a bilinear map $[,] : L \times L \longrightarrow L$ satisfying the following axioms: the $\mathbb{Z}_2$-grading, the anti-supersymmetry, that is, for all $a,b \in \mathcal{H}(L), [a,b] = -(-1)^{|a||b|}[b,a]$ and the super Jacobi identity.

Proof. First, we prove that $[SDer(S)_i, SDer(S)_j] \subseteq SDer(S)_{i+j}, \forall i,j \in \mathbb{Z}_2$. Pick d_1,d_2 be homogeneous superderivations of degrees $|d_1|$ and $|d_2|$ respectively. Since d_1 and d_2 are homogeneous linear mappings $|d_1d_2| = |d_1| + |d_2|$ and then $|[d_1,d_2]| = |d_1| + |d_2|$ that implies $[SDer(S)_i, SDer(S)_j] \subseteq SDer(S)_{i+j}$. It remains to show that $[d_1,d_2] \in SDer(S)$. Let $x,y \in \mathcal{H}(S)$. Some computations give

$$\begin{aligned}[d_1,d_2](x\bullet y) &= (d_1d_2)(x\bullet y) - (-1)^{|d_1||d_2|}(d_2d_1)(x\bullet y)\\ &= [d_2,d_1](x)\bullet y + (-1)^{(|d_1|+|d_2|)|x|}x\bullet[d_1,d_2](y).\end{aligned}$$

Therefore, $[d_1,d_2] \in SDer(S)$.

Next, we show the anti-supersymmetry. We have,

$$\begin{aligned}[d_1,d_2] &:= d_1d_2 - (-1)^{|d_1||d_2|}d_2d_1\\ &= -(-1)^{|d_1||d_2|}d_2d_1 + (-1)^{2|d_1||d_2|}d_1d_2\\ &= -(-1)^{|d_1||d_2|}\Big(d_2d_1 - (-1)^{|d_1||d_2|}d_1d_2\Big)\\ &= -(-1)^{|d_1||d_2|}[d_2,d_1].\end{aligned}$$

Finally, we show the supe Jacobi identity. Pick $d_1,d_2,d_3 \in SDer(S)$. We have

$$\circlearrowleft_{(d_1,d_2,d_3)} \big(-1\big)^{|d_1||d_3|}[d_1,[d_2,d_3]]$$

$$
\begin{aligned}
&= \circlearrowleft_{(d_1,d_2,d_3)} \big(-1\big)^{|d_1||d_3|}[d_1, d_2d_3 - (-1)^{|d_2||d_3|}d_3d_2] \\
&= \circlearrowleft_{(d_1,d_2,d_3)} \big(-1\big)^{|d_1||d_3|}\Big([d_1, d_2d_3] - (-1)^{|d_2||d_3|}[d_1, d_3d_2]\Big) \\
&= \circlearrowleft_{(d_1,d_2,d_3)} \big(-1\big)^{|d_1||d_3|}\Big(d_1d_2d_3 - (-1)^{|d_1|(|d_2|+|d_3|)}d_2d_3d_1 \\
&\quad -(-1)^{|d_2||d_3|}d_1d_3d_2 + (-1)^{|d_1|(|d_2|+|d_3|)+|d_2||d_3|}d_3d_2d_1\Big) \\
&= \circlearrowleft_{(d_1,d_2,d_3)} \Big(\big(-1\big)^{|d_1||d_3|}d_1d_2d_3 - (-1)^{|d_1||d_2|}d_2d_3d_1 \\
&\quad -(-1)^{|d_1||d_3|+|d_2||d_3|}d_1d_3d_2 + (-1)^{|d_1||d_2|+|d_2||d_3|}d_3d_2d_1\Big) \\
&= \big(-1\big)^{|d_1||d_3|}d_1d_2d_3 - (-1)^{|d_1||d_2|}d_2d_3d_1 \\
&\quad -(-1)^{|d_1||d_3|+|d_2||d_3|}d_1d_3d_2 + (-1)^{|d_1||d_2|+|d_2||d_3|}d_3d_2d_1 \\
&\quad +\big(-1\big)^{|d_1||d_2|}d_2d_3d_1 - (-1)^{|d_2||d_3|}d_3d_1d_2 \\
&\quad -(-1)^{|d_1||d_2|+|d_1||d_3|}d_2d_1d_3 + (-1)^{|d_2||d_3|+|d_1||d_3|}d_1d_3d_2 \\
&\quad +\big(-1\big)^{|d_2||d_3|}d_3d_1d_2 - (-1)^{|d_3||d_1|}d_1d_2d_3 \\
&\quad -(-1)^{|d_2||d_3|+|d_2||d_1|}d_3d_2d_1 + (-1)^{|d_3||d_1|+|d_2||d_1|}d_2d_1d_3 \\
&= 0.
\end{aligned}
$$

Thus, the super Jacobi identity holds on $SDer(S)$.

Hence, since the bracket is even bilinear, preserves the $\mathbb{Z}_2$-grading, satisfies the anti-supersymmetry, and fulfills the super Jacobi identity, the superspace $(SDer(S), [,])$ is a Lie superalgebra. □

Proposition 3. *Let $(S, \bullet)$ be a J-J superalgebra and $d_1; d_2 \in SADer(S)$. Then $\{d_1, d_2\} \in SADer(S)$ if and only if for all $x, y \in \mathcal{H}(S)$*

$$\{d_1, d_2\}(x \bullet y) = \big(-1\big)^{|d_1|(|x|+|d_2|)}d_2(x) \bullet d_1(y) + \big(-1\big)^{|d_2||x|}d_1(x) \bullet d_2(y). \tag{9}$$

Proof. Let $d_1, d_2 \in SADer(S)$ and $x, y \in \mathcal{H}(S)$. Then, we have

$$
\begin{aligned}
&\{d_1, d_2\}(x \bullet y) \\
&= d_1d_2(x \bullet y) + (-1)^{|d_1||d_2|}d_2d_1(x \bullet y) \\
&\overset{(7)}{=} d_1\Big(-d_2(x) \bullet y - \big(-1\big)^{|d_2||x|}x \bullet d_2(y)\Big) \\
&\quad +(-1)^{|d_1||d_2|}d_2\Big(-d_1(x) \bullet y - \big(-1\big)^{|d_1||x|}x \bullet d_1(y)\Big) \\
&= d_1\Big(-d_2(x) \bullet y\Big) - \big(-1\big)^{|d_2||x|}d_1\Big(x \bullet d_2(y)\Big) \\
&\quad +(-1)^{|d_1||d_2|}d_2\Big(-d_1(x) \bullet y\Big) - \big(-1\big)^{|d_1||x|}(-1)^{|d_1||d_2|}d_2\Big(x \bullet d_1(y)\Big) \\
&= d_1(d_2(x)) \bullet y + \big(-1\big)^{|d_1||d_2(x)|}d_2(x) \bullet d_1(y) \\
&\quad +\big(-1\big)^{|d_2||x|}d_1(x) \bullet d_2(y)\Big) + \big(-1\big)^{|d_2||x|+|d_1||x|}x \bullet d_1(d_2(y))
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{|d_1||d_2|}d_2(d_1(x))\bullet y+\big(-1\big)^{|d_1||d_2|+|d_2||d_1(x)|}d_1(x)\bullet d_2(y)\\
&+\big(-1\big)^{|d_1||d_2|+|d_1||x|}d_2(x)\bullet d_1(y)\\
&+\big(-1\big)^{|d_1||d_2|+|d_1||x|+|d_2||x|}x\bullet d_2(d_1(y))\\
\overset{(7)}{=}\ &\{d_1,d_2\}(x)\bullet y+\big(-1\big)^{|\{d_1,d_2\}||x|}x\bullet\{d_1,d_2\}(y)\\
&+2\big(-1\big)^{|d_1|(|x|+|d_2|)}d_2(x)\bullet d_1(y)\\
&+2\big(-1\big)^{|d_2||x|}d_1(x)\bullet d_2(y).
\end{aligned}
$$

Hence, by (10) we obtain that $\{d_1,d_2\}\in SADer(S)$ if and only if

$$
\begin{aligned}
\{d_1,d_2\}(x\bullet y) \ &= \ -\{d_1,d_2\}(x\bullet y)+2\big(-1\big)^{|d_1|(|x|+|d_2|)}d_2(x)\bullet d_1(y)\\
&\quad +2\big(-1\big)^{|d_2||x|}d_1(x)\bullet d_2(y)
\end{aligned}
$$

i.e. the relation (9) holds. □

In a similar way, we can easily prove the following:

Proposition 4. *Let $(S,\bullet)$ be a J-J superalgebra.*

(i) For all $d_1,d_2\in SADer(S)$, $\{d_1,d_2\}\in SDer(S)$ if and only if

$$\big(-1\big)^{|d_1|(|x|+|d_2|)}d_2(x)\bullet d_1(y)+\big(-1\big)^{|d_2||x|}d_1(x)\bullet d_2(y)=0 \tag{10}$$

holds for all $x,y\in\mathcal{H}(S)$.

(ii) For all $d_1\in SDer(S)$ and $d_2\in SADer(S)$, $\{d_1,d_2\}\in SDer(S)$ if and only if

$$\{d_1,d_2\}(x\bullet y)=-\big(-1\big)^{|d_1|(|x|+|d_2|)}d_2(x)\bullet d_1(y)-\big(-1\big)^{|d_2||x|}d_1(x)\bullet d_2(y)$$

holds for all $x,y\in\mathcal{H}(S)$.

(iii) For all $d_1\in SDer(S)$ and $d_2\in SADer(S)$, $\{d_1,d_2\}\in SADer(S)$ if and only if (10) holds.

(iv) For all $d_1,d_2\in SDer(S)$, $\{d_1,d_2\}\in SDer(S)$ if and only if (10) holds.

(v) For all $d_1,d_2\in SDer(S)$, $\{d_1,d_2\}\in SADer(S)$ if and only if (9) holds.

4 Cohomologies of J-J superalgebras and Extensions

4.1 Differential operators on J-J superalgebras

Definition 6. *Let S and V be two vector superspaces ($\mathbb{Z}_2$-graded spaces). A homogeneous n-linear map $f:S\times S\times\cdots\times S\longrightarrow V$ is called*
1. supersymmetric if

$$f(x_1,\ldots,x_{i-1},x_i,x_{i+1},x_{i+2},\ldots,x_n)=(-1)^{|x_i||x_{i+1}|}\,f(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)$$

2. anti-supersymmetric if

$$\begin{aligned} &f(x_1,\dots,x_{i-1},x_i,x_{i+1},x_{i+2},\dots,x_n)\\ &=-(-1)^{|x_i||x_{i+1}|}f(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n) \end{aligned} \tag{11}$$

In the sequel, for all $n\in\mathbb{N}$,

1. let $\mathcal{C}^n(S,V)$ denote the space of homogeneous n-linear maps from $S\times\cdots\times S$ to V;
2. $\mathcal{A}^n(S,V)=\{f\in\mathcal{C}^n(S,V), f \text{ anti-supersymmetric}\}$;
3. $\mathcal{S}^n(S,V)=\{f\in\mathcal{C}^n(S,V), f \text{ supersymmetric}\}$;
4. $\mathcal{C}^0(S,V):=\mathcal{S}^0(S,V):=\mathcal{A}^0(S,V):=V$.

Definition 7. *Let $(S,\bullet)$ be a J-J superalgebra and (V,π) be an even representation of it. We define two sequences of even linear maps $\{\delta^n\}_{n\in\mathbb{N}}$ and $\{\rho^n\}_{n\in\mathbb{N}}$ as follows:*

$$0\longrightarrow\mathcal{C}^0(S,V)\xrightarrow{\delta^0}\mathcal{C}^1(S,V)\xrightarrow{\delta^1}\mathcal{C}^2(S,V)\xrightarrow{\delta^2}\cdots\xrightarrow{\delta^{n-1}}\mathcal{C}^n(S,V)\xrightarrow{\delta^n}\mathcal{C}^{n+1}(S,V)\xrightarrow{\delta^{n+1}}\cdots$$

and

$$\mathcal{A}^n(S,V)\xrightarrow{\rho^n}\mathcal{C}^{n+1}(S,V)$$

by

$$\delta^0(v)x=\rho^0(v)x:=(-1)^{|x||v|}\pi(x)v, \tag{12}$$

$$\begin{aligned} \delta^n(\varphi)(x_1,\dots,x_{n+1}):=&\sum_{i=1}^{n+1}(-1)^{(|\varphi|+|x_1|+\cdots+|x_{i-1}|)|x_i|}\pi(x_i)\varphi(x_1,\dots,\hat{x}_i,\\ &\dots,x_{n+1})+\sum_{1\le i<j\le n+1}(-1)^{(|x_1|+\cdots+|x_{i-1}|)|x_i|+(|x_1|+\cdots+|x_{j-1}|)|x_j|+|x_i||x_j|}\varphi(x_i\bullet x_j,\\ &x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \rho^n(\varphi)(x_1,\dots,x_{n+1}):=&\sum_{i=1}^{n+1}(-1)^{(|\varphi|+|x_1|+\cdots+|x_{i-1}|)|x_i|}\pi(x_i)\varphi(x_1,\dots,\hat{x}_i,\\ &\dots,x_{n+1})-\sum_{1\le i<j\le n+1}(-1)^{(|x_1|+\cdots+|x_{i-1}|)|x_i|+(|x_1|+\cdots+|x_{j-1}|)|x_j|+|x_i||x_j|}\varphi(x_i\bullet x_j,\\ &x_1,\dots,\hat{x}_i,\dots,\hat{x}_j,\dots,x_{n+1}) \end{aligned} \tag{14}$$

where the sign $\hat{x}$ indicates that the element x must be omitted.

We have the following result.

Proposition 5. *If $n = 1, 2$, then $\delta^n \circ \rho^{n-1} \equiv 0$.*

Proof. To see what happens for $n = 1$ and $n = 2$, we have first:

$$\begin{aligned}
& (\delta^1 \circ \rho^0)\varphi(x_1, x_2) \\
= \; & (\rho^0\varphi)(x_1 \bullet x_2) + (-1)^{|x_1||\rho^0\varphi|}\, \pi(x_1)(\rho^0\varphi)(x_2) \\
& + (-1)^{|\rho^0\varphi||x_2|+|x_1||x_2|}\, \pi(x_2)(\rho^0\varphi)(x_1) \\
\overset{(14)}{=} \; & (-1)^{|x_1\bullet x_2||\varphi|}\, \pi(x_1 \bullet x_2)\varphi + (-1)^{|x_1||\rho^0\varphi|}\, \pi(x_1)\Big((-1)^{|x_2||\varphi|}\, \pi(x_2)\varphi\Big) \\
& + (-1)^{|\rho^0\varphi||x_2|+|x_1||x_2|}\, \pi(x_2)\Big((-1)^{|x_1||\varphi|}\, \pi(x_1)\varphi\Big) \\
\overset{(5)}{=} \; & 0.
\end{aligned}$$

Next, we compute

$$\begin{aligned}
& (\delta^2 \circ \rho^1)\varphi(x_1, x_2, x_3) \\
= \; & (-1)^{|\rho^1\varphi||x_1|}\, \pi(x_1)(\rho^1\varphi)(x_2, x_3) + (-1)^{|\rho^1\varphi||x_2|+|x_1||x_2|}\, \pi(x_2)(\rho^1\varphi)(x_1, x_3) \\
& + (-1)^{|\rho^1\varphi||x_3|+|x_1||x_3|+|x_2||x_3|}\, \pi(x_3)(\rho^1\varphi)(x_1, x_2) + (\rho^1\varphi)(x_1 \bullet x_2, x_3) \\
& + (-1)^{|x_2||x_3|}\, (\rho^1\varphi)(x_1 \bullet x_3, x_2) + (-1)^{|x_1||x_2|+|x_1||x_3|}\, (\rho^1\varphi)(x_2 \bullet x_3, x_1). \qquad (15)
\end{aligned}$$

Observe that

$$\begin{aligned}
\rho^1\varphi(x_1 \bullet x_2, x_3) &= (-1)^{|x_1\bullet x_2||\varphi|}\, \pi(x_1 \bullet x_2)\varphi(x_3) \\
&\quad + (-1)^{|\varphi||x_3|+|x_1\bullet x_2||x_3|}\, \pi(x_3)\varphi(x_1 \bullet x_2) - \varphi((x_1 \bullet x_2) \bullet x_3), \\
\rho^1\varphi(x_1 \bullet x_3, x_2) &= (-1)^{|x_1\bullet x_3||\varphi|}\, \pi(x_1 \bullet x_3)\varphi(x_2) \\
&\quad + (-1)^{|\varphi||x_2|+|x_1\bullet x_3||x_2|}\, \pi(x_2)\varphi(x_1 \bullet x_3) - \varphi((x_1 \bullet x_3) \bullet x_2), \\
\rho^1\varphi(x_2 \bullet x_3, x_1) &= (-1)^{|x_2\bullet x_3||\varphi|}\, \pi(x_2 \bullet x_3)\varphi(x_1) \\
&\quad + (-1)^{|\varphi||x_1|+|x_2\bullet x_3||x_1|}\, \pi(x_1)\varphi(x_2 \bullet x_3) - \varphi((x_2 \bullet x_3) \bullet x_1).
\end{aligned}$$

Hence, the relation (15) becomes

$$\begin{aligned}
& (\delta^2 \circ \rho^1)\varphi(x_1, x_2, x_3) \\
= \; & (-1)^{|\varphi||x_1|+|\varphi||x_2|}\, \Big(\pi(x_1 \bullet x_2) + (-1)^{|\rho^1\varphi||x_1|+|\varphi||x_1|}\, \pi(x_1)\pi(x_2) \\
& + (-1)^{|\rho^1\varphi||x_2|+|\varphi||x_2|+|x_1||x_2|}\, \pi(x_2)\pi(x_1)\Big)\varphi(x_3) \\
& + (-1)^{|\varphi||x_1|+|\varphi||x_3|+||x_2|||x_3|}\, \Big(\pi(x_1 \bullet x_3) + (-1)^{|\rho^1\varphi||x_1|+|\varphi||x_1|}\, \pi(x_1)\pi(x_3) \\
& + (-1)^{|\rho^1\varphi||x_3|+|\varphi||x_3|+|x_1||x_3|}\, \pi(x_3)\pi(x_1)\Big)\varphi(x_2) \\
& + (-1)^{|\varphi||x_2|+|\varphi||x_3|+||x_1|||x_2|+||x_1|||x_3|}\, \Big(\pi(x_2 \bullet x_3) + (-1)^{|\rho^1\varphi||x_2|+|\varphi||x_2|}\, \pi(x_2)\pi(x_3) \\
& + (-1)^{|\rho^1\varphi||x_3|+|\varphi||x_3|+|x_2||x_3|}\, \pi(x_3)\pi(x_2)\Big)\varphi(x_1)
\end{aligned}$$

$$
\begin{aligned}
&+\Big((-1)^{|\varphi||x_1|} - (-1)^{|\rho^1\varphi||x_1|} \Big)\pi(x_1)\varphi(x_2 \bullet x_3) \\
&+\Big((-1)^{|\varphi||x_2|+|x_1||x_2|} - (-1)^{|\rho^1\varphi||x_2|+|x_1||x_2|} \Big)\pi(x_2)\varphi(x_1 \bullet x_3) \\
&+\Big((-1)^{|\varphi||x_3|+|x_1||x_3|+|x_2||x_3|} - (-1)^{|\rho^1\varphi||x_3|+|x_1||x_3|+|x_2||x_3|} \Big)\pi(x_3)\varphi(x_1 \bullet x_2) \\
&-\varphi\Big((x_1 \bullet x_2) \bullet x_3 + (-1)^{|x_2||x_3|} (x_1 \bullet x_3) \bullet x_2 \\
&+ (-1)^{|x_1||x_2|+|x_1||x_3|} (x_2 \bullet x_3) \bullet x_1 \Big) \\
= \;& 0
\end{aligned}
$$

by (5) and the fact that ρ^1 is even i.e., $|\rho^1\varphi| = |\varphi|$. □

We now obtain the following generalization.

Theorem 1. *If $n > 0$ then the operators δ^n and ρ^{n-1} satisfy $\delta^n \circ \rho^{n-1} \equiv 0$.*

Proof. To simplify the expressions in the rest of this proof, let put for all $p \in \mathbb{N} \setminus \{0\}$,

$$|x_{1,p}| := (|x_1| + \cdots + |x_{p-1}|)|x_p| \text{ and } |x_{1,1}| = |x_1|. \tag{16}$$

Then, for any integer $n \geq 1$ and for any $\varphi \in \mathcal{A}^{n-1}(S, V)$, we have

$$\delta^n \circ \rho^{n-1}\varphi(x_1, x_2, \ldots, x_{n+1}) = A + B$$

where

$$
\begin{aligned}
A &= \sum_{i=1}^{n+1} (-1)^{(|\rho^{n-1}\varphi||x_i|+|x_{1,i}|)}\, \pi(x_i)(\rho^{n-1}\varphi)(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}), \\
B &= \sum_{1\leq i<j\leq n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|)}\, (\rho^{n-1}\varphi)(x_i \bullet x_j, x_1; \ldots; \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}).
\end{aligned}
$$

Observe that for all fixed $i \in \mathbb{N}$ with $1 \leq i \leq n+1$, we have

$$
\begin{aligned}
&(\rho^{n-1}\varphi)(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\
&= \sum_{j=1, i<j}^{n+1} (-1)^{(|x_{1,j}|+|\varphi||x_j|+|x_i||x_j|)}\, \pi(x_j)\varphi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}) \\
&+ \sum_{j=1, i>j}^{n+1} (-1)^{(|x_{1,j}|+|\varphi||x_j|)}\, \pi(x_j)\varphi(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\
&- \sum_{1\leq j<k\leq n+1, i<j} (-1)^{(|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|+|x_j||x_k|)}\, \varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_i, \\
&\ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}) - \sum_{1\leq j<k\leq n+1, k<i} (-1)^{(|x_{1,j}|+|x_{1,k}|+|x_j||x_k|)}\, \varphi(x_j \bullet x_k, x_1, \ldots, \\
&\hat{x}_j, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{n+1}) - \sum_{1\leq j<k\leq n+1, j<i<k} (-1)^{(|x_{1,j}|+|x_{1,k}|+|x_j||x_k|+|x_i||x_k|)}
\end{aligned}
$$

$$\varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_{n+1}).$$

Therefore,

$$A = A_1 - A_2 - A_3 - A_4, \tag{17}$$

where

$$\begin{aligned}
A_1 &= \sum_{1\le i<j\le n+1} (-1)^{(|x_{1,i}|+|\rho^{n-1}\varphi||x_i|+|x_{1,j}|+|\varphi||x_j|)} \Big((-1)^{|x_i||x_j|}\, \pi(x_i)\pi(x_j) + \pi(x_j)\pi(x_i \Big) \\
&\quad \varphi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}), \\
A_2 &= \sum_{1\le i<j<k\le n+1} (-1)^{(|x_{1,i}|+|\rho^{n-1}\varphi||x_i|+|x_{1,j}|+|x_{1,k}|+|x_i||x_j|+|x_i||x_k|+|x_j||x_k|)}\, \pi(x_i) \\
&\quad \varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}), \\
A_3 &= \sum_{1\le j<k<i\le n+1} (-1)^{(|x_{1,i}|+|\rho^{n-1}\varphi||x_i|+|x_{1,j}|+|x_{1,k}|+|x_j||x_k|)}\, \pi(x_i) \\
&\quad \varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{n+1}), \\
A_4 &= \sum_{1\le j<i<k\le n+1} (-1)^{(|x_{1,i}|+|\rho^{n-1}\varphi||x_i|+|x_{1,j}|+|x_{1,k}|+|x_i||x_k|+|x_j||x_k|)}\, \pi(x_i) \\
&\quad \varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_{n+1}).
\end{aligned}$$

Next, for all fixed $(i, j) \in \{1, \ldots, n+1\}^2$ with $i < j$, we have:

$$\begin{aligned}
&\rho^{n-1}\varphi(x_i \bullet x_j, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}) \\
&= (-1)^{|\varphi||x_i \bullet x_j|}\pi(x_i \bullet x_j)\varphi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}) \\
&\quad + \sum_{1\le i\le n+1, i<j<k} (-1)^{((|\varphi|+|x_j \bullet x_k|)|x_i|+|x_{1,i}|)}\, \pi(x_i)\varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \\
&\ldots, \hat{x}_k, \ldots, x_{n+1}) + \sum_{1\le i\le n+1, j<i<k} (-1)^{((|\varphi|+|x_j \bullet x_k|)|x_i|+|x_{1,i}|+|x_i||x_j|)}\, \pi(x_i)\varphi(x_j \bullet x_k, \\
&x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, x_{n+1}) \\
&\quad + \sum_{1\le i\le n+1, j<k<i} (-1)^{((|\varphi|+|x_j \bullet x_k|)|x_i|+|x_{1,i}|+|x_i||x_j|+|x_i||x_k|)}\, \pi(x_i)\varphi(x_j \bullet x_k, x_1, \ldots, \hat{x}_j \\
&, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \ldots, x_{n+1}) - \sum_{1\le k\le n+1,\ k<i} (-1)^{(|x_{1,k}|)}\, \varphi((x_i \bullet x_j) \bullet x_k, x_1, \ldots, \hat{x}_k, \\
&\ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}) - \sum_{1\le k\le n+1,\ i<k<j} (-1)^{(|x_{1,k}|+|x_i||x_k|)}\, \varphi((x_i \bullet x_j) \bullet x_k, x_1, \ldots, \\
&\hat{x}_i, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_{n+1}) - \sum_{1\le k\le n+1,\ i<j<k} (-1)^{(|x_{1,k}|+|x_i||x_k|+|x_j||x_k|)} \\
&\times\varphi((x_i \bullet x_j) \bullet x_k, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}) \\
&- \sum_{1\le k<l\le n+1,\ i<j<k<l} (-1)^{(|x_{1,k}|+|x_i||x_k|+|x_j||x_k|+|x_{1,l}|)}
\end{aligned}$$

$$\times (-1)^{(|x_i||x_l|+|x_j||x_l|+|x_k||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,$$
$$\ldots,\hat{x}_k,\ldots,\hat{x}_l+\ldots x_{n+1}) - \sum_{1\leq k<l\leq n+1,\ i<k<l<j} (-1)^{(|x_{1,k}|+|x_i||x_k|+|x_{1,l}|)}$$
$$(-1)^{(|x_i||x_l|+|x_k||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k,$$
$$\ldots,\hat{x}_l,\ldots,\hat{x}_j\ldots,x_{n+1}) - \sum_{1\leq k<l\leq n+1,\ k<i<j<l} (-1)^{(|x_{1,k}|+|x_{1,l}|)}$$
$$\times (-1)^{(|x_k||x_l|+|x_i||x_l|+|x_j||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,\ldots,\hat{x}_k,\ldots,\hat{x}_i,$$
$$\ldots,\hat{x}_j,\ldots,\hat{x}_l\ldots,x_{n+1}) - \sum_{1\leq k<l\leq n+1,\ k<l<i<j} (-1)^{(|x_{1,k}|+|x_{1,l}|)}$$
$$\times (-1)^{(|x_k||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,\ldots,\hat{x}_k,\ldots,\hat{x}_l,\ldots,\hat{x}_i,\ldots,\hat{x}_j\ldots,x_{n+1})$$

$$- \sum_{1\leq k<l\leq n+1,\ k<i<l<j} (-1)^{(|x_{1,k}|+|x_{1,l}|}\,(-1)^{|x_k||x_l|+|x_i||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,$$
$$\ldots,\hat{x}_k,\ldots,\hat{x}_i,\ldots,\hat{x}_l,\ldots,\hat{x}_j\ldots,x_{n+1}) - \sum_{1\leq k<l\leq n+1,\ i<k<j<l} (-1)^{(|x_{1,k}|)}$$
$$\times (-1)^{(|x_i||x_k|+|x_{1,l}|+|x_k||x_l|+|x_i||x_l|+|x_j||x_l|)}\,\varphi(x_i\bullet x_j, x_k\bullet x_l, x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_k,$$
$$\ldots,\hat{x}_j,\ldots,\hat{x}_l\ldots,x_{n+1}).$$

Therefore,

$$\begin{aligned} B \quad &= B_1+B_2+B_3+B_4 \\ &\quad -(B_5+B_6+B_7+B_8+B_9+B_{10}+B_{11}+B_{12}+B_{13}), \end{aligned} \tag{18}$$

where

$$\begin{aligned}
B_1 &= \sum_{1\leq i<j\leq n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|\varphi|(|x_i|+|x_j|))}\Big((-1)^{|x_i||x_j|}\,\pi(x_i\bullet x_j)\Big)\\
&\quad \varphi(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_{n+1}),\\
B_2 &= \sum_{1\leq i<j<k\leq n+1} (-1)^{(|x_{1,j}|+|x_{1,k}|+|x_j||x_k|+(|\varphi|+|x_j\bullet x_k|)|x_i|+|x_{1,i}|)}\,\pi(x_i)\\
&\quad \varphi(x_j\bullet x_k, x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,\hat{x}_k,\ldots,x_{n+1}),\\
B_3 &= \sum_{1\leq j<k<i\leq n+1} (-1)^{(|x_{1,j}|+|x_{1,k}|+|x_j||x_k|+(|\varphi|+|x_j\bullet x_k|)|x_i|+|x_{1,i}|+|x_i||x_j|+|x_i||x_k|)}\,\pi(x_i)\\
&\quad \varphi(x_j\bullet x_k, x_1,\ldots,\hat{x}_j,\ldots,\hat{x}_k,\ldots,\hat{x}_i,\ldots,x_{n+1}),\\
B_4 &= \sum_{1\leq j<i<k\leq n+1} (-1)^{(|x_{1,j}|+|x_{1,k}|+|x_j||x_k|+(|\varphi|+|x_j\bullet x_k|)|x_i|+|x_{1,i}|+|x_i||x_j|)}\,\pi(x_i)\\
&\quad \varphi(x_j\bullet x_k, x_1,\ldots,\hat{x}_j,\ldots,\hat{x}_i,\ldots,\hat{x}_k,\ldots,x_{n+1}),
\end{aligned}$$

$$B_5 = \sum_{1\leq k<i<j\leq n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|)}\,\varphi((x_i\bullet x_j)\bullet x_k, x_1, ..., \hat{x}_k,$$

$$
\begin{aligned}
& \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n+1}), \\
B_6 \;=\; & \sum_{1\le i<k<j\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|)}\, \varphi((x_i \bullet x_j) \bullet x_k, x_1, \\
& \ldots, \hat{x}_i, \ldots, \hat{x}_k, \ldots, \hat{x}_j, \ldots, x_{n+1}), \\
B_7 \;=\; & \sum_{1\le i<j<k\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|+|x_j||x_k|)}\, \varphi((x_i \bullet x_j) \bullet x_k, x_1, \\
& \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}), \\
B_8 \;=\; & \sum_{1\le i<j<k<l\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|+|x_j||x_k|+|x_{1,l}|)} \\
& (-1)^{(|x_i||x_l|+|x_j||x_l|+|x_k||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \\
& \ldots, \hat{x}_k, \ldots, \hat{x}_l + \cdots + x_{n+1}), \\
B_9 \;=\; & \sum_{1\le i<k<l<j\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|+|x_{1,l}|)} \\
& (-1)^{(|x_i||x_l|+|x_k||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \\
& \ldots, \hat{x}_l, \ldots, \hat{x}_j \ldots, x_{n+1}),
\end{aligned}
$$

$$
\begin{aligned}
B_{10} \;=\; & \sum_{1\le k<i<j<l\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_{1,l}|)} \\
& (-1)^{(|x_k||x_l|+|x_i||x_l|+|x_j||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \\
& \ldots, \hat{x}_j, \ldots, \hat{x}_l \ldots, x_{n+1}), \\
B_{11} \;=\; & \sum_{1\le k<l<i<j\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_{1,l}|)} \\
& (-1)^{(|x_k||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_l, \\
& \ldots, \hat{x}_i, \ldots, \hat{x}_j \ldots, x_{n+1}), \\
B_{12} \;=\; & \sum_{1\le k<i<l<j\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_{1,l}|)} \\
& (-1)^{(|x_k||x_l|+|x_i||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_k, \ldots, \hat{x}_i, \\
& \ldots, \hat{x}_l, \ldots, \hat{x}_j \ldots, x_{n+1}), \\
B_{13} \;=\; & \sum_{1\le i<k<j<l\le n+1} (-1)^{(|x_{1,i}|+|x_{1,j}|+|x_i||x_j|+|x_{1,k}|+|x_i||x_k|+|x_{1,l}|)} \\
& (-1)^{(|x_k||x_l|+|x_i||x_l|+|x_j||x_l|)}\, \varphi(x_i \bullet x_j, x_k \bullet x_l, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_k, \\
& \ldots, \hat{x}_j, \ldots, \hat{x}_l \ldots, x_{n+1}).
\end{aligned}
$$

We know that for all homogeneous elements x and y, we have $|x \bullet y| = |x| + |y|$. Furthermore, for any integer $n \geq 0$, ρ^n is an even linear map i.e., $|\rho^{n-1}\varphi| = |\varphi|$. Hence, $A_1 + B_2 = 0$ by (5). Next, observe that

$$B_2 - A_2 = B_3 - A_3 = B_4 - A_4 = 0 \text{ since } |\rho^{n-1}\varphi| = |\varphi| \ .$$

In addition, $B_5 + B_6 + B_7 = 0$ by (2) and $B_8 + B_9 + B_{10} + B_{11} + B_{12} + B_{13} = 0$ by (11). Hence $\delta^n \circ \rho^{n-1} = 0$. as desired. □

Let $(S, \bullet)$ be a J-J superalgebra and (V, π) be a representation on it. We denote by $\mathcal{Z}^n(S,V) := Ker(\delta^n)$ the superspace of $n-$cocycles and by $\mathcal{B}^n(S,V) := \Big\{ f \in \mathcal{C}^n(S,V); \exists f' \in \mathcal{A}^{n-1}(S;V) : \rho^{n-1} f' = f \Big\}$ the superspace of coboundaries. In particular, for $V = S$, we denote these sets by $\mathcal{Z}^n(S)$ and $\mathcal{B}^n(S)$ respectively. Then, the nth cohomology superspace of S with coefficients in V is the quotient superspace

$$\mathcal{H}^n(S,V) := \mathcal{Z}^n(S,V) \diagup \mathcal{B}^n(S,V). \tag{19}$$

In particular, for $V = S$, the nth cohomology superspace of S with coefficients in S is the quotient superspace

$$\mathcal{H}^n(S) := \mathcal{Z}^n(S) \diagup \mathcal{B}^n(S). \tag{20}$$

4.2 Cohomology of J-J superalgebras in degree 0 and in degree 1

By convention, we set $\mathcal{B}^0(S,V) = 0$ and therefore, in degree zero we have

$$\begin{aligned} \mathcal{H}^0(S,V) &= \mathcal{Z}^0(S,V) \\ &= Ker\Big(\delta^0 : V \longrightarrow \mathcal{C}^1(S,V)\Big) \\ &= \Big\{ v \in \mathcal{H}(V) : (-1)^{|v||x|} \pi(x)v = 0, \forall x \in \mathcal{H}(S) \Big\}. \end{aligned}$$

In particular, for $V = S$, we get

$$\mathcal{H}^0(S) = \{ x \in \mathcal{H}(S) : x \bullet y = 0, \forall y \in \mathcal{H}(S) \}.$$

Let us call $\mathcal{H}^0(S)$ the annihilator of S and denote it by $Ann(S)$.

Now, let us investigate the first cohomology space. We have first,

$$\begin{aligned} \mathcal{Z}^1(S,V) &= Ker(\delta^1 : \mathcal{C}^1(S,V) \to \mathcal{C}^2(S,V)) \\ &= \Big\{ \varphi \in \mathcal{C}^1(S,V), \pi(x_1)\varphi(x_2) + (-1)^{|x_1||x_2|} \pi(x_2)\varphi(x_1) \\ &\quad + \varphi(x_1 \bullet x_2) = 0, \forall x_1, x_2 \in \mathcal{H}(S) \Big\} \\ &= SADer(S,V) \end{aligned}$$

and next,

$$\begin{aligned} \mathcal{B}^1(S,V) &= Im(\rho^0 : V \to \mathcal{C}^1(S,V)) \\ &= InnSADer(S,V) \end{aligned}$$

since $\rho^0 v(x) = (-1)^{|v||x|} \pi(x)v = d_v(x), \forall x \in \mathcal{H}(S)$. Hence, we have

$$\mathcal{H}^1(S,V) := SADer(S,V) \diagup InnSADer(S,V).$$

This space is called the space of outer anti-superderivations of S with coefficients in V denoted by $OSADer(S,V)$. In particular for $V=S$, we get space of outer anti-superderivations of S denoted by $OSADer(S)$ i.e.,

$$\mathcal{H}^1(S) := SADer(S)/InnSADer(S) = OSADer(S).$$

Let us provide some concrete examples of cohomology spaces in degree 1. For each example of a given J-J superalgebra, we compute the even and odd antiderivations separately.

Example 4. Let us consider the J-J superalgebra J in Example 1. Since an even linear map preserves the grading and an odd linear map swaps the components, we obtain:

$$SADer(J)_0 := \left\{ \begin{pmatrix} -2a_1 & 0 \\ 0 & a_1 \end{pmatrix}, a_1 \in \mathbb{R} \right\} \text{ and } SADer(J)_1 := \left\{ \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, a_2 \in \mathbb{R} \right\}.$$

Hence, $SADer(J) = SADer(J)_0 \oplus SADer(J)_1 := \left\{ \begin{pmatrix} -2a_1 & a_2 \\ 0 & a_1 \end{pmatrix}, a_1,\ a_2 \in \mathbb{R} \right\}$. Similarly, we obtain $InnSADer(J) := \left\{ \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix}, a_2 \in \mathbb{R} \right\}$. It follows that

$$\mathcal{H}^1(J) := \left\{ \begin{pmatrix} -2a_1 & 0 \\ 0 & a_1 \end{pmatrix} + InnSADer(J), a_1 \in \mathbb{R} \right\}.$$

Example 5. Let us consider the J-J superalgebra S in Example 2. Similarly as in Example 4 we have

$$SADer(S)_0 := \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & -2a & 0 & 0 \\ 0 & 0 & a & -q \\ 0 & 0 & q & a \end{pmatrix}, a,b,q \in \mathbb{R} \right\}$$

$$SADer(S)_1 := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & q & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, q,s \in \mathbb{R} \right\}.$$

Hence, the dimension of $SADer(S) = SADer(S)_0 \oplus SADer(S)_1$ is 5. In addition, $InnSADer(S) := \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & y & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x,y,z \in \mathbb{R} \right\}$ and its dimension is 3. It follows that $\mathcal{H}^1(S)$ is 2-dimensional vector superspace and

$$\mathcal{H}^1(S) := \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -2a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{pmatrix} + InnSADer(S), a,b,c,d,r,s \in \mathbb{R} \right\}.$$

4.3 Cohomology superspace of J-J superalgebras and abelian extension

In this section, we show that associated to any abelian extension, there is a representation and a 2-cocycle.

Definition 8. *Let S be a J-J superalgebra and V vector superspace. An abelian extension of S by V is an exact sequence*

$$0 \longrightarrow V \xrightarrow{i} \tilde{S} \xrightarrow{p} S \longrightarrow 0 \tag{21}$$

such that

(i) i and p are even morphisms,

(ii) $i(u) \bullet_{\tilde{S}} i(v) = 0$ for all $u, v \in V$,

(iii) $i(V) \bullet_{\tilde{S}} \tilde{S} \subseteq i(V)$.

In this case, $\tilde{S}$ is called an extension of S by V and is denoted by $E_{\tilde{S}}$. A section s of $p : \tilde{S} \to S$ consists of an even linear map $s : S \to \tilde{S}$ such that $p \circ s = Id_S$.

Proposition 6. *Let $(S, \bullet)$ be a J-J superalgebra and (V, π) be a representation on it and $c : S \times S \to V$ be an even bilinear map. Define the product $\bullet_c$ on $S \oplus V$ by*

$$(x, u) \bullet_c (y, v) := (x \bullet y, \pi(x)v + (-1)^{|x||y|}\pi(y)u + c(x, y)) \tag{22}$$

and set $|(x, u)| := |x|$ for all $x \in \mathcal{H}(S)$ and $u \in \mathcal{H}(V)$. Then,

$$S \ltimes_c V := (S \oplus V, \bullet_c)$$

is a J-J superalgebra if and only if c is a supersymetric 2-cocycle of S with coefficients in V.

Proof. The supersymetry of $\bullet_c$ is equivalent to the one of c. Next, pick $x, y, z \in \mathcal{H}(S)$ and $u, v, w \in \mathcal{H}(V)$. Then, observe first that

$$\Big((x, u) \bullet_c (y, v)\Big) \bullet_c (z, w) = \Big((x \bullet y) \bullet z, \pi(x \bullet y)w + (-1)^{|z|(|x|+|y|)}\pi(z)\pi(x)v$$
$$+(-1)^{|z|(|x|+|y|)+|x||y|}\pi(z)\pi(y)u + (-1)^{|z|(|x|+|y|)}\pi(z)c(x, y) + c(x \bullet y, z)\Big).$$

Then, we have

$$\Big((x, u) \bullet_c (y, v)\Big) \bullet_c (z, w) + (-1)^{|x|(|y|+|z|)}\Big((y, v) \bullet_c (z, w)\Big) \bullet_c (x, u)$$
$$+(-1)^{|z|(|x|+|y|)}\Big((z, w) \bullet_c (x, u)\Big) \bullet_c (y, v)$$
$$= \Big((x \bullet y) \bullet z, \pi(x \bullet y)w + (-1)^{|z|(|x|+|y|)}\pi(z)\pi(x)v$$
$$+(-1)^{|z|(|x|+|y|)+|x||y|}\pi(z)\pi(y)u + (-1)^{|z|(|x|+|y|)}\pi(z)c(x, y) + c(x \bullet y, z)\Big)$$

$$
\begin{aligned}
&+\Big((-1)^{|x|(|y|+|z|)}(y\bullet z)\bullet x, (-1)^{|x|(|y|+|z|)}\pi(y\bullet z)u+\pi(x)\pi(y)w\\
&+(-1)^{|y||z|}\pi(x)\pi(z)v+\pi(x)c(y,z)+(-1)^{|x|(|y|+|z|)}c(y\bullet z,x)\Big)\\
&+\Big((-1)^{|z|(|x|+|y|)}(z\bullet x)\bullet y, (-1)^{|z|(|x|+|y|)}\pi(z\bullet x)v+(-1)^{|x|(|y|+|z|)}\pi(y)\pi(z)u\\
&+(-1)^{|x||y|}\pi(y)\pi(x)w+(-1)^{|x|(|y|+|z|)}\pi(y)c(z,x)+(-1)^{|z|(|x|+|y|)}c(z\bullet x,y)\Big)\\
&=(0,\delta^2c(x,y,z))\ \text{(by (3) in } S\text{, (5) and the supersymetry of } c\ ).
\end{aligned}
$$

It follows that, (3) holds for $\bullet_c$ if and only if $c\in\mathcal{Z}^2(S,V)$. □

Now, let us give the following example of abelian extension of J-J superalgebras.

Example 6. Let $(S,\bullet)$ be a J-J superalgebra and (V,π) be a representation on it. Consider an even supersymetric 2-cocyce $c\in\mathcal{Z}^2(S,V)$ and the associated J-J superalgebra $(S\oplus V,\bullet_c)$ of the previous proposition. Then, $0\longrightarrow V\xrightarrow{i}S\oplus V\xrightarrow{p}S\longrightarrow 0$ is an abelian extension of S by V where $i(v)=(0,v)$ and $p(x+v)=x$, for all $x\in\mathcal{H}(S)$ and $v\in\mathcal{H}(V)$.

Corollary 2. *Let $(S,\bullet)$ be a J-J superalgebra and (V,π) be a representation on it. Then, $S\ltimes V:=(S\oplus V,\tilde{\bullet})$ is J-J superalgebra where*

$$(x,u)\tilde{\bullet}(y,v):=(x\bullet y,\pi(x)v+(-1)^{|x||y|}\pi(y)u) \tag{23}$$

and $|(x,u)|:=|x|$ for all $x\in\mathcal{H}(S)$ and $u\in\mathcal{H}(V)$. This J-J superalgebra is called the semi-direct product of S by V.

Definition 9. *$E_{\tilde{S}}: 0\longrightarrow V\xrightarrow{i}\tilde{S}\xrightarrow{p}S\longrightarrow 0$ et $E_{\hat{S}}: 0\longrightarrow V\xrightarrow{i}\hat{S}\xrightarrow{p}S\longrightarrow 0$ are said to be equivalent if there is a J-J superalgebra isomorphism $\varphi:\tilde{S}\longrightarrow\hat{S}$ such that the following diagram commutes*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{i} & \tilde{S} & \xrightarrow{p} & S & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle id_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id_S} & & \\
0 & \longrightarrow & V & \xrightarrow{j} & \hat{S} & \xrightarrow{q} & S & \longrightarrow & 0
\end{array}
\tag{24}
$$

We denote by $Ext(S,V)$, the set of equivalence classes of extensions of S by V.

Proposition 7. *Let $(S,\bullet)$ be a J-J superalgebra and (V,π) be a representation of it. If c and c' are two supersymetric even 2-cocycles equivalent, then the extensions $S\ltimes_c V$ and $S\ltimes'_c V$ of the J-J superalgebra by V, are equivalent.*

Proof. Suppose that c and c' are two supersymetric even 2-cocycles equivalent. Then, there is an even $h\in\mathcal{C}^1(S,V)$ such that $c'=c+\rho^1h$. Define the linear map $\varphi:S\oplus V\to S\oplus V$ by $\varphi(x,u):=(x,-h(x)+u)$. It is obvious that h is an isomorphism. Moreover, we have:

$$
\begin{aligned}
&\varphi(x,u)\bullet'_c\,\varphi(y,v)\\
&=(x\bullet y,-\pi(x)h(y)+\pi(x)v-(-1)^{|x||y|}\pi(y)h(x)+(-1)^{|x||y|}\pi(y)u+c'(x,y))
\end{aligned}
$$

$$= \varphi((x,u) \bullet_c (y,v)).$$

Hence, $\varphi : S \oplus V \to S \oplus V$ is an isomorphism of J-J superalgebras. Therefore, the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & S \ltimes_c V & \xrightarrow{p} & S & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id_V} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id_S} & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & S \ltimes'_c V & \xrightarrow{p} & S & \longrightarrow & 0 \end{array}$$

gives, the extensions $S \ltimes_c V$ and $S \ltimes'_c V$ are equivalent. □

Let $\tilde{S}$ be an abelian extension of S by V and $s : S \to \tilde{S}$ be a section. Define the even map $\rho : S \to End(V)$ by

$$\rho(x)u := s(x) \bullet_{\tilde{S}} u, \forall (x,u) \in S \times V. \tag{25}$$

Theorem 2. *Let V and S be J-J superalgebras. With the above notations, (V,ρ) is a representation of S and does not depend on the choice of the section s. Moreover, equivalent abelian extensions give the same representation.*

Proof. First, by choosing another section $s' : S \to \tilde{S}$ and $x \in S$ and by noting, we show that ρ is independent of the choice of s.

Next, we prove that (V,ρ) is a representation of $(S, \bullet_S)$. Pick $x, y \in S$, $u \in V$. Since $s(x) \bullet_{\tilde{S}} s(y) - s(x \bullet_S y) \in V$ we obtain first

$$\begin{aligned} \rho(x \bullet_S y)u &= s(x \bullet_S y) \bullet_{\tilde{S}} u = (s(x) \bullet_{\tilde{S}} s(y)) \bullet_{\tilde{S}} u. \\ &= -\big(-1\big)^{|s(x)|(|s(y)|+|u|)}(s(y) \bullet_{\tilde{S}} u) \bullet_{\tilde{S}} s(x) \\ &\quad -\big(-1\big)^{|u|(|s(x)|+|s(y)|)}(u \bullet_{\tilde{S}} s(x)) \bullet_{\tilde{S}} s(y) \\ &= -s(x) \bullet_{\tilde{S}} (s(y) \bullet_{\tilde{S}} u) \\ &\quad -\big(-1\big)^{|s(x)||s(y)|} s(y) \bullet_{\tilde{S}} (s(x) \bullet_{\tilde{S}} u) \text{ (by supersymetry of } \bullet_{\tilde{S}}) \\ &\quad -\rho(x)\rho(y)u - \big(-1\big)^{|x||y|}\rho(y)\rho(x)u. \end{aligned}$$

Thus, (V,ρ) is a representation of S.

Finally, we will show that equivalent abelian extensions give the same ρ. Suppose that $E_{\tilde{S}}$ and $E_{\hat{S}}$ are equivalent abelian extensions, and $\varphi : \tilde{S} \to \hat{S}$ is the J-J superalgebras morphism satisfying $\varphi \circ i = j$ and $q \circ \varphi = p$. Choosing linear sections s and s' of p and q, we obtain $q\varphi s(x) = ps(x) = x = qs'(x)$. Hence, $\varphi s(x) - s'(x) \in Ker(q) \cong V$. Moreover,

$$s(x) \bullet_{\tilde{S}} u = \varphi s(x) \bullet_{\tilde{S}} u = s'(x) \bullet_{\tilde{S}} u.$$

The proof is complete. □

Let $s : S \to \tilde{S}$ be a section of the abelian extension. Define the following map ω given by:

$$\omega(x_1, x_2) := s(x_1) \bullet_{\tilde{S}} s(x_2) - s(x_1 \bullet_S x_2), \forall x_1, x_2 \in S. \tag{26}$$

Observe that since s is an even linear map, the map ω is an even bilinear.

Theorem 3. *Let* $0 \longrightarrow V \xrightarrow{i} \tilde{S} \xrightarrow{p} S \longrightarrow 0$ *be an abelian extension of* S *by* V*. Then* ω *defined by (26) is a* 2*-cocycle of* S *with coefficients in* V*, where the representation* ρ *is given by (25).*

Proof. Pick $x_1, x_2, x_3 \in S$, then we have:

$$
\begin{aligned}
& \delta^2\omega(x_1,x_2,x_3) \\
= \; & (-1)^{|\omega||x_1|}\,\rho(x_1)\omega(x_2,x_3) + (-1)^{|\omega||x_2|+|x_1||x_2|}\,\rho(x_2)\omega(x_1,x_3) \\
& + (-1)^{|\omega||x_3|+|x_1||x_3|+|x_2||x_3|}\,\rho(x_3)\omega(x_1,x_2) + \omega(x_1 \bullet_S x_2, x_3) \\
& + (-1)^{|x_2||x_3|}\,\omega(x_1 \bullet_S x_3, x_2) + (-1)^{|x_1||x_2|+|x_1||x_3|}\,\omega(x_2 \bullet_S x_3, x_1) \\
= \; & \Big((-1)^{|\omega||x_1|}s(x_1) \bullet_{\tilde{S}} (s(x_2) \bullet_{\tilde{S}} s(x_3)) + (-1)^{|\omega||x_2|+|x_1||x_2|}s(x_2) \bullet_{\tilde{S}} \\
& (s(x_1) \bullet_{\tilde{S}} s(x_3)) + (-1)^{|\omega||x_2|+|x_1||x_3|+|x_2||x_3|}s(x_3) \bullet_{\tilde{S}} (s(x_1) \bullet_{\tilde{S}} s(x_2))\Big) \\
& -\Big(s((x_1 \bullet_S x_2) \bullet_S x_3) + (-1)^{|x_1||x_3|+|x_1|x_2|}s((x_2 \bullet_S x_3) \bullet_S x_1) \\
& +(-1)^{|x_2||x_3|}s((x_1 \bullet_S x_3) \bullet_S x_2)\Big) + \Big(s(x_1 \bullet_S x_2) \bullet_{\tilde{S}} s(x_3) \\
& -(-1)^{|x_3|(|\omega|+|x_1|+|x_2|)}s(x_3) \bullet_{\tilde{S}} s(x_1 \bullet_S x_2)\Big) + \Big((-1)^{|x_2||x_3|}s(x_1 \bullet_S x_3) \bullet_{\tilde{S}} s(x_2) \\
& -(-1)^{|x_2|(|\omega|+|x_1|)}s(x_2) \bullet_{\tilde{S}} s(x_1 \bullet_S x_3)\Big) + \Big((-1)^{|x_1|(|x_2|+|x_3|)}s(x_2 \bullet_S x_3) \bullet_{\tilde{S}} s(x_1) \\
& -(-1)^{|\omega||x_1|)}s(x_1) \bullet_{\tilde{S}} s(x_2 \bullet_S x_3)\Big) = 0
\end{aligned}
$$

where the last equality follows from the super Jacobi identity and the fact that ω is even i.e., $|\omega| = 0$. Hence, ω is a 2-cocycle. □

Outlook

This paper identifies several important directions for future research on the cohomology of Jacobi-Jordan superalgebras. A primary objective is to determine whether the cohomology groups $H^n(S)$ can be interpreted as Ext groups within an appropriate homological framework, thereby connecting the theory to broader tools in homological algebra. Another key direction is to extend the cohomological classification of abelian extensions to central extensions, including the study of obstruction classes, automorphisms, and deformation phenomena. More generally, these investigations aim to establish a comprehensive homological theory that strengthens the foundations of Jacobi-Jordan superalgebras and promotes further advances in their representation theory, deformation theory, and structural analysis.

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