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Approximate symmetry structure of the heat flow on perturbed flat revolution spaces

Akram Mohammadpouri*, Zahra Ghaffarzade

Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran

Email(s): pouri@tabrizu.ac.ir, z.ghaffarzade@phd.tabrizu.ac.ir

Abstract. This paper studies heat conduction on slightly deformed flat surfaces of revolution: planes, cylinders, and cones. The reference geometry is defined by an arc-length-parameterized curve, and the perturbed surface arises from a small deformation of that curve. Expanding the metric tensor to first order yields the Laplace–Beltrami operator at linear accuracy. Approximate Lie symmetries are computed for each canonical surface, and the analysis shows how infinitesimal generators change under the unit-speed constraint. While exact surfaces admit rotational Killing fields, real systems like deformed membranes and nanostructures involve small imperfections. The linear expansion reveals that the perturbation preserves the rotational symmetry exactly, while it breaks the exactness of other symmetries (such as axial translations and scaling), reducing them to approximate symmetries. This offers a geometric framework for approximate symmetry reduction. The results apply to heat conduction in engineering components and to pattern formation in biological membranes.

Keywords: Approximate Lie point symmetries; cone; cylinder; flat surfaces; heat equation; Laplace–Beltrami operator; perturbed surfaces of revolution
AMS Subject Classification 2010: 35A30, 53A05, 58J70.

1 Introduction

Differential equations often admit symmetries that can be exploited to simplify their structure. The systematic study of such symmetries originated with the pioneering work of Sophus Lie during the nineteenth century, and this method has now become an indispensable tool in modern mathematics and physics. Its applications include simplifying ordinary and partial differential equations, constructing invariant solutions, the order reduction of ODEs, and dimensional reduction of PDE systems [11, 12, 25, 26].

*Corresponding author
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Within differential geometry, Lie symmetries play a fundamental role in solving problems such as finding geodesics, identifying minimal and maximal surfaces, and studying surfaces of constant mean curvature [1, 3, 5, 18, 19]. The profound connection between Lie symmetries and Noether symmetries further amplifies the value of this method in physics, since these symmetries influence Lagrangian and dynamical systems even when the problem is not directly related to geodesic equations [18, 19]. Building upon these geometric applications of Lie symmetries, the first author has made significant contributions to the field, particularly in the study of minimal and maximal surfaces. In one study, she and colleagues [18] analysed Noether and Killing symmetries of the minimal-area Lagrangian within vacuum pp-wave geometries, obtaining an interesting connection between these symmetries and the intrinsic isometries of these spaces. In another study, the first author and colleagues [19] addressed maximal surfaces through the lens of Lie symmetry analysis, producing a complete classification of one-dimensional symmetry subalgebras in the form of an optimal system.

Related studies on symmetry analysis and geometric properties of space-times include the Cartan geometry of Melvin space-time [6], the geometry of the straight spinning string space-time [7], and the Noether symmetries of the BTZ black hole metric [8].

Among the various classes of surfaces from geometric and physical perspectives, surfaces of revolution hold a special position. Such surfaces are obtained by revolving a planar generating curve around a fixed axis and exhibit an inherent axial symmetry. Fundamental examples include the cylinder, cone, sphere, torus, pseudosphere, and catenoid. The symmetry properties of these surfaces have made them a thriving research area in computer graphics, engineering design, biological membrane studies, and differential geometry [10, 21].

The symmetry properties of the heat flow on rotational spaces have attracted considerable attention in the literature. In this direction, Jhangeer and Naeem [17] focused on similarity reduction techniques for the toroidal case, while Nass [23] examined harmonic functions on surfaces of revolution using Lie symmetries. A major advance came with the thesis of Mpungu [Table 1.1, p. 8] [21], which offered a unified group classification based on Ovsiannikov's methodology [26]. In this work, the generic symmetry algebra is shown to be three-dimensional, with exceptional cases admitting algebras of dimensions four, five, and nine. This classification scheme thus provides a complete inventory of revolution surfaces classified by the symmetry algebra of the heat equation. Subsequent work by Nass, Mpungu, and Nuruddeen [24] has shown that the biharmonic heat equation on these surfaces possesses the same symmetry structure as the standard heat equation. Nevertheless, it is important to note that all of these findings pertain to pristine, perfectly axisymmetric revolution surfaces.

In real physical systems, small imperfections inevitably exist in the generating curve of the surface. Manufacturing tolerances, natural variations, or slight deformations break the exact rotational symmetry. For example, a cylindrical shell may exhibit weak axial variations in its radius; a nearly flat interface may possess small undulations; a conical nanostructure may deviate slightly from a perfectly straight generating line. Although small, these perturbations can significantly alter heat diffusion behavior and may reduce or completely destroy the original isometries. Related reaction-diffusion models on planar and spherical geometries, which share the same geometric setting, have been studied in [2]. The traditional Lie symmetry method, which assumes exact symmetry, is inadequate for studying such systems. This difficulty has motivated the development of approximate symmetry methods.

A framework for handling approximate Lie symmetries in the presence of a small perturbation parameter was introduced in [4]. In this approach, both the dependent variables and the infinitesimal generator vectors are expanded in powers of the perturbation parameter. The technique has been successfully em-

Corrected Proof

ployed across a broad range of perturbed differential equations [13–15]. Subsequent contributions have refined and extended this methodology. The work in [9] offered deeper insight into the hierarchical organisation of approximate symmetry algebras. Among the studies that extended this framework, Zhang and Lu [31] employed approximate symmetry techniques to obtain homotopy-series approximate solutions for systems of perturbed PDEs, demonstrating its applicability to equations such as the Cahn–Hilliard equation. Nadjafikhah and Mokhtari [22] extended the approximate symmetry approach to Hamiltonian systems, developing the concept of approximate Hamiltonian symmetry groups for perturbed evolution equations. Furthermore, the first author and colleagues [20] investigated the approximate invariance for the KdV–KS model with small perturbations using this approach, further confirming the power of approximate symmetry methods for nonlinear perturbed systems. Nevertheless, the existing literature contains no investigation of approximate symmetry methods for the heat equation on perturbed rotational surfaces. Although various software packages exist for finding Lie symmetries [28–30], these programmes face serious limitations in high dimensions, especially when a perturbation is added to the equation. This circumstance further highlights the need for analytical and geometric approaches.

Our analysis focuses on the heat conduction problem defined on perturbed versions of three standard flat revolution surfaces: the cylinder, the plane, and the cone. In this study, the surfaces under consideration are obtained by rotating a curve of unit speed and then imposing a small deviation on this generating profile. By expanding the metric tensor to linear order in the perturbation parameter, we obtain the Laplace–Beltrami operator exact up to linear order. Using this linearised operator, we identify the approximate infinitesimal transformations of the perturbed heat equation for each of the three flat surfaces. In each case, we show how the infinitesimal generators are modified under the unit-speed constraint of the generating curve. Our linear expansion reveals that the rotational symmetry is preserved exactly by the perturbation, while other symmetries (such as axial translations and scaling) are reduced to approximate symmetries.

The layout of this manuscript is organized in the following way. Section 2 reviews the essential geometric tools, namely the parametrization of perturbed surfaces of revolution and the expansion of the metric tensor. Section 3 applies the standard infinitesimal symmetry technique in the context of the perturbed heat operator and derives the determining equations. Section 4 classifies the approximate Lie point symmetries for the perturbed cylinder, plane, and cone, respectively. Section 5 discusses the physical implications of the symmetry reduction and finally Section 6 presents the conclusions of the present study.

2 The geometry of perturbed flat surfaces of revolution

As a preliminary step towards perturbed revolution surfaces, consider the unit-speed generating curve $\alpha_0(x) = (\rho_0(x), e^{\mathrm{f}_0(x)})$, whose rotation about an axis yields the reference surface

$$\chi_0(x,y) = (\rho_0(x), e^{\mathrm{f}_0(x)}\cos y, e^{\mathrm{f}_0(x)}\sin y).$$

The perturbed unit-speed profile is expressed as

$$\alpha(x) = (\rho_0(x) + \varepsilon\rho_1(x), e^{\mathrm{f}_0(x)+\varepsilon \mathrm{f}_1(x)}),$$

Corrected Proof

and we obtain a perturbed surface of revolution that consists of an exact part plus an approximate part, namely

$$\chi = \chi_0 + \varepsilon\chi_1 + \mathscr{O}(\varepsilon^2) = (\rho_0, e^{\mathrm{f}_0}\cos y, e^{\mathrm{f}_0}\sin y) + \varepsilon(\rho_1, \mathrm{f}_1 e^{\mathrm{f}_0}\cos y, \mathrm{f}_1 e^{\mathrm{f}_0}\sin y) + \mathscr{O}(\varepsilon^2),$$

where

$$f_0'^2 e^{2f_0} + \rho_0'^2 = 1, \tag{1a}$$

$$(f_1 f_0' + f_1') f_0' e^{2f_0} + \rho_0'\rho_1' = 0. \tag{1b}$$

The appearance of the term $\mathrm{f}_1 e^{\mathrm{f}_0}$ in χ_1 follows directly from the Maclaurin expansion of the exponential function,

$$e^{\varepsilon \mathrm{f}_1} = 1 + \varepsilon\,\mathrm{f}_1 + \mathscr{O}(\varepsilon^2).$$

Moreover, the perturbed profile is required to remain unit-speed up to the first order in ε, i.e., $\langle \alpha'(x), \alpha'(x)\rangle = 1 + \mathscr{O}(\varepsilon^2)$. Expanding this relation to the first order in ε yields the constraint relations (1).

The decoupling of the metric components follows directly from the unit-speed constraint (1). The $\mathscr{O}(\varepsilon)$ terms in g_{11} cancel exactly, leaving $g_{11} = 1 + \mathscr{O}(\varepsilon^2)$. The cross term g_{12} vanishes due to the orthogonality of angular and radial perturbations, giving $g_{12} = \mathscr{O}(\varepsilon^2)$. Finally, the Maclaurin expansion of $e^{2(f_0+\varepsilon f_1)}$ separates the background from the perturbation, yielding $g_{22} = e^{2f_0}(1 + 2\varepsilon f_1) + \mathscr{O}(\varepsilon^2)$.

Thus, the perturbed metric can be written as the sum of a background metric and a small perturbative term, namely

$$ds^2 = dx^2 + e^{2\mathrm{f}_0(x)}\big(1 + 2\varepsilon \mathrm{f}_1(x)\big)dy^2 + \mathscr{O}(\varepsilon^2).$$

Thus, the inverse metric components are

$$g^{11} = 1 + \mathscr{O}(\varepsilon^2), \qquad g^{12} = \mathscr{O}(\varepsilon^2), \qquad g^{22} = e^{-2f_0}(1 - 2\varepsilon f_1) + \mathscr{O}(\varepsilon^2).$$

Substituting these into $\Delta u = \dfrac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j u)$ with $\sqrt{|g|} = e^{f_0}(1 + \varepsilon f_1) + \mathscr{O}(\varepsilon^2)$ yields

$$\Delta \mathrm{u} = \mathrm{f}_0'(x)\mathrm{u}_x + \mathrm{u}_{xx} + e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy} + \varepsilon\big(\mathrm{f}_1'(x)\mathrm{u}_x - 2\mathrm{f}_1(x)e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy}\big) + \mathscr{O}(\varepsilon^2).$$

Thus, for a surface arising from a perturbed revolution geometry, the perturbative heat equation $\Delta \mathrm{u} = \mathrm{u}_t$ can be expressed as

$$\mathrm{f}_0'(x)\mathrm{u}_x + \mathrm{u}_{xx} + e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy} - \mathrm{u}_t + \varepsilon\big(\mathrm{f}_1'(x)\mathrm{u}_x - 2\mathrm{f}_1(x)e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy}\big) = \mathscr{O}(\varepsilon^2). \tag{2}$$

We now proceed to determine all perturbed flat surfaces that arise from a surface of revolution. In the first case, where $\mathrm{f}_0(x) = \ln b$, the rotational surface reduces to a cylinder, which reads

$$\chi_0(x,y) = (a \pm x, b\cos y, b\sin y); \quad \alpha_0(0) = (a,b).$$

For the perturbed cylinder, substitution into Eq. (1) yields

$$\chi(x,y) = (a \pm x, b\cos y, b\sin y) + b\varepsilon(e, \mathrm{f}_1\cos y, \mathrm{f}_1\sin y) + \mathscr{O}(\varepsilon^2), \alpha(0) = (a,b) + b\varepsilon(e, \mathrm{f}_1(0)) + \mathscr{O}(\varepsilon^2), \tag{3}$$

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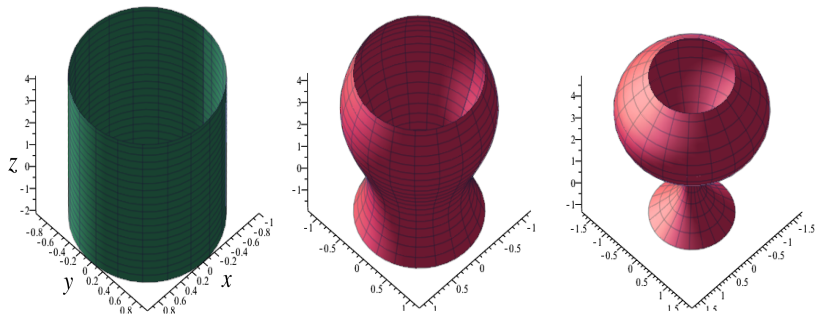


Figure 1: Visualization of the perturbed cylinder for $\varepsilon = 0$ (unperturbed), $\varepsilon = 0.2$, and $\varepsilon = 0.8$. The parameter ε controls the amplitude of the radial profile perturbation. Because the deformation is independent of the azimuthal angle, it changes only the radial profile and the cylinder remains rotationally symmetric about its axis.

where f_1 remains an arbitrary function.

Under this perturbation, the heat equation transforms into

$$\mathrm{u}_{xx} + \frac{1}{b^2}\mathrm{u}_{yy} - \mathrm{u}_t + \varepsilon\left(\mathrm{f}_1'(x)\mathrm{u}_x - \frac{2}{b^2}\mathrm{f}_1(x)\mathrm{u}_{yy}\right) = \mathscr{O}(\varepsilon^2). \tag{4}$$

Figure 1 illustrates the perturbation of the cylinder.

In the second case, where $\mathrm{f}_0 = \ln(b + x\sin\Theta) = \ln(l(x+p))$, with $l = \sin\Theta$, $p = b/\sin\Theta$, the rotational surface reduces to a cone, described by

$$\chi_0(x,y) = \big(a + x\cos\Theta,\ (b + x\sin\Theta)\cos y,\ (b + x\sin\Theta)\sin y\big), \qquad \sin\Theta > 0.$$

Here $l = \sin\Theta$ is the fixed geometric parameter of the cone, and the condition $\sin\Theta \neq 0$ excludes the cylinder case, since $\sin\Theta = 0$ corresponds to a cylinder. This parameter does not appear in the cylinder or plane cases. We have explicitly stated that the geometric parametrization and descriptions are expressed in terms of the cone angle Θ, while for notational simplicity in the subsequent differential equations, we use $l = \sin\Theta$ and $p = b/\sin\Theta = b/l$.

For the perturbed cone, assuming ρ_1' is constant and substituting into Eq. (1b), we obtain:

$$f_1 = \frac{c}{x+p} + m,$$

where c and m are arbitrary constants. To analyze the effect of the perturbation, we focus on the two simplest choices that capture the distinct behaviors of this general form: the constant case $f_1 = m$ (which corresponds to $c = 0$) and the decaying case $f_1 = \frac{1}{x+p}$ (which corresponds to $m = 0$, $c = 1$). This classification is exhaustive within the chosen ansatz, i.e., under the assumption $\rho_1' = \text{const}$.

Case 1: $\mathrm{f}_1 = m$. In this case, we obtain

$$\begin{aligned}\chi(x,y) &= \big(a + x\cos\Theta, (b + x\sin\Theta)\cos y, (b + x\sin\Theta)\sin y\big)\\ &\quad + m\varepsilon\big(c - x\sin\Theta\tan\Theta, (b + x\sin\Theta)\cos y, (b + x\sin\Theta)\sin y\big) + \mathscr{O}(\varepsilon^2),\\ \alpha(0) &= \big(a, b\cos y, b\sin y\big) + m\varepsilon\big(c, b\cos y, b\sin y\big) + \mathscr{O}(\varepsilon^2).\end{aligned} \tag{5}$$

Corrected Proof

To explicitly verify that the perturbed generating curve of the cone satisfies the unit-speed constraint, we differentiate the curve

$$\alpha(x) = \Big(a + x\cos\Theta + m\varepsilon(c - x\sin\Theta\tan\Theta),\ (b + x\sin\Theta)(1 + m\varepsilon)\Big) + \mathscr{O}(\varepsilon^2)$$

directly with respect to x. This yields

$$\alpha'(x) = \Big(\cos\Theta - m\varepsilon\sin\Theta\tan\Theta,\ \sin\Theta(1 + m\varepsilon)\Big) + \mathscr{O}(\varepsilon^2).$$

The squared speed is then computed as

$$\langle\alpha'(x), \alpha'(x)\rangle = \big(\cos\Theta - m\varepsilon\sin\Theta\tan\Theta\big)^2 + \big(\sin\Theta(1 + m\varepsilon)\big)^2 + \mathscr{O}(\varepsilon^2)$$

yields

$$\langle\alpha'(x), \alpha'(x)\rangle = \cos^2\Theta + \sin^2\Theta + \mathscr{O}(\varepsilon^2) = 1 + \mathscr{O}(\varepsilon^2).$$

This confirms the unit-speed condition and ensures that the parametrization and the metric are fully consistent.

The effect of this perturbation on the heat equation is given by

$$\frac{1}{x+p}\mathrm{u}_x + \mathrm{u}_{xx} + \frac{1}{l^2(x+p)^2}\mathrm{u}_{yy} - \mathrm{u}_t - \frac{2m\varepsilon}{l^2(x+p)^2}\mathrm{u}_{yy} = \mathscr{O}(\varepsilon^2). \tag{6}$$

Case 2: $\mathrm{f}_1 = \dfrac{1}{x+p}$. In this case, we obtain

$$\chi(x,y) = \big(a + x\cos\Theta, (b + x\sin\Theta)\cos y, (b + x\sin\Theta)\sin y\big) + \sin\Theta\varepsilon\big(e, \cos y, \sin y\big) + \mathscr{O}(\varepsilon^2), \tag{7}$$

with $\alpha(0) = \big(a, b\cos y, b\sin y\big) + \sin\Theta\varepsilon\big(e, \cos y, \sin y\big) + \mathscr{O}(\varepsilon^2)$.

To ensure full transparency in the derivation, we explicitly show how the general first-order operator in Eq. (10) reduces to this specific case. Substituting $f_1 = \frac{1}{x+p}$ and its derivative $f_1' = -\frac{1}{(x+p)^2}$, and using $e^{-2f_0(x)} = \frac{1}{l^2(x+p)^2}$, the coefficients of u_x and u_{yy} in the perturbation term are found to be exactly $-\frac{1}{(x+p)^2}$ and $-\frac{2}{l^2(x+p)^3}$, respectively. This confirms that no derivative or sign has been misplaced. Therefore, the perturbed heat equation takes the form:

$$\frac{1}{x+p}\mathrm{u}_x + \mathrm{u}_{xx} + \frac{1}{l^2(x+p)^2}\mathrm{u}_{yy} - \mathrm{u}_t - \frac{\varepsilon}{(x+p)^2}\Big(\mathrm{u}_x + \frac{2}{l^2(x+p)}\mathrm{u}_{yy}\Big) = \mathscr{O}(\varepsilon^2). \tag{8}$$

Figure 2 illustrates the perturbation of the cone. For the conical case examined earlier, setting $l = 1$ and $p = b$ leads to a perturbed plane surface as follows:

$$\chi(x,y) = \big(a, (b+x)\cos y, (b+x)\sin y\big) + \varepsilon\big(e, \cos y, \sin y\big) + \mathscr{O}(\varepsilon^2),$$

while $\alpha(0)$ becomes

$$\alpha(0) = \big(a, b\cos y, b\sin y\big) + \sin\Theta\varepsilon\big(e, \cos y, \sin y\big) + \mathscr{O}(\varepsilon^2).$$

Furthermore, setting $l = 1$ and $b = p$ in Eq. (8) directly simplifies the coefficients to yield the corresponding heat equation for the perturbed plane:

$$\frac{1}{x+b}\mathrm{u}_x + \mathrm{u}_{xx} + \frac{1}{(x+b)^2}\mathrm{u}_{yy} - \mathrm{u}_t - \frac{\varepsilon}{(x+b)^2}\Big(\mathrm{u}_x + \frac{2}{x+b}\mathrm{u}_{yy}\Big) = \mathscr{O}(\varepsilon^2). \tag{9}$$

Figure 3 illustrates the perturbation of the plane.

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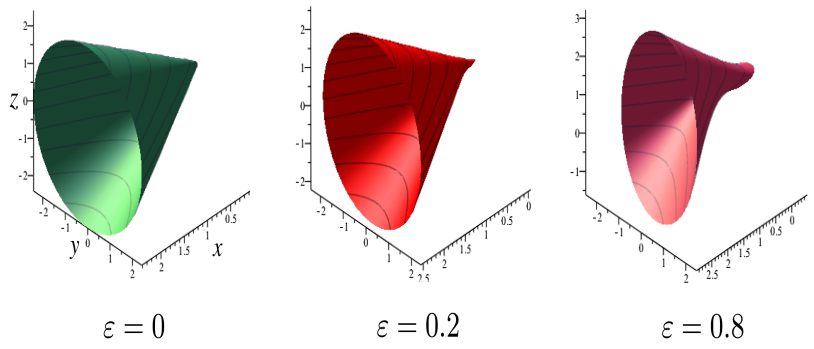


Figure 2: Visualization of the perturbed cone for $\varepsilon = 0$ (unperturbed), $\varepsilon = 0.2$, and $\varepsilon = 0.8$. The perturbation introduces a slight axial shift and a small deformation of the generating curve. Since these changes do not depend on the azimuthal angle, the rotational symmetry of the cone about its axis is preserved.

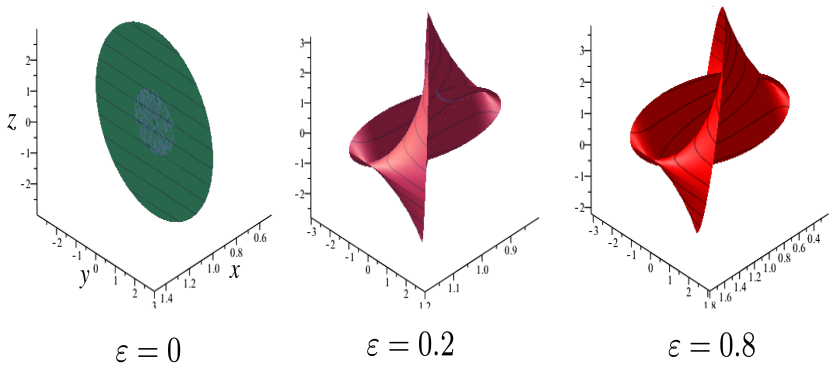


Figure 3: Visualization of the perturbed plane for $\varepsilon = 0$ (unperturbed), $\varepsilon = 0.2$, and $\varepsilon = 0.8$. The parameter ε controls the amplitude of the perturbation; increasing ε deforms the otherwise flat surface.

3 Approximate Lie symmetry analysis of a perturbed heat equation

In this section, we examine approximate symmetries for the perturbed heat equation given by (10), following the framework of approximate Lie symmetry analysis [4]. Special attention is given to the derivation of determining equations for nontrivial approximate symmetries depending on the structure of the perturbative contribution of (10). To explore the invariance characteristics of the perturbed system, we take a Lie group of infinitesimal symmetries in the perturbative sense:

$$\begin{aligned}
\mathbf{x}(x,y,t,\mathrm{u},\varepsilon,\delta) &= \mathbf{x}(x,y,t,\mathrm{u},\delta) + \varepsilon\xi(x,y,t,\mathrm{u},\varepsilon,\delta) + \mathscr{O}(\varepsilon^2),\\
\mathbf{y}(x,y,t,\mathrm{u},\varepsilon,\delta) &= \mathbf{y}(x,y,t,\mathrm{u},\delta) + \varepsilon\nu(x,y,t,\mathrm{u},\varepsilon,\delta) + \mathscr{O}(\varepsilon^2),\\
\mathbf{t}(x,y,t,\mathrm{u},\varepsilon,\delta) &= \mathbf{t}(x,y,t,\mathrm{u},\delta) + \varepsilon\tau(x,y,t,\mathrm{u},\varepsilon,\delta) + \mathscr{O}(\varepsilon^2),\\
\mathbf{u}(x,y,t,\mathrm{u},\varepsilon,\delta) &= \mathbf{u}(x,y,t,\mathrm{u},\delta) + \varepsilon\varphi(x,y,t,\mathrm{u},\varepsilon,\delta) + \mathscr{O}(\varepsilon^2),
\end{aligned}$$

Correct Proof

in which δ and ε serve as the approximate parameters. Accordingly, the approximate symmetry of Eq. (10) takes the form

$$\mathbf{X} = \mathbf{X}_0 + \varepsilon \mathbf{X}_1 + \mathscr{O}(\varepsilon^2) = \xi_0 \partial_x + \nu_0 \partial_y + \tau_0 \partial_t + \varphi_0 \partial_{\mathrm{u}} + \varepsilon(\xi_1 \partial_x + \nu_1 \partial_y + \tau_1 \partial_t + \varphi_1 \partial_{\mathrm{u}}) + \mathscr{O}(\varepsilon^2).$$

Following the standard framework for approximate symmetries (see, e.g., [16,27]), the approximate prolongation (to second order) of the generator, $\mathbf{X}^{[2]}$, is introduced to capture the transformation properties of the derivatives in Eq. (10). For simplicity of calculations, we set in Eq. (10):

$$F_0 = \mathrm{f}_0'(x)\mathrm{u}_x + \mathrm{u}_{xx} + e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy} - \mathrm{u}_t, \quad F_1 = \mathrm{f}_1'(x)\mathrm{u}_x - 2\mathrm{f}_1(x)e^{-2\mathrm{f}_0(x)}\mathrm{u}_{yy}.$$

The requirement for invariance reads as follows:

$$\mathbf{X}^{[2]}(F_0 + \varepsilon F_1)\Big|_{F_0+\varepsilon F_1=0} = \mathscr{O}(\varepsilon^2). \tag{10}$$

Expanding the generator as $\mathbf{X}^{[2]} = \mathbf{X}_0^{[2]} + \varepsilon \mathbf{X}_1^{[2]} + \mathscr{O}(\varepsilon^2)$ and applying it to $F_0 + \varepsilon F_1$ gives:

$$\mathbf{X}^{[2]}(F_0 + \varepsilon F_1) = \mathbf{X}_0^{[2]} F_0 + \varepsilon \left(\mathbf{X}_1^{[2]} F_0 + \mathbf{X}_0^{[2]} F_1 \right) + \mathscr{O}(\varepsilon^2).$$

The zeroth-order term (coefficient of ε^0) must vanish on the unperturbed solution manifold, yielding the exact symmetry condition:

$$\mathbf{X}_0^{[2]} F_0\big|_{F_0=0} = 0. \tag{11}$$

Since $\mathbf{X}_0$ is an exact symmetry, its second prolongation satisfies the identity $\mathbf{X}_0^{[2]} F_0 = \lambda F_0$, where λ is a suitable differential function (the exact symmetry factor).

Substituting this identity back into the perturbed expression yields:

$$\mathbf{X}^{[2]}(F_0 + \varepsilon F_1) = \lambda F_0 + \varepsilon \left(\mathbf{X}_1^{[2]} F_0 + \mathbf{X}_0^{[2]} F_1 \right) + \mathscr{O}(\varepsilon^2).$$

Now, we evaluate this expression on the perturbed solution manifold $F_0 + \varepsilon F_1 = 0$ (which implies $F_0 = -\varepsilon F_1$). The term λF_0 then contributes as $-\varepsilon \lambda F_1$. Factoring out $1/\varepsilon$ and projecting onto the unperturbed surface $F_0 = 0$, the first-order term yields the corrected determining equation:

$$\left[\mathbf{X}_1^{[2]} F_0 + \mathbf{X}_0^{[2]} F_1 - \lambda F_1 \right] \Big|_{F_0=0} = 0. \tag{12}$$

Consequently, the auxiliary function is rigorously defined as:

$$\mathscr{H} = \left[\mathbf{X}_0^{[2]} F_1 - \lambda F_1 \right] \Big|_{F_0=0}. \tag{13}$$

Alternatively, this function can be expressed by evaluating directly on the full perturbed system as:

$$\mathscr{H}_{\text{full}} = \frac{1}{\varepsilon} \mathbf{X}_0^{[2]}(F_0 + \varepsilon F_1)\Big|_{F_0+\varepsilon F_1=0}. \tag{14}$$

Because the definition in Eq. (13) is evaluated on the unperturbed manifold $F_0 = 0$, while the formulation in Eq. (14) is evaluated on the perturbed manifold $F_0 + \varepsilon F_1 = 0$, the two expressions are not identically

Corrected Proof

equal for a fixed, nonzero ε. Instead, they coincide in the limit $\varepsilon \to 0$. This equivalence is mathematically expressed as:

$$\mathscr{H} = \lim_{\varepsilon\to 0} \mathscr{H}_{\text{full}} = \lim_{\varepsilon\to 0} \left[\frac{1}{\varepsilon}\mathbf{X}_0^{[2]}(F_0+\varepsilon F_1)\right]_{F_0+\varepsilon F_1=0}.$$

The limiting relation is understood along a family of perturbed solution points that converges to a solution point of the unperturbed equation as $\varepsilon \to 0$. Assuming the relevant functions are continuous (and the perturbed solution points depend continuously on ε), the evaluations of $X_0^{[2]}F_1 - \lambda F_1$ on the perturbed and unperturbed solution manifolds converge to the same value. Thus, the equivalence is asymptotic rather than an exact identity for fixed nonzero ε.

Therefore, using Eq. (12), $\mathbf{X}_1$ is determined by solving the following deformation determining equation:

$$\mathbf{X}_1^{[2]}F_0\big|_{F_0=0} + \mathscr{H} = 0. \tag{15}$$

The set of determining equations corresponding to (11) was obtained in [Sec. 2.2.1, pp. 26–28] [21]. Equate the u-derivative appearing in the determining Eq. (15), which leads to:

$$\begin{aligned}
&\mathscr{E}_1 : \left(\mathrm{f}_1''\mathrm{f}_1 + 2\mathrm{f}_0'\mathrm{f}_1'\mathrm{f}_1 - \mathrm{f}_1'^2\right)\xi_0 - \mathrm{f}_1'\mathrm{f}_1\xi_{0x} - 2\mathrm{f}_1^2 e^{-2\mathrm{f}_0}\xi_{0yy} + 2\mathrm{f}_1\mathrm{f}_1'\nu_{0y} = 0,\\
&\mathscr{E}_2 : \xi_1\mathrm{f}_0'' + \mathrm{f}_0'\xi_{1x} + \xi_{1t} + 2\varphi_{1\mathrm{u}x} - \xi_{1xx} - e^{-2\mathrm{f}_0}\xi_{1yy} = 0,\\
&\mathscr{E}_3 : -\tau_{1t} + 2\xi_{1x} = 0,\\
&\mathscr{E}_4 : \mathrm{f}_0'\xi_1 + \nu_{1y} - \xi_{1x} = 0,\\
&\mathscr{E}_5 : e^{-2\mathrm{f}_0}\left(\varphi_{1yy} - 2\mathrm{f}_1\varphi_{0yy}\right) + \mathrm{f}_1'\varphi_{0x} + \mathrm{f}_0'\varphi_{1x} + \varphi_{1xx} - \varphi_{1t} = 0,\\
&\mathscr{E}_6 : e^{-2\mathrm{f}_0}\left(2\mathrm{f}_1(2\varphi_{0\mathrm{u}y} - \nu_{0yy}) - 2\varphi_{1\mathrm{u}y} + \nu_{1yy}\right) + \mathrm{f}_1'\nu_{0x} + \mathrm{f}_0'\nu_{1x} + \nu_{1xx} - \nu_{1t} = 0,\\
&\mathscr{E}_7 : e^{-2\mathrm{f}_0}(2\mathrm{f}_1\xi_{0y} - \xi_{1y}) - \nu_{1x} = 0,\\
&\mathscr{E}_8 : \xi_{1\mathrm{u}} = 0,\\
&\mathscr{E}_9 : \nu_{1\mathrm{u}} = 0,\\
&\mathscr{E}_{10} : \tau_{1x} = 0,\\
&\mathscr{E}_{11} : \tau_{1y} = 0,\\
&\mathscr{E}_{12} : \tau_{1\mathrm{u}} = 0.
\end{aligned}$$

4 Symmetry analysis on perturbed flat surfaces

Having established the general framework in the previous sections, we now extend the analysis to the approximate symmetry algebra of the perturbed heat flow equation for all flat revolution surfaces. In particular, these consist of three canonical cases: a perturbed cylinder, a perturbed cone, and a perturbed plane. The determining equations are derived for each surface, allowing us to explicitly find the approximate symmetry generators.

Arbitrary functions in the solution. Throughout Sections 4 and 5, several arbitrary functions of the independent variables appear in the solutions, such as $g(y,t)$, $k(x,y,t)$, $H(y,t)$, and $\delta(y)$. These functions parameterize the residual freedom in the solution of the linear background heat equation and its perturbed correction, and each is governed by an auxiliary PDE or ODE.

Corrected Proof

4.1 The case of a perturbed cylinder

The perturbed cylinder was already defined in (3). Moreover, in [Sec. 3.2.1, pp. 34–35] [21], the exact form of the infinitesimal generator $\mathbf{X}$, namely $\mathbf{X}_0$ for Eq. (4), was derived, leading to the coefficients ξ_0, τ_0, ν_0, φ_0 expressed as follows:

$$\xi_0 = \big(k_1 t + \frac{k_2}{2}\big)x + k_4 y + k_5 t + k_6, \qquad \tau_0 = k_1 t^2 + k_2 t + k_3,$$
$$\nu_0 = -\frac{k_4}{b^2}x + \big(k_1 t + \frac{k_2}{2}\big)y + k_7 t + k_8, \quad \varphi_0 = \big(-k_1 t - \frac{1}{2}(\frac{k_1}{2}x + k_5)x - \frac{1}{2}b^2(\frac{k_1}{2}y + k_7)y + k_9\big)\mathrm{u} + h(x,y,t),$$

in which h obeys the exact heat equation of the form

$$h_t = h_{xx} + \frac{1}{b^2}h_{yy}. \tag{16}$$

Therefore, upon substituting the above relations along with $\mathrm{f}_0 = \ln b$ into the determining equations derived in the preceding section (namely, the $\mathscr{E}_i$'s), the determining equations reduce to the following form:

$$\begin{aligned}
&\mathscr{E}_{1,1} : \big(\mathrm{f}_1\mathrm{f}_1'' - \mathrm{f}_1'^2\big)\big((k_1 t + \tfrac{k_2}{2})x + k_4 y + k_5 t + k_6\big) + \mathrm{f}_1\mathrm{f}_1'(k_1 t + \tfrac{k_2}{2}) = 0,\\
&\mathscr{E}_{1,2} : \xi_{1t} + 2\varphi_{1\mathrm{u}x} - \xi_{1xx} - \frac{1}{b^2}\xi_{1yy} = 0,\\
&\mathscr{E}_{1,3} : 2\xi_{1x} - \tau_{1t} = 0,\\
&\mathscr{E}_{1,4} : \nu_{1y} - \xi_{1x} = 0,\\
&\mathscr{E}_{1,5} : k_1\mathrm{u}\mathrm{f}_1 - \frac{2}{b^2}\mathrm{f}_1 h_{yy} + \mathrm{f}_1' h_x - \frac{1}{2}\mathrm{f}_1'(k_1 x + k_5)\mathrm{u} + \frac{1}{b^2}\varphi_{1yy} + \varphi_{1xx} - \varphi_{1t} = 0,\\
&\mathscr{E}_{1,6} : \frac{k_4}{b^2}\mathrm{f}_1' + 2\mathrm{f}_1\big(\tfrac{k_1}{2}y + k_7\big) - \nu_{1xx} + \frac{1}{b^2}\big(2\varphi_{1\mathrm{u}y} - \nu_{1yy}\big) + \nu_{1t} = 0,\\
&\mathscr{E}_{1,7} : \frac{2k_4}{b^2}\mathrm{f}_1 - \frac{1}{b^2}\xi_{1y} - \nu_{1x} = 0,\\
&\text{and} \quad \xi_1 = \xi_1(x,y,t), \quad \nu_1 = \nu_1(x,y,t), \quad \tau_1 = \tau_1(t), \quad (\text{via } \mathscr{E}_8 - \mathscr{E}_{12}).
\end{aligned}$$

As previously noted, f_1 must satisfy $\mathscr{E}_{1,1}$. For the symmetries to hold for all f_1, we require $k_1 = k_2 = k_4 = k_5 = k_6 = 0$. However, $\mathscr{E}_{1,1}$ also holds for the special form $\mathrm{f}_1 = m =$ constant. Therefore, we examine two separate cases: (i) f_1 arbitrary (the most general situation); (ii) $\mathrm{f}_1 = m$ (constant).

Case 1: Arbitrary f_1. As we noted a little earlier, in this case we must have $k_1 = k_2 = k_4 = k_5 = k_6 = 0$. Since $\tau_1 = \tau_1(t)$, differentiating equation $\mathscr{E}_{1,3}$ in the independent variables results in $\xi_{1xx} = \xi_{1xy} = 0$; therefore, differentiation of $\mathscr{E}_{1,4}$ in the x and y directions provides $\nu_{1xy} = \nu_{1yy} = 0$. These relations indicate that ξ is linear with respect to x, ν is linear with respect to y, the coefficient of x in ξ is independent of y, and the coefficient of y in ν is independent of x; consequently, the general forms of ξ_1 and ν_1 can be expressed as $\xi_1 = A(t)x + B(y,t)$, $\nu_1 = A(t)y + C(x,t)$. Substituting these expressions into the determining equation $\mathscr{E}_{1,7}$ leads to the relation

$$B_y(y,t) = -b^2 C_x(x,t). \tag{17}$$

Corrected Proof

Using (17) and inserting the previously obtained relations into $\mathscr{E}_{1,2}$ gives $\varphi_{1ux} = -\frac{1}{2}(xA' + B_t)$. Integrating this with respect to x yields $\varphi_{1u} = g(y,t) - \frac{1}{4}x^2A' - \frac{1}{2}xB_t$. Integrating once more with respect to u gives the expression for φ_1 as follows:

$$\varphi_1 = \big(g(y,t) - \frac{1}{4}x^2A' - \frac{1}{2}xB_t\big)\mathrm{u} + k(x,y,t). \tag{18}$$

By inserting the previously obtained relations into $\mathscr{E}_{1,6}$ and using (17), the result is the following equation:

$$2k_7\mathrm{f}_1 + \frac{1}{b^2}\big(2g_y - xB_{ty}\big) + yA' + C_t = 0, \tag{19}$$

and integration of (19) with respect to y yields

$$g(y,t) = \frac{b^2}{2}yD(t) - \frac{b^2}{4}y^2A' + F(t). \tag{20}$$

Finally, the expression for $g(y,t)$ when substituted into (19) leads to the consistency relation

$$2k_7\mathrm{f}_1 - \frac{1}{b^2}xB_{ty} + D + C_t = 0. \tag{21}$$

Given that the above relation must be satisfied for any f_1, we consequently obtain $k_7 = 0$. Differentiation of the above relation with respect to x immediately gives $C_{tx} = B_{ty} = B_{yy} = 0$. Turning to $(\mathscr{E}_{1,5})_{\mathrm{u}}$, one readily finds $A''(t) = 0$, which implies $A(t) = e_1t + e_2$ with $e_1, e_2 \in \mathbb{R}$. Furthermore, we obtain $B_{tt}(y,t) = 0$, leading to $B(y,t) = e_3t + e_4y + e_5$ for some constants $e_3, e_4, e_5 \in \mathbb{R}$.

We also have $g_t(y,t) = -e_1$, which together with Eq. (20) forces $D'(t) = 0$. Hence we may take $D(t) = e_6$ ($e_6 \in \mathbb{R}$). The condition $F'(t) = -e_1$ then follows, yielding $F(t) = -e_1t + e_7$ where $e_7 \in \mathbb{R}$. Substituting these results into (21) and integrating with respect to t yields the explicit form of $C(x,t)$:

$$C(x,t) = -\frac{e_4}{b^4}x - e_6t + e_8, \qquad e_8 \in \mathbb{R}.$$

Since $\tau_1 = \tau_1(t)$, from $\mathscr{E}_{1,3}$ we conclude that τ_1 is a quadratic function of t. Finally, by substituting the expression φ_1 from Eq. (18) into $\mathscr{E}_{1,5}$, we conclude that

$$\mathrm{f}_1'h_x - \frac{2}{b^2}\mathrm{f}_1h_{yy} + k_{xx} + \frac{1}{b^2}k_{yy} - k_t = 0. \tag{22}$$

Consequently, upon a suitable renumbering of the constants, the infinitesimal coefficients take the following form:

$$\begin{aligned}
&\xi_0 = 0,\ \tau_0 = l_1,\ \eta_0 = l_2,\ \varphi_0 = l_3\mathrm{u} + h(x,y,t),\\
&\xi_1 = (l_4t + l_5)x + l_6t + l_7y + l_8, \qquad \eta_1 = (l_4t + l_5)y - l_9t - \frac{l_7}{b^2}x + l_{10},\\
&\tau_1 = l_4t^2 + 2l_5t + l_{11}, \qquad \varphi_1 = (\frac{l_9b^2}{2}y - \frac{l_4}{4}x^2 - \frac{l_6}{2}x - \frac{b^2l_4}{4}y^2 - l_4t + l_{12})\mathrm{u} + k(x,y,t),
\end{aligned}$$

where h and k satisfy Eqs. (16) and (22), respectively.

Corrected Proof

Thus, when f_1 is arbitrary, the approximate symmetry algebra of (4) becomes infinite-dimensional and is spanned by

$$
\begin{aligned}
&Y_1=\partial_y,\ Y_2=\partial_t,\ Y_3=\mathrm{u}\,\partial_{\mathrm{u}},\ Y_4=\varepsilon\big(tx\,\partial_x+ty\,\partial_y+t^2\partial_t-(t+\frac{1}{4}x^2+\frac{b^2}{4}y^2)\mathrm{u}\,\partial_{\mathrm{u}}\big),\\
&Y_5=\varepsilon(\frac{1}{2}x\,\partial_x+\frac{1}{2}y\,\partial_y+t\,\partial_t),\ Y_6=\varepsilon(t\,\partial_x-\frac{x}{2}\mathrm{u}\,\partial_{\mathrm{u}}),\ Y_7=\varepsilon(y\,\partial_x-\frac{1}{b^2}x\,\partial_y),\ Y_8=\varepsilon\,\partial_x,\\
&Y_9=\varepsilon(t\,\partial_y-\frac{b^2}{2}y\mathrm{u}\,\partial_{\mathrm{u}}),\ Y_{10}=\varepsilon Y_1,\ Y_{11}=\varepsilon Y_2,\ Y_{12}=\varepsilon Y_3,\ Y_h=h\partial_{\mathrm{u}},\ Y_{\varepsilon k}=\varepsilon k\partial_{\mathrm{u}}.
\end{aligned}
\tag{23}
$$

The infinite-dimensional nature of the approximate symmetry algebra arises from the arbitrary function $k(x,y,t)$ appearing in Eq. (22). Since k is only constrained by the linear PDE (22) and is otherwise arbitrary, it generates an infinite family of independent infinitesimal generators $Y_{ek}=\varepsilon k\partial_u$.

Comparing the generators of the unperturbed cylinder with the approximate ones, we observe that three symmetries are preserved exactly: the rotational symmetry ∂_y, time translation ∂_t, and scaling $u\partial_u$. This is because the perturbation $f_1(x)$ is independent of y and t, and the equation remains linear in u. The other exact generators, such as axial translation ∂_x and scaling $\frac{1}{2}x\partial_x+\frac{1}{2}y\partial_y+t\partial_t$, are modified and appear as approximate symmetries at order $\mathscr{O}(\varepsilon)$. For instance, the action of $x\partial_x$ on the perturbation term yields non-zero contributions, which do not vanish identically.

The first 12×12 block of Table 1 forms a subalgebra generated by $Y_1,\dots,Y_{12}$ for arbitrary f_1.

Verification of a representative generator. To verify the validity of the approximate symmetry generators, we explicitly check one of them, namely $\mathbf{Y}_4$. Let $\mathbf{Y}_4=\mathbf{Y}_{4,0}+\varepsilon\mathbf{Y}_{4,1}$, where $\mathbf{Y}_{4,0}=0$ is the exact (zeroth-order) part and $\mathbf{Y}_{4,1}=tx\,\partial_x+ty\,\partial_y+t^2\partial_t-(t+\frac{1}{4}x^2+\frac{b^2}{4}y^2)\mathrm{u}\,\partial_{\mathrm{u}}$ is the first-order part. Based on the established formulas for approximate symmetries, since $\mathbf{Y}_{4,0}=0$, the auxiliary function $\mathscr{H}$ is identically zero. Since $\mathbf{Y}_4$ belongs to the approximate part of the perturbed heat equation for the cylinder (where $f_0=\ln b$), we have $F_0=u_{xx}+\frac{1}{b^2}u_{yy}-u_t$. Therefore, under invariance condition (15), it suffices to prove that

$$
\mathbf{Y}^{[2]}_{4,1}\big(u_{xx}+\frac{1}{b^2}u_{yy}-u_t\big)\big|_{u_{xx}+\frac{1}{b^2}u_{yy}-u_t=0}=0.
\tag{22}
$$

According to the definition of the second-order prolongation, one can write

$$
\mathbf{Y}^{[2]}_{4,1}\big(\mathrm{u}_{xx}+\frac{1}{b^2}\mathrm{u}_{yy}-\mathrm{u}_t\big)=\big(\varphi_1^{xx}\frac{\partial}{\partial u_{xx}}+\varphi_1^{yy}\frac{\partial}{\partial u_{yy}}+\varphi_1^{t}\frac{\partial}{\partial u_t}\big)\big(\mathrm{u}_{xx}+\frac{1}{b^2}\mathrm{u}_{yy}-\mathrm{u}_t\big)=\varphi_1^{xx}+\frac{1}{b^2}\varphi_1^{yy}-\varphi_1^{t},
$$

where φ_1^{xx}, φ_1^{yy}, and φ_1^{t} are the prolongation coefficients defined as follows [12]:

$$
\begin{aligned}
\varphi_1^{t}&=\varphi_{1t}-\xi_{1t}\,\mathrm{u}_x-\nu_{1t}\,\mathrm{u}_y+(\varphi_{1\mathrm{u}}-\tau_{1t})\,\mathrm{u}_t,\\
\varphi_1^{yy}&=\varphi_{1yy}-\xi_{1yy}\,\mathrm{u}_x+(2\varphi_{1\mathrm{u}y}-\nu_{1yy})\,\mathrm{u}_y-2\xi_{1y}\,\mathrm{u}_{xy}+(\varphi_{1\mathrm{u}}-2\nu_{1y})\,\mathrm{u}_{yy},\\
\varphi_1^{xx}&=\varphi_{1xx}+(2\varphi_{1\mathrm{u}x}-\xi_{1xx})\,\mathrm{u}_x-\nu_{1xx}\,\mathrm{u}_y+(\varphi_{1\mathrm{u}}-2\xi_{1x})\,\mathrm{u}_{1xx}-2\nu_{1x}\,\mathrm{u}_{xy}.
\end{aligned}
$$

For $\mathbf{Y}_{4,1}$, the corresponding infinitesimal coefficients are given by

$$
\xi_1=tx,\qquad \nu_1=ty,\qquad \tau_1=t^2,\qquad \varphi_1=-\big(t+\frac{1}{4}x^2+\frac{b^2}{4}y^2\big)\mathrm{u}.
$$

Corrected Proof

So the second prolongation of $\mathbf{Y}_{4,1}$ yields the following coefficients:

$$\varphi_1^t = -u - xu_x - yu_y - \big(3t + \frac{x^2}{4} + \frac{b^2y^2}{4}\big)u_t,$$
$$\varphi_1^{xx} = -\frac{1}{2}u - xu_x - \big(3t + \frac{x^2}{4} + \frac{b^2y^2}{4}\big)u_{xx},$$
$$\varphi_1^{yy} = -\frac{b^2}{2}u - b^2yu_y - \big(3t + \frac{x^2}{4} + \frac{b^2y^2}{4}\big)u_{yy}.$$

So we have

$$\mathbf{Y}_{4,1}^{[2]}\big(\mathrm{u}_{xx} + \frac{1}{b^2}\mathrm{u}_{yy} - \mathrm{u}_t\big) = \varphi_1^{xx} + \frac{1}{b^2}\varphi_1^{yy} - \varphi_1^t = -\big(\frac{1}{2}u + xu_x + \big(3t + \frac{x^2}{4} + \frac{b^2}{4}y^2\big)u_{xx}\big)$$
$$-\frac{1}{b^2}\big(\frac{b^2}{2}u + b^2yu_y + \big(3t + \frac{x^2}{4} + \frac{b^2}{4}y^2\big)u_{yy}\big) + u + xu_x + yu_y + \big(3t + \frac{x^2}{4} + \frac{b^2}{4}y^2\big)u_t$$
$$= -\big(3t + \frac{x^2}{4} + \frac{b^2}{4}y^2\big)\big(u_{xx} + \frac{1}{b^2}u_{yy} - u_t\big).$$

It is easily seen that the condition $F_0 = \mathrm{u}_{xx} + \frac{1}{b^2}\mathrm{u}_{yy} - \mathrm{u}_t = 0$ implies that (22) is satisfied.

Case 2: Constant $\mathrm{f}_1 = \mathbf{m}$. We now consider the constant case $\mathrm{f}_1 = m$. As in the previous case, we deduce that the general forms of ξ_1 and ν_1 can be expressed as $\xi_1 = A(t)x + B(y,t)$, $\nu_1 = A(t)y + C(x,t)$. Substituting these expressions into the determining equation $\mathscr{E}_{1,7}$ leads to the relation $B_y(y,t) = 2mk_4 - b^2C_x(x,t)$.

Substituting the above relations into $\mathscr{E}_{1,2}$ yields the explicit form of φ_1:

$$\varphi_1 = \big(g(y,t) - \frac{1}{4}x^2A' - \frac{1}{2}xB_t\big)\mathrm{u} + k(x,y,t), \tag{24}$$

and substituting the appropriate relations obtained above into $\mathscr{E}_{1,6}$ yields the following equation:

$$2m(k_1y + k_7) + \frac{1}{b^2}\big(2g_y - xB_{ty}\big) + yA' + C_t = 0, \tag{25}$$

and integrating (25) with respect to y, one gets

$$g(y,t) = \frac{b^2}{2}D(t)y - \frac{b^2}{4}y^2A'(t) - \frac{b^2mk_1}{2}y^2 + F(t). \tag{26}$$

Finally, substituting the explicit form of $g(y,t)$ into (25) yields the consistency condition

$$2mk_7 - \frac{x}{b^2}B_{ty} + D + C_t = 0. \tag{27}$$

Differentiation of the above relation with respect to x immediately gives $C_{tx} = B_{ty} = B_{yy} = 0$. Turning to $(\mathscr{E}_{1,5})_{\mathrm{u}}$, one readily finds $A''(t) = 0$, which implies $A(t) = e_1t + e_2$ with $e_1, e_2 \in \mathbb{R}$. Also, we obtain $B_{tt}(y,t) = 0$, leading to $B(y,t) = e_3t + e_4y + e_5$ for some constants $e_3, e_4, e_5 \in \mathbb{R}$. Furthermore, we have $g_t(y,t) = -e_1$, which together with Eq. (25) forces $D'(t) = 0$. Hence we may take $D(t) = e_6$ ($e_6 \in \mathbb{R}$). The condition $F'(t) = -e_1$ then follows, yielding $F(t) = -e_1t + e_7$ where $e_7 \in \mathbb{R}$. Plugging these results into (27) and integrating with respect to t leads to the expression for $C(x,t)$:

$$C(x,t) = \frac{2mk_4 - e_6}{b^2}x - 2mk_7t - e_3t + e_8, \qquad e_8 \in \mathbb{R}.$$

Corrected Proof

Finally, the insertion of the expression φ_1 from Eq. (24) into $\mathscr{E}_{1,5}$ leads to

$$k_{xx}+\frac{1}{b^2}k_{yy}-k_t=\frac{2m}{b^2}h_{yy}. \tag{28}$$

After a suitable relabeling of the constants, the infinitesimal coefficients take the following form:

$$\begin{aligned}
&\xi_0=(l_1t+\frac{l_2}{2})x+l_3y+l_4t+l_5,\ \tau_0=l_1t^2+l_2t+l_6,\ \nu_0=(l_1t+\tfrac{l_2}{2})y-\frac{l_3}{b^2}x+l_7t+l_8,\\
&\varphi_0=(-l_1t-\frac{1}{2}(\frac{l_1}{2}x+l_4)x-\frac{1}{2}b^2(\frac{l_1}{2}y+l_7)y+l_9)\mathrm{u}+h(x,y,t),\ \xi_1=(l_{10}t+l_{11})x+l_{12}t+l_{13}y+l_{14},\\
&\nu_1=(l_{10}t+l_{11})y-2ml_7t-l_{15}t+\frac{2ml_4-l_{13}}{b^2}x+l_{16},\ \tau_1=l_{10}t^2+2l_{11}t+l_{17},\\
&\varphi_1=(-\frac{l_{10}}{4}x^2-\frac{l_{12}}{2}x-\frac{b^2l_{10}}{4}y^2-\frac{b^2l_1}{2}my^2-\frac{b^2l_{15}}{2}y-l_{10}t+l_{18})\mathrm{u}+k(x,y,t),
\end{aligned}$$

where h and k satisfy Eqs. (16) and (28), respectively. Thus, when $f_1=m$, the generators $Y_1,\cdots,Y_{12}$ from (23) together with the following additional $Y_{13},\ldots,Y_{18}$ generate a subalgebra of the approximate symmetry algebra of (4).

$$\begin{aligned}
&Y_{13}=tx\partial_x+ty\partial_y+t^2\partial_t-(t+\frac{1}{4}x^2+\frac{b^2}{4}y^2)u\partial_u-\frac{b^2m}{2}\varepsilon y^2u\partial_u,\\
&Y_{14}=\frac{1}{2}x\partial_x+\frac{1}{2}y\partial_y+t\partial_t,\quad Y_{15}=y\partial_x-\frac{1}{b^2}x\partial_y+\frac{2m}{b^2}\varepsilon x\partial_y,\\
&Y_{16}=t\partial_x-\frac{1}{2}xu\partial_u,\quad Y_{17}=\partial_x,\quad Y_{18}=t\partial_y-\frac{b^2}{2}yu\partial_u-2m\varepsilon t\partial_y,\\
&Y_h=h\partial_u,\qquad Y_{\varepsilon k}=\varepsilon k\partial_u.
\end{aligned}$$

Table 1 presents the Lie subalgebra consisting of all linear combinations of $Y_1,\ldots,Y_{18}$ for $\mathrm{f}_1=m$. It reports several commutator entries as $O(\varepsilon^2)$ rather than as one of the listed generators. This does not mean the spanning set is incomplete: those commutators vanish at order ε and first contribute at $O(\varepsilon^2)$. Since the approximate symmetry algebra is defined modulo $O(\varepsilon^2)$, such entries are consistent with the closure property, and the $O(\varepsilon^2)$ terms need not be expressible in terms of the listed generators. The same remark applies to the entries of Table 2.

4.2 The case of a perturbed cone

In [Sec. 3.2.2, pp. 37–39] [21], the exact expression of the infinitesimal generator $\mathbf{X}$, associated with Eq. (8), was obtained, yielding the coefficients ξ_0, τ_0, ν_0, and φ_0 in the following form:

$$\begin{aligned}
&\xi_0=(k_1t+\frac{k_2}{2})x+\big(k_4\sin(ly)+k_5\cos(ly)+pk_1\big)t+k_6\sin(ly)+k_7\cos(ly)+\frac{k_2}{2}p,\\
&\tau_0=k_1t^2+k_2t+k_3,\qquad \nu_0=\frac{k_4\cos(ly)-k_5\sin(ly)}{l(x+p)}t+k_8+\frac{k_6\cos(ly)-k_7\sin(ly)}{l(x+p)},\\
&\varphi_0=-\frac{1}{2}\big(\frac{k_1}{2}x+k_4\sin(ly)+k_5\cos(ly)+pk_1\big)x-k_1t-\frac{1}{2}p\big(k_4\sin(ly)+k_5\cos(ly)\big)+k_9u+h(x,y,t).
\end{aligned} \tag{29}$$

Corrected Proof

Table 1: Approximate commutators $[Y_i, Y_j]$ for $i, j = 1, \dots, 18$, ($\mathscr{O} = \mathscr{O}(\varepsilon^2)$)

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	Y_{16}	Y_{17}	Y_{18}
Y_1	0	0	0	Y_9	$\frac{1}{2}Y_{10}$	0	0	0	$-\frac{b^2}{2}Y_{12}$	0	0	0	$Y_{18}+2mY_9$	$\frac{1}{2}Y_1$	0	0	0	$-\frac{1}{2}b^2Y_3$
Y_2		0	0	$Y_{12}-2Y_5$	$-Y_{11}$	$-Y_8$	0	0	$-Y_{10}$	0	0	0	$2Y_{14}-Y_3$	Y_2	0	Y_{17}	0	Y_1-2mY_{10}
Y_3			0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Y_4				0	$\mathscr{O}$	0	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	0	0	$-Y_4$	$\mathscr{O}$	0	$-Y_6$	$\mathscr{O}$
Y_5					0	$\mathscr{O}$	0	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	0	Y_4	0	0	$\frac{1}{2}Y_6$	$-\frac{1}{2}Y_8$	$-\frac{1}{2}Y_9$
Y_6						0	$\mathscr{O}$	$\mathscr{O}$	0	0	$\mathscr{O}$	0	0	$-\frac{1}{2}Y_6$	$-\frac{1}{b}Y_9$	0	$\frac{1}{2}Y_{12}$	0
Y_7							0	$\mathscr{O}$	$\mathscr{O}$	0	0	0	$\mathscr{O}$	0	0	$\frac{1}{b}Y_9$	$\frac{1}{b^2}Y_{10}+\mathscr{O}$	$-Y_6+\mathscr{O}$
Y_8								0	0	0	0	0	Y_6	$\frac{1}{2}Y_8$	$-\frac{1}{b^2}Y_{10}+\mathscr{O}$	$-Y_{12}$	0	0
Y_9									0	$\mathscr{O}$	$\mathscr{O}$	0	$\mathscr{O}$	$-\frac{1}{2}Y_9$	$Y_6+\mathscr{O}$	0	0	0
Y_{10}										0	0	0	$Y_9+\mathscr{O}$	$\frac{1}{2}Y_{10}$	0	0	0	$-\frac{b^2}{2}Y_{12}$
Y_{11}											0	0	$2Y_5-Y_{12}$	Y_{11}	0	Y_8	0	$Y_{10}+\mathscr{O}$
Y_{12}												0	0	0	0	0	0	0
Y_{13}													0	$-Y_{13}$	$\mathscr{O}$	0	$-Y_{15}$	$\mathscr{O}$
Y_{14}														0	0	$\frac{1}{2}Y_{16}$	$-\frac{1}{2}Y_{17}$	$\frac{1}{2}Y_{18}$
Y_{15}															0	$\frac{1}{b}Y_{18}$	$\frac{1}{b^2}Y_1-\frac{2m}{b^2}Y_{10}$	$2mY_6-Y_{16}$
Y_{16}																0	$\frac{1}{2}Y_3$	0
Y_{17}																	0	0
Y_{18}																		0

with the function h satisfying the exact heat equation

$$h_t = \frac{1}{x+p}h_x + h_{xx} + \frac{1}{l^2(x+p)^2}h_{yy}. \tag{30}$$

In the following, we analyze the reduced determining equations for the two simplest choices of the perturbation function obtained earlier, namely $f_1 = m$ and $f_1 = 1/(x+p)$, which capture the constant and decaying behaviors, respectively.

Case 1: Constant $\mathbf{f_1 = m}$. Hence, inserting the relations (29) together with $\mathrm{f}_0 = \ln\big(l(x+p)\big)$ and $\mathrm{f}_1 = m$ into the determining equations from the preceding section (the $\mathscr{E}_i$'s), we obtain the reduced system:

$$\begin{aligned}
&\mathscr{E}_{2,1} :\Leftrightarrow k_4 = k_5 = k_6 = k_7 = 0,\\
&\mathscr{E}_{2,2} : \frac{1}{(x+p)^2}\xi_1 + \xi_{1t} + 2\varphi_{1ux} - \xi_{1xx} - \frac{1}{l^2(x+p)^2}\xi_{1yy} = \frac{1}{(x+p)^2},\\
&\mathscr{E}_{2,3} : 2\xi_{1x} - \tau_{1t} = 0,\\
&\mathscr{E}_{2,4} : \frac{1}{x+p}\xi_1 + \nu_{1y} - \xi_{1x} = 0,\\
&\mathscr{E}_{2,5} : \frac{1}{x+p}\varphi_{1x} + \varphi_{1xx} + \frac{1}{l^2(x+p)^2}\varphi_{1yy} - \varphi_{1t} - \frac{2m}{l^2(x+p)^2}h_{yy} = 0,\\
&\mathscr{E}_{2,6} : \frac{1}{l^2(x+p)^2}(2\varphi_{1uy} - \nu_{1yy}) + \nu_{1t} - \frac{1}{x+p}\nu_{1x} - \nu_{1xx} = 0,\\
&\mathscr{E}_{2,7} : \frac{1}{l^2(x+p)^2}\xi_{1y} + \nu_{1x} = 0,\\
&\text{and} \quad \xi_1 = \xi_1(x,y,t), \quad \nu_1 = \nu_1(x,y,t), \quad \tau_1 = \tau_1(t).
\end{aligned}$$

From differentiation of $\mathscr{E}_{2,3}$ in the variables x and y, we get $\xi_1 = A(t)x + B(y,t)$. Next, upon differentiating $\mathscr{E}_{2,4}$ with respect to x and subsequently integrating the resulting expression with respect to y, we obtain

$$\frac{p}{x+p}A(t)y - \frac{1}{(x+p)^2}\int B(y,t)\,dy + C(x,t) = -\nu_{1x}.$$

Corrected Proof

Integrating once more, this time with respect to x, we obtain

$$v_1 = \frac{p}{x+p}A(t)y - \frac{1}{x+p}\int B(y,t)\,dy - \int C(x,t)\,dx. \tag{31}$$

Inserting this result into $\mathscr{E}_{2,7}$ gives the relation

$$-\frac{1}{l^2}B_y + pAy - \int B dy + (x+p)^2 C = 0. \tag{32}$$

Differentiating (32) with respect to x yields $2(x+p)C + (x+p)^2 C_x = 0$. Solving this ordinary differential equation gives $C(t,x) = \dfrac{F(t)}{(x+p)^2}$, with $F(t)$ being an arbitrary function.

Differentiating (32) twice with respect to y leads to $B_{yyy} + l^2 B_y = 0$. Solving this equation yields

$$B(t,y) = \frac{C_1(t)}{l}\sin(ly) - \frac{C_2(t)}{l}\cos(ly) + G(t).$$

Substituting the obtained relations into $\mathscr{E}_{2,2}$ and solving the resulting equation, a straightforward computation yields the following expression for φ_1:

$$\varphi_1 = \Big(\frac{p}{2(x+p)}A - \frac{G}{2(x+p)} - \frac{1}{4}x^2A' - \frac{x}{2}\big(\frac{C_1'}{l}\sin(ly) - \frac{C_2'}{l}\cos(ly) + G'\big) + H(y,t)\Big)\mathrm{u} + k(x,y,t). \tag{33}$$

Now, substituting the above relations into $\mathscr{E}_{2,6}$ gives

$$\begin{aligned} &-\frac{p}{(x+p)^3}Ay + \frac{1}{(x+p)^3}\int B\,dy - \frac{x}{l^2(x+p)^2}\big(C_1'\cos(ly) + C_2'\sin(ly)\big) \\ &+ \frac{2}{l^2(x+p)^2}H_y + \frac{1}{l^2(x+p)^3}B_t - \frac{1}{x+p}\frac{d}{dt}\int B\,dy - \frac{F}{(x+p)^3} + \frac{p}{x+p}A'y + \frac{F'}{x+p} + W'(t) = 0, \end{aligned} \tag{34}$$

consequently, we deduce that

$$H_y(y,t) = -\frac{p}{2}\big(C_1'(t)\cos(ly) + C_2'(t)\sin(ly)\big) + \frac{1}{2}k(y,t).$$

Now, applying $H_y(y,t)$ to (34) and comparing the coefficients of $\dfrac{1}{(x+p)^3}$ in (34) yields $pA(t) = G(t)$ and $Z(t) = F(t)$. Moreover, from the comparison of constant terms, we deduce $W'(t) = 0$, and then $W(t) = e_1, e_1 \in \mathbb{R}$. Substituting $k = 0$ and integrating H_y with respect to y yields the following expression for H:

$$H(y,t) = \frac{p}{2l}\big(C_2'(t)\cos(ly) - C_1'(t)\sin(ly)\big) + \mathrm{l}(t).$$

Inserting all the obtained expressions into $\mathscr{E}_{2,5}$ leads to

$$\frac{x}{2}\big(C_1''\sin(ly) - C_2''\cos(ly)\big) + \frac{p}{2l}\big(C_1''\sin(ly) - C_2''\cos(ly)\big) - \frac{1}{4}x^2A'' - \frac{p}{2}xA'' + \frac{1}{2}A' + \mathrm{l}' = 0,$$

Corrected Proof

which implies $A''(t)=C_1''(t)=C_2''(t)=0$, $\mathrm{l}'(t)=-\frac{1}{2}A'(t)$. Thus $A(t)=e_2t+e_3$, $C_1(t)=le_4t+le_5$, $C_2(t)=le_6t+le_7$, $\mathrm{l}(t)=-\frac{1}{2}e_2t+e_8$, where $e_i, 2\leq i\leq 8$ are arbitrary constants. Finally, upon inserting the expression of φ_1 given in (33) into $\mathscr{E}_{2,5}$, we obtain

$$k_t+\frac{2m}{l^2(x+p)^2}h_{yy}=\frac{1}{x+p}k_x+k_{xx}+\frac{1}{l^2(x+p)^2}k_{yy}. \tag{35}$$

By renaming the constants suitably, we may present the infinitesimal coefficients in the following form:

$$\begin{aligned}
&\xi_0=(l_1t+\frac{l_2}{2})x+pl_1t+\frac{l_2}{2}p,\quad \tau_0=l_1t^2+l_2t+l_3,\quad \nu_0=l_4,\quad \varphi_0=\big(-l_1t-\frac{l_1}{2}(\frac{1}{2}x+p)x+l_5\big)\mathrm{u}+h,\\
&\xi_1=(l_6t+l_7)x+(l_8t+l_9)\sin ly-(l_{10}t+l_{11})\cos ly+p(l_6t+l_7),\qquad \tau_1=l_6t^2+2l_7t+l_{12},\\
&\nu_1=\frac{1}{l(x+p)}\big((l_8t+l_9)\cos ly+(l_{10}t+l_{11})\sin ly\big),\\
&\varphi_1=\big(\frac{1}{2}(x+p)(l_{10}\cos ly-l_8\sin ly)-\frac{1}{2}\big(\frac{l_6}{2}x^2+pl_6x\big)+l_6t+l_{13}\big)\mathrm{u}+k.
\end{aligned}$$

where h and k follow from Eqs. (30) and (35), respectively. Accordingly, setting $\mathrm{f}_1=m$ yields an infinite-dimensional approximate symmetry algebra for (6), which is spanned by

$$\begin{aligned}
&Z_1=(x+p)\partial_x+2t\partial_t,\qquad Z_2=(tx+pt)\partial_x+t^2\partial_t-(t+\frac{1}{4}x^2+\frac{p}{2}x)\mathrm{u}\partial_{\mathrm{u}},\\
&Z_3=\partial_t,\qquad Z_4=\partial_y,\qquad Z_5=\mathrm{u}\partial_{\mathrm{u}},\qquad Z_6=\varepsilon Z_1,\\
&Z_7=\varepsilon\big(t\sin ly\,\partial_x+\frac{t}{l(x+p)}\cos ly\,\partial_y-\frac{1}{2}(x+p)\mathrm{u}\sin ly\,\partial_{\mathrm{u}}\big),\qquad Z_8=\varepsilon\big(\sin ly\,\partial_x+\frac{1}{l(x+p)}\cos ly\,\partial_y\big),\\
&Z_9=\varepsilon\big(-t\cos ly\,\partial_x+\frac{t}{l(x+p)}\sin ly\,\partial_y+\frac{1}{2}(x+p)\mathrm{u}\cos ly\,\partial_{\mathrm{u}}\big),\quad Z_{10}=\varepsilon\big(-\cos ly\,\partial_x+\frac{1}{l(x+p)}\sin ly\,\partial_y\big),\\
&Z_{11}=\varepsilon Z_5,\qquad Z_{12}=\varepsilon Z_3,\qquad Z_{13}=\varepsilon Z_2,\qquad Z_h=h\,\partial_{\mathrm{u}},\qquad Z_{\varepsilon k}=\varepsilon k\,\partial_{\mathrm{u}}.
\end{aligned} \tag{36}$$

Analogous to the cylinder case, the generators ∂_y, ∂_t, and $u\partial_u$ are preserved exactly (appearing as Z_3, Z_4, and Z_5), while the remaining generators are modified to approximate symmetries at $\mathscr{O}(\varepsilon)$.

For $\mathrm{f}_1=m$, the collection $\{Z_1,\dots,Z_{13}\}$ constitutes the subalgebra whose commutators are listed in Table 2.

Case 2: $\mathrm{f}_1=\dfrac{1}{\mathrm{x}+\mathrm{p}}$. The proof for this case follows the same steps as the previous one, under the same assumption $\rho_1'=\mathrm{const}$, and we omit the repetitive algebraic details. In this case, however, in addition to the relations $k_4=k_5=k_6=k_7=0$, the relation $k_1=0$ is also obtained from equation $\mathscr{E}_1$. Moreover, from equation $\mathscr{E}_5$ we conclude that

$$k_t+\frac{2}{l^2(x+p)^3}h_{yy}=\frac{1}{x+p}k_x+k_{xx}+\frac{1}{l^2(x+p)^2}k_{yy}. \tag{37}$$

Consequently, the infinitesimal coefficients assume the following form:

$$\xi_0=\frac{l_1}{2}x+\frac{pl_1}{2},\quad \tau_0=l_1t+l_2,\quad \nu_0=l_3,\quad \varphi_0=l_4\mathrm{u}+h,$$

Corrected Proof

Table 2: Commutators $[Z_i, Z_j]$ for $i, j = 1, \ldots, 13$, ($\mathscr{O} = \mathscr{O}(\varepsilon^2)$)

$[Z_i, Z_j]$	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8	Z_9	Z_{10}	Z_{11}	$2Z_{12}$	Z_{13}
Z_1	0	$2Z_2 - \frac{1}{2}p^2 Z_5$	$-2Z_3$	0	0	0	Z_7	$-Z_8$	Z_9	$-Z_{10}$	0	$-Z_{12}$	$Z_{13} - \frac{1}{2}p^2 Z_{11}$
Z_2		0	$Z_5 - Z_1$	0	0	$\frac{1}{2}p^2 Z_{11} - 2Z_{13}$	0	$-Z_7$	0	$-Z_9$	0	$Z_{11} - Z_6$	0
Z_3			0	0	0	$2Z_{12}$	Z_8	0	Z_{10}	0	0	0	$Z_6 - Z_{11}$
Z_4				0	0	0	$-lZ_9$	lZ_8	lZ_7	lZ_8	0	0	0
Z_5					0	0	0	0	0	0	0	0	0
Z_6						0	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	0	$\mathscr{O}$	$\mathscr{O}$
Z_7							0	$\mathscr{O}$	$\mathscr{O}$	$\mathscr{O}$	0	$\mathscr{O}$	0
Z_8								0	$\mathscr{O}$	$\mathscr{O}$	0	0	$\mathscr{O}$
Z_9									0	$\mathscr{O}$	0	$\mathscr{O}$	0
Z_{10}										0	0	0	$\mathscr{O}$
Z_{11}											0	$\mathscr{O}$	0
Z_{12}												0	$\mathscr{O}$
Z_{13}													0

$$\xi_1 = (l_5 t + l_6)x + (l_7 t + l_8)\sin ly - (l_9 t + l_{10})\cos ly + p(l_5 t + l_6),$$
$$\nu_1 = \frac{1}{l(x+p)}\big((l_7 t + l_8)\cos ly + (l_9 t + l_{10})\sin ly\big),$$
$$\tau_1 = l_5 t^2 + 2l_6 t + l_{11},$$
$$\varphi_1 = \big(\frac{1}{2}(x+p)(l_9\cos ly - l_7 \sin ly) - \frac{1}{2}\big(\frac{l_5}{2}x^2 + pl_5 x\big) + l_5 t + l_{12}\big)\mathrm{u} + k,$$

where h satisfies the heat equation, and k is determined by (37).

Thus, when $f_1 = \dfrac{1}{x+p}$, the generators $Z_1, Z_3, \cdots, Z_{13}$ from (36) generate a subalgebra of the symmetry algebra in the approximate sense for (8). The submatrix consisting of the first row, rows 3 through 13, and the corresponding columns of Table 1 forms a subalgebra generated by $Z_1, Z_3, \cdots, Z_{13}$ for $\mathrm{f}_1 = \dfrac{1}{x+p}$.

4.3 The case of a perturbed plane

As was stated earlier, the present case is nothing but the conical case with $l = 1$ and $b = p$. Hence, the infinitesimal generators of Eq. (9) coincide precisely with those of the previous case, namely the perturbed cone case with $f = \dfrac{1}{x+b}$, and are listed below:

$$W_1 = (x+b)\partial_x + 2t\partial_t, \quad W_2 = \partial_t, \quad W_3 = \partial_y, \quad W_4 = \mathrm{u}\partial_\mathrm{u}, \quad W_5 = \varepsilon W_1,$$
$$W_6 = \varepsilon\big(t\sin y\,\partial_x + \frac{t}{x+b}\cos y\,\partial_y - \frac{1}{2}(x+b)\mathrm{u}\sin y\,\partial_\mathrm{u}\big), \quad W_7 = \varepsilon\big(\sin y\,\partial_x + \frac{1}{x+b}\cos y\,\partial_y\big),$$
$$W_8 = \varepsilon\big(-t\cos y\,\partial_x + \frac{t}{x+b}\sin y\,\partial_y + \frac{1}{2}(x+b)\mathrm{u}\cos y\,\partial_\mathrm{u}\big), \quad W_9 = \varepsilon\big(-\cos y\,\partial_x + \frac{1}{x+b}\sin y\,\partial_y\big),$$
$$W_{10} = \varepsilon W_4, \quad W_{11} = \varepsilon W_3, \quad W_{12} = \varepsilon W_2, \quad W_h = h\partial_\mathrm{u}, \quad W_{\varepsilon k} = \varepsilon k\partial_\mathrm{u}.$$

Similarly, for the perturbed plane, the generators ∂_y, ∂_t, and $u\partial_u$ remain exact symmetries (appearing as W_2, W_3, and W_4, respectively). The other generators are modified and appear as approximate symmetries at order $\mathscr{O}(\varepsilon)$.

As a final consistency check, we examine the limit $\varepsilon \to 0$. Setting $\varepsilon = 0$ in the approximate generators Y_i, Z_i, and W_i reduces them to a subalgebra of the classical rotational symmetry algebra of the flat

Corrected Proof

surfaces of revolution classified by Mpungu [21]. The approximate generators are constructed modulo $O(\varepsilon^2)$, i.e. as asymptotic symmetries in the limit $\varepsilon \to 0$, whereas the classical algebra of Mpungu applies only to the unperturbed equation and cannot solve the perturbed one. Consequently, the generators obtained here recover precisely the part of the classical algebra that survives under the perturbation. This partial matching provides a nontrivial consistency check on the correctness of the approximate symmetry structure.

5 Symmetry reductions of the heat equation in its perturbed form

The preceding section dealt with the approximate group symmetry of the perturbed heat flow equation on a perturbed flat surface obtained by revolution. In the present section, the work is finalized by applying symmetry reductions and then the reduced PDEs are solved to obtain some explicit solutions for these equations on the given surface.

5.1 Approximate solutions for equation (4) when $\mathrm{f}_1 = m$ via Y_{18}

Assume that $\mathrm{u}(x,y,t) = \mathrm{u}_0(x,y,t) + \varepsilon\,\mathrm{u}_1(x,y,t) + \mathscr{O}(\varepsilon^2)$ is an approximate solution of Eq. (4) in the case $\mathrm{f}_1 = m$, valid to first order in ε. Substituting this expansion into (4) and equating the coefficients according to powers of ε separately yields the following system of PDEs:

$$\begin{cases} \mathrm{u}_{0xx} + \dfrac{1}{b^2}\mathrm{u}_{0yy} - \mathrm{u}_{0t} = 0, \\ \mathrm{u}_{1xx} + \dfrac{1}{b^2}\mathrm{u}_{1yy} - \mathrm{u}_{1t} - b^2\mathrm{u}_{0yy} = 0. \end{cases} \tag{38}$$

The characteristic system corresponding to the generator Y_{18} is written as

$$Y_{18}\big(\mathrm{u} - \mathrm{u}(x,y,t)\big) = \mathscr{O}(\varepsilon^2).$$

Using the above expansion in this characteristic equation and sorting factors according to order of ε yields the following coupled system:

$$\begin{cases} t\mathrm{u}_{0y} + \dfrac{b^2}{2}y\mathrm{u}_0 = 0, \\ 2mt\mathrm{u}_{0y} - t\mathrm{u}_{1y} - \dfrac{b^2}{2}y\mathrm{u}_1 = 0. \end{cases} \tag{39}$$

By solving the first equation of this system, it is readily seen that $\mathrm{u}_0 = \psi(x,t)\,e^{-\frac{b^2y^2}{4t}}$. Putting u_0 into the first equation of system (38), it is observed that $\psi(x,t)$ satisfies the following PDE:

$$\psi_{xx} - \psi_t = \frac{1}{2t}\psi. \tag{40}$$

Corrected Proof

By applying the symmetry generator $X = t\partial_x - \frac{x}{2}\psi\partial_\psi$ to reduce Eq. (40), we arrive at the solution $\psi = \frac{c}{t}e^{-\frac{x^2}{4t}}$, where c is any real constant. Consequently, we find

$$\mathrm{u}_0 = \frac{c}{t}e^{-\frac{x^2+b^2y^2}{4t}}.$$

Next, by inserting u_0 into the second ODE of system (39), we write the solution u_1 as

$$\mathrm{u}_1 = \varphi(x,t)e^{-\frac{b^2y^2}{4t}} - \frac{mcb^2y^2}{2t^2}e^{-\frac{x^2+b^2y^2}{4t}},$$

where $\varphi(x,t)$ is a function yet to be found from the second equation of system (39). Equivalently, it reads $\varphi_{xx} - \varphi_t = \frac{1}{2t}\varphi$. Thus, $\varphi = \frac{c}{t}e^{-\frac{x^2}{4t}}$. Consequently, one of the perturbed solutions of the heat Eq. (4) in the case $\mathrm{f}_1 = m$ is given by

$$\mathrm{u} = e^{-\frac{x^2+b^2y^2}{4t}}\left(\frac{c}{t} + \varepsilon\left(\frac{c}{t} - \frac{mcb^2y^2}{2t^2}\right)\right) + \mathscr{O}(\varepsilon^2).$$

Exact solution and validity of the first-order approximation. For the cylinder case with $f_0 = \ln b$ and $f_1 = m$, the exact profile is given by $w(x) = be^{\varepsilon m}$. Substituting this exact profile into the heat equation yields the exact solution

$$u_{\text{exact}} = \frac{C(\varepsilon)}{t}e^{-\frac{x^2+(be^{\varepsilon m})^2y^2}{4t}},$$

where $C(\varepsilon) = c(1+\varepsilon)$. Expanding this exact solution as a Taylor series in ε around $\varepsilon = 0$ precisely recovers our first-order approximate solution $u_0 + \varepsilon u_1$. This confirms that the first-order truncation is a valid asymptotic approximation in the regime of small ε ($\varepsilon \ll 1$), which is the inherent assumption of our perturbative framework.

5.2 Approximate solutions for equation (6) via $Z_2 + Z_9$

Let us assume that equation (6) admits an approximate solution, valid up to linear order in ε, namely

$$\mathrm{u} = \mathrm{u}_0 + \varepsilon\,\mathrm{u}_1 + \mathscr{O}(\varepsilon^2).$$

Plugging the preceding expansion into (6) and then balancing the coefficients of ε^0 and ε^1 separately yields the following system:

$$\begin{cases} \frac{1}{x+p}\mathrm{u}_{0x} + \mathrm{u}_{0xx} + \frac{1}{l^2(x+p)^2}\mathrm{u}_{0yy} - \mathrm{u}_{0t} = 0, \\ \frac{1}{x+p}\mathrm{u}_{1x} + \mathrm{u}_{1xx} + \frac{1}{l^2(x+p)^2}\mathrm{u}_{1yy} - \mathrm{u}_{1t} - \frac{2m}{l^2(x+p)^2}\mathrm{u}_{0yy} = 0. \end{cases} \tag{41}$$

Corrected Proof

The characteristic system corresponding to the generator Z_2+Z_9 is given by

$$(Z_2+Z_9)\big(\mathrm{u}-\mathrm{u}(x,y,t)\big)=\mathscr{O}(\varepsilon^2).$$

Upon inserting the preceding expansion into this characteristic equation and separating coefficients according to powers of ε, we find the following coupled set of ODEs:

$$\begin{cases} t(x+p)\mathrm{u}_{0x}+t^2\mathrm{u}_{0t}+(\frac{1}{4}x^2+\frac{p}{2}x+t)\mathrm{u}_0=0, \\ t(x+p)\mathrm{u}_{1x}+t^2\mathrm{u}_{1t}+(\frac{1}{4}x^2+\frac{p}{2}x+t)\mathrm{u}_1-t\mathrm{u}_{0x}\cos ly+\dfrac{t}{l(x+p)}\mathrm{u}_{0y}\sin ly-\dfrac{x+p}{2}\mathrm{u}_0\cos ly=0. \end{cases} \tag{42}$$

From the first equation of the system, the corresponding characteristic equations take the form

$$\frac{dx}{t(x+p)}=\frac{dt}{t^2}=-\frac{d\mathrm{u}_0}{\big(\frac{1}{4}x^2+\frac{p}{2}x+t\big)\,\mathrm{u}_0}.$$

Solving these characteristic equations yields the general solution

$$\mathrm{u}_0=\frac{\psi(z,y)}{t}e^{-\frac{x^2+2p(x+p)}{4t}},$$

where z is the reduced variable given by $z=\dfrac{x+p}{t}$. Now, substituting u_0 back into the first equation of system (41), it is observed that $\psi(z,y)$ must satisfy the following linear PDE:

$$\frac{1}{z}\psi_z+\psi_{zz}+\frac{1}{l^2z^2}\psi_{yy}-\frac{p^2}{4}\psi=0. \tag{43}$$

Upon substituting u_0 into the second Eq. of (42), the associated characteristic equation simplifies to

$$\frac{dx}{z}=\frac{dt}{1}=\frac{d\mathrm{u}_1}{(\cos ly-\frac{1}{lz}\sin ly)\frac{\psi_z}{t^3}e^{-\frac{x^2+2p(x+p)}{4t}}-(\frac{1}{t}+\frac{z^2t^2-p^2}{4t^2})\mathrm{u}_1},$$

the solution u_1 is obtained in the following form:

$$\mathrm{u}_1=\Big(\frac{\varphi(z,y)}{t}+\frac{1}{t^2}\big(\frac{\psi_y}{lz}\sin ly-\psi_z\cos ly\big)\Big)e^{-\frac{z^2t^2+p^2}{4t}},$$

where $\varphi(z,y)$ is a function to be determined, which must satisfy the second equation of system (42). In addition, applying Eq. (43) and its partial derivatives with respect to y and z, and then performing some relatively lengthy but straightforward calculations, leads us to the following relation:

$$\frac{\sin ly}{lz}\psi_y-\psi_z\cos ly-\frac{2m}{l^2z^2}\psi_{yy}+\frac{1}{z}\varphi_z+\varphi_{zz}+\frac{1}{l^2z^2}\varphi_{yy}-\frac{p^2}{4}\varphi=0. \tag{44}$$

Corrected Proof

We now solve Eq. (43) and substitute the result $\psi(z,y)$ into Eq. (44) to find $\varphi(z,y)$. With the aim of solving Eq. (43), we employ the separation of variables technique and assume $\psi(z,y) = \alpha(z)\beta(y)$. Substituting this into Eq. (43), we arrive at the following result:

$$\frac{l^2 z}{\alpha}\alpha_z + \frac{l^2 z^2}{\alpha}\alpha_{zz} - \frac{p^2 l^2 z^2}{4} + \frac{1}{\beta}\beta_{yy} = 0.$$

It is readily seen that $\frac{l^2 z}{\alpha}\alpha_z + \frac{l^2 z^2}{\alpha}\alpha_{zz} - \frac{p^2 l^2 z^2}{4}$ is a function of z, while $\frac{1}{\beta}\beta_{yy}$ is a function of y. This is only possible if both expressions are equal to a constant with opposite signs. Therefore, we may write:

$$\begin{cases} z\alpha_z + z^2\alpha_{zz} - \frac{p^2}{4}z^2\alpha = -\frac{q}{l^2}\alpha, \\ \frac{\beta_{yy}}{\beta} = q, \end{cases} \tag{45}$$

where q is an arbitrary constant required for solving Eq. (44). We set $q = 2m$. We now examine the effect of m on the solution derived from the first equation of (45). In the physical domain, the variable z satisfies $z \geq 0$. Therefore, the solution must remain regular at the origin $z = 0$. Since the Bessel functions of the second kind diverge at $z = 0$, the regularity condition requires setting their coefficients to zero. Consequently, only the Bessel functions of the first kind are admissible in each case. Similarly, the angular function $\beta(y)$ must remain positive on the physical domain, $\beta(y) > 0$, so that the temperature field stays finite and positive everywhere. This condition will be used below when determining the correction $\delta(y)$.

Case 1: $m = 0$**.** For $m = 0$, the equation becomes the modified Bessel equation of order zero, whose general solution reads

$$\alpha(z) = c_1 I_0\left(\frac{p}{2}z\right) + c_2 K_0\left(\frac{p}{2}z\right),$$

where I_0 and K_0 denote the modified Bessel functions of the first and second kind, respectively. Since K_0 diverges at $z = 0$, the regularity condition imposes $c_2 = 0$. Thus, the physically admissible solution reads

$$\alpha(z) = c_1 I_0\left(\frac{p}{2}z\right).$$

Case 2: $m > 0$**.** For $m > 0$, the equation becomes the standard Bessel equation of order $\frac{\sqrt{2m}}{l}$. The solution is

$$\alpha(z) = c_1 J_{\frac{\sqrt{2m}}{l}}\left(\frac{p}{2}z\right) + c_2 Y_{\frac{\sqrt{2m}}{l}}\left(\frac{p}{2}z\right),$$

where $J_{\frac{\sqrt{2m}}{l}}$ and $Y_{\frac{\sqrt{2m}}{l}}$ are the Bessel functions of the first and second kind, respectively. The physically admissible solution is

$$\alpha(z) = c_1 J_{\frac{\sqrt{2m}}{l}}\left(\frac{p}{2}z\right).$$

Corrected Proof

Case 3: $m<0$**.** When $m<0$, the equation transforms into the modified Bessel equation of order $\frac{\sqrt{-2m}}{l}$. The corresponding solution is

$$\alpha(z)=c_1 I_{\frac{\sqrt{-2m}}{l}}\left(\frac{p}{2}z\right)+c_2 K_{\frac{\sqrt{-2m}}{l}}\left(\frac{p}{2}z\right),$$

where $I_{\frac{\sqrt{-2m}}{l}}$ and $K_{\frac{\sqrt{-2m}}{l}}$ are the modified Bessel functions of the first and second kind of order $\frac{\sqrt{-2m}}{l}$. The corresponding solution is

$$\alpha(z)=c_1 I_{\frac{\sqrt{-2m}}{l}}\left(\frac{p}{2}z\right).$$

Thus, in all three cases, the regularity condition at the origin eliminates the Bessel functions of the second kind, leaving only the first-kind Bessel functions as the physically admissible solutions.

In all three cases, c_1 and c_2 denote arbitrary constants.

We propose the following solution for Eq. (44):

$$\varphi=\psi\left(\frac{z}{2}\cos ly+\delta(y)\right)=\alpha(z)\beta(y)\left(\frac{z}{2}\cos ly+\delta(y)\right). \tag{46}$$

Substituting the proposed form of φ into Eq. (44), we obtain

$$-\frac{2m}{l^2z^2}\psi_{yy}+\frac{1}{2}z\cos(ly)\left(\frac{1}{z}\psi_z+\psi_{zz}+\frac{1}{l^2z^2}\psi_{yy}-\frac{p^2}{4}\psi\right)+\frac{1}{l^2z^2}\left(2\delta_y\psi_y+\psi\delta_{yy}\right)=0.$$

Using Eq. (43) and the second part of Eq. (45), the above equation reduces to the following ordinary differential equation for $\delta(y)$:

$$\delta_{yy}+\frac{2\beta_y}{\beta}\delta_y=-4m^2.$$

The general exact solution for $\delta(y)$ simplifies to

$$\delta(y)=\int\left(\frac{e_1-4m^2\int\beta^2(y)\,dy}{\beta^2(y)}\right)dy+e_2, \tag{47}$$

where e_1 and e_2 are arbitrary constants. The constants e_1 and e_2 are fixed by the requirement that $\delta(y)$ remains regular in the physical domain. Since $\beta(y)$ appears in the denominator of (47), we require $\beta(y)>0$ throughout the physical domain. From a physical viewpoint, $\beta(y)$ controls the angular profile of the leading-order temperature, $u_0\propto\alpha(z)\beta(y)$. If $\beta(y)$ became zero at some point, the temperature would vanish there at leading order, while the correction $\delta(y)$ would blow up, producing an infinite temperature gradient and an infinite heat flux. This is not physically acceptable. Therefore, the constants in $\beta(y)$ are chosen so that $\beta(y)>0$, which keeps the temperature and its gradient finite and positive everywhere. This is the same regularity requirement that was already imposed on $\alpha(z)$ at $z=0$, where the Bessel functions of the second kind were discarded because they diverge. We now verify that this condition can be satisfied in all three cases:

- **Case** $m=0$: $\beta(y)=e_3y+e_4$. Choosing $e_3=0$ and $e_4>0$ gives $\beta(y)=e_4>0$ for all y. More generally, if $e_3\neq0$, the condition $\beta(y)>0$ restricts the physical domain to the interval where $e_3y+e_4>0$.

  In this case, the equation for $\delta(y)$ splits into two distinct subcases depending on the choice of constants in $\beta(y)$:

Corrected Proof

– **Subcase 1** ($e_3 = 0$)**:** Here, $\beta(y) = e_4$ is a positive constant. The differential equation for δ reduces to $\delta_{yy} = 0$, which yields the general linear solution:

$$\delta(y) = C_1 y + C_2.$$

Since $\beta = e_4 > 0$ for all y, the physical domain can be the entire real line $\mathbb{R}$; however, to ensure the temperature field remains bounded as $y \to \pm\infty$, the linear term must be suppressed by setting $C_1 = 0$, leaving $\delta(y) = C_2$.

– **Subcase 2** ($e_3 \neq 0$)**:** Here, $\beta(y) = e_3 y + e_4$ is linear. Substituting this into (47) yields the exact solution:

$$\delta(y) = \frac{e_1}{e_3 y + e_4} + e_2.$$

To ensure $\beta(y) > 0$, the physical domain is restricted to the interval where $e_3 y + e_4 > 0$.

- **Case** $m > 0$: $\beta(y) = e_3 e^{\sqrt{2m}y} + e_4 e^{-\sqrt{2m}y}$. Choosing $e_3, e_4 > 0$ gives $\beta(y) > 0$ for all $y \in \mathbb{R}$.
- **Case** $m < 0$: $\beta(y) = e_3 \cos(\sqrt{-2m}y) + e_4 \sin(\sqrt{-2m}y)$. This function is oscillatory and has zeros on $\mathbb{R}$. However, it is strictly positive on any interval between two consecutive zeros, so the physical domain is restricted to such an interval, where $\beta(y) > 0$.

Substituting $\beta(y)$ into (47) yields the exact integral representation for $\delta(y)$. Consequently, one of the perturbed solutions of the heat Eq. (6) is given by

$$\mathrm{u} = e^{-\frac{x^2 + 2p(x+p)}{4t}} \Big[\frac{\alpha\left(\frac{x+p}{t}\right)\beta(y)}{t} + \varepsilon \Big(\frac{\alpha\left(\frac{x+p}{t}\right)\beta(y)}{t} \left(\frac{z}{2} \cos ly + \delta(y) \right) + \frac{1}{t^2} \Big(\frac{\alpha\left(\frac{x+p}{t}\right)\beta_y(y)}{l\left(\frac{x+p}{t}\right)} \sin ly - \alpha_{\frac{x+p}{t}}\left(\frac{x+p}{t}\right) \cos ly \Big) \Big) \Big] + \mathscr{O}(\varepsilon^2).$$

6 Conclusions

The present study has provided a systematic treatment of approximate Lie symmetries for the heat equation defined on perturbed flat revolution surfaces, namely the cylinder, the plane, and the cone. The perturbation was introduced through a small deformation of the unit-speed generating curve, and the associated metric tensor was expanded to first order, yielding a linearised version of the Laplace-Beltrami operator. For each of the three canonical geometries, the corresponding approximate symmetry algebras were derived explicitly. A key finding is that the rotational Killing symmetry (∂_y) is preserved exactly in all three cases due to the axial nature of the perturbation. However, the deformation breaks the exactness of the other symmetries, such as axial translations and scaling, reducing them to approximate symmetries valid to first order in ε.

For the cylindrical surface, two distinct symmetry regimes were identified according to the behaviour of the perturbation function f_1: an infinite-dimensional algebra arises when f_1 is arbitrary, whereas the special choice $\mathrm{f}_1 = m$ (constant) yields a finite-dimensional subalgebra. Comparable classifications were established for the conical case, with the planar geometry recovered as a limiting configuration. First-order approximate solutions were constructed analytically, expressed in terms of standard and modified

Corrected Proof

Bessel functions, and were shown to exhibit three qualitatively different behaviours depending on the sign of the parameter m.

From an applied perspective, the proposed geometric framework is particularly relevant to heat transfer problems in engineering structures that are nearly cylindrical, nearly planar, or nearly conical. It also offers a useful tool for modelling diffusion phenomena on weakly deformed biological interfaces. Unlike conventional numerical approaches, which often obscure the structural role of symmetries, the present analytical method provides clear insight into how geometric imperfections affect the invariance properties of the underlying equations and, consequently, the behaviour of their solutions.

Several directions for future investigation can be envisaged. These include the extension of the present first-order analysis to higher perturbative orders, the application of the same methodology to surfaces with non-zero curvature, and the adaptation of the framework to other parabolic systems such as the Schrödinger or Fokker-Planck equations. The potential connection between approximate symmetries and adiabatic invariants also merits further exploration.

In summary, this work establishes a rigorous geometric foundation for approximate symmetry reduction on perturbed revolution surfaces and offers practical analytical tools for studying diffusion processes in nearly symmetric physical environments.

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Conflicts of interest

The authors have no conflicts of interest.

Author contributions

The original idea was proposed by A. Mohammadpouri. Both authors contributed equally to this work and approved the final manuscript.

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We declare that an AI language model was used solely to polish the English language, check grammar, and improve the overall readability of the text. It was not involved in the mathematical derivations, the theoretical framework, or the interpretation of the results. The core research, all algebraic calculations, and the scientific conclusions are entirely the authors' own work.

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