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Some new types of hybrid ideals in L-algebras

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Abstract. The main goal of this paper is to introduce hybrid positive implicative, hybrid implicative, and hybrid commutative ideals of L-algebras. In the following, some characterizations of these hybrid ideals are investigated, and it is proved that the quotient algebra induced by hybrid commutative ideals in CKL-algebras is a commutative BCK-algebra. Moreover, the relationships between hybrid positive implicative ideals, hybrid implicative ideals, and hybrid commutative ideals are discussed. It is demonstrated that in a bounded CKL-algebra with double negation, a hybrid ideal is a hybrid positive implicative ideal if and only if it is both a hybrid implicative ideal and a hybrid commutative ideal.

Keywords: L-algebra, Hybrid positive implicative ideal, Hybrid implicative ideal, Hybrid commutative ideal.
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1 Introduction

L-algebras, which are related to algebraic logic and quantum structures, were introduced by Rump [13]. Many examples show that L-algebras are very useful. Yang and Rump [14] characterized pseudo-MV-algebras and Bosbach's non-commutative bricks as L-algebras. Wu and Yang [16] proved that orthomodular lattices form a special class of L-algebras in different ways. It was shown that every lattice-ordered effect algebra has an underlying L-algebra structure in Wu et al. [15]. Also, other mathematicians studied the relationship between basic algebras and L-algebras.

After the introduction of fuzzy sets by Zadeh [17], the theory has been applied in different areas of computer and management sciences and computer engineering, such as artificial intelligence, expert systems, operations research, pattern recognition, robotics, control engineering, and others. Molodtsov, in [8], introduced the concept of soft sets to deal with uncertainties while

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avoiding the difficulties that appear with the usual theoretical approaches. Non-classical logic has become a formal and useful tool for computer science to deal with uncertain information.

The notion of hybrid structure was introduced by Jun, Song, and Muhiuddin [7], as a parallel circuit of fuzzy and soft sets. The notion of hybrid structure was introduced into a set of parameters on an initial universe set, and it was applied to BCK/BCI algebras and linear spaces. Hybrid ideals in semigroups were introduced by S. Anis et al. [2]. Moreover, the notion of hybrid structures on EQ-algebras and some types of hybrid (pre)filters in EQ-algebras were introduced by Paad [10–12]. In the following, the concept of hybrid structures on L-algebras and hybrid subalgebras of L-algebras was introduced by [12], and some of their properties were studied. This subject motivated us to define new kinds of hybrid ideals in L-algebras, by means of which we can study more properties of L-algebras and also conduct a deeper study of the relationship between BCK-algebras and L-algebras.

This paper is organized as follows:

- Section 2 reviews the fundamental definitions and properties of L-algebras and their hybrid structures.
- Section 3 introduces the concept of hybrid positive implicative ideals in L-algebras and provides key characterizations of these structures.
- In Section 4, we define hybrid implicative ideals in L-algebras and examine their relationship with hybrid positive implicative ideals.
- Finally, Section 5 presents the notion of hybrid commutative ideals in L-algebras and establishes that the quotient algebras induced by hybrid commutative ideals in CKL-algebras form commutative BCK-algebras. Additionally, we explore the relationships among hybrid positive implicative ideals, hybrid implicative ideals, and hybrid commutative ideals.

2 Preliminaries

In this section, we present fundamental definitions and key results. For further details, please refer to the cited references.

Definition 1 ([3,4]). *An L-algebra is an algebraic structure* $(L; \to, 1)$ *of type* $(2, 1)$ *satisfying*
$(L1)$ $x \to x = x \to 1 = 1$, *and* $1 \to x = x$,
$(L2)$ $(x \to y) \to (x \to z) = (y \to x) \to (y \to z)$,
$(L3)$ *if* $x \to y = y \to x = 1$, *then* $x = y$, *for any* $x, y, z \in L$.
Condition $(L1)$ *states that* 1 *is a logical unit, while* $(L2)$ *is related to the quantum Yang-Baxter equation. Note that a logical unit is always unique. By* $(L3)$, *the relation* $x \leq y$ *if and only if* $x \to y = 1$, *defines a partial order for any L-algebra* L. *If* L *admits a smallest element* 0, *then* L *is called a bounded L-algebra and for an element* x *of an L-algebra with* 0, *we define operation* $'$ *on* L *such that* $x' = x \to 0$. *If the mapping* $x \longmapsto x'$ *is bijective, the bounded L-algebra* L *is called an L-algebra with negation. The inverse mapping is denoted by* $x \longmapsto \tilde{x}$. *If for any* $x \in L$, $\tilde{x} = x'$, *then* L *is called an L-algebra with double negation.*

Definition 2 ([13]). *An L-algebra $(L;\to,1)$ which satisfies the following condition*

$$x\to(y\to x)=1,\quad (K)$$

for any $x,y\in L$ is called a KL-algebra. A CKL-algebra is an L-algebra $(L;\to,1)$ such that

$$x\to(y\to z)=y\to(x\to z),\quad (C)$$

Moreover, if L satisfies condition (C), then

$$x\to((x\to y)\to y)=1,\quad (D).$$

An L-algebra L is called commutative if L is a KL-algebra and, for all $x,y\in L$

$$(x\to y)\to y=(y\to x)\to x.$$

Proposition 1 ([14]). *Let $(L;\to,1)$ be an L-algebra. Then $x\le y$ implies $z\to x\le z\to y$, for any $x,y,z\in L$.*

Proposition 2 ([14]). *Let $(L;\to,1)$ be an L-algebra. Then the following statements are equivalent for any $x,y,z\in L$:*
(i) $x\le y\to x$,
(ii) if $x\le y$, then $y\to z\le x\to z$,
(iii) $((x\to y)\to z)\to z\le((x\to y)\to z)\to((y\to x)\to z)$.

Proposition 3 ([1]). *Let $(L;\to,1)$ be a CKL-algebra. Then the following properties hold for any $x,y,z\in L$:*
(i) if $x\le y$, then $z\to x\le z\to x$,
(ii) if $x\le y$, then $y\to z\le x\to z$,
(iii) $(z\to y)\le(y\to x)\to(z\to x)$,
(iv) $(z\to y)\le(x\to z)\to(x\to y)$,
(v) If L has double negation, then $x\to y=y'\to x'$,

In the following, we shall use conventions I for the unit interval, X for the set of parameters and $\mathbf{P}(\mathbf{U})$ for the power set of an initial universe set U.

Definition 3 ([7]). *A hybrid structure in X over U is defined to be a mapping*

$$\tilde{\varphi}_\lambda:=(\tilde{\varphi},\lambda):X\longrightarrow\mathbf{P}(\mathbf{U})\times I, x\longrightarrow(\tilde{\varphi}(x),\lambda(x))$$

where $\tilde{\varphi}:X\longrightarrow\mathbf{P}(\mathbf{U})$ and $\lambda:X\longrightarrow I$ are mappings.
Moreover, the sets $\tilde{\varphi}_\lambda[\alpha,t]:=\{x\in X\mid\tilde{\varphi}(x)\supseteq\alpha,\lambda(x)\le t\}$ is called $[\alpha,t]$-hybrid cut of $\tilde{\varphi}_\lambda$.

Let us denote by $\mathbf{H}(\mathbf{L})$ the set of all hybrid structures in L over U. We define an order $\ll$ in $\mathbf{H}(\mathbf{L})$ as follows:

$$(\forall\tilde{\varphi}_\lambda,\tilde{\psi}_\nu\in\mathbf{H}(\mathbf{L}))\ \ (\tilde{\varphi}_\lambda\ll\tilde{\psi}_\nu)\Leftrightarrow\tilde{\varphi}\tilde{\subseteq}\tilde{\psi},\ \text{and}\ \lambda\succeq\nu$$

where $\tilde{\varphi}\tilde{\subseteq}\tilde{\psi}$ means that $\tilde{\varphi}(x)\subseteq\tilde{\psi}(x)$ and $\lambda\succeq\nu$ means that $\lambda(x)\ge\nu(x)$, for all $x\in L$.

Definition 4 ([12]). *Let L be an L-algebra. Then a hybrid structure $\tilde{\varphi}_\lambda$ in L over U is called a* hybrid ideal *of L over U if it satisfies the followings for any $x, y \in L$:*
$(HI0)$ $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$, $\lambda(1) \leq \lambda(x)$,
$(HI1)$ $\tilde{\varphi}(y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to y)$, $\lambda(y) \leq \lambda(x) \vee \lambda(x \to y)$.
$(HI2)$ $\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(x)$, $\lambda((x \to y) \to y) \leq \lambda(x)$.
$(HI3)$ $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x)$, $\lambda(y \to x) \vee \lambda(y \to (x \to y)) \leq \lambda(x)$.
Denote $\mathbf{HID(L)}$ *the set of all hybrid ideals of L over U. Note that by $(HI3)$ every hybrid ideal of L over U is a hybrid subalgebra of L over U.*

Remark 1 ([12]). *Let L be an L-algebra and $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then*
(i) if L satisfies condition (K), then $(HI3)$ can be omitted.
(ii) If L satisfies condition (D), then $(HI2)$ can be omitted.
(iii) If L satisfies condition (C), then $(HI2)$ and $(HI3)$ can be omitted.

Proposition 4 ([12]). *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then for any $x, y, z \in L$:*
(i) if $x \leq y$, then $\tilde{\varphi}(x) \subseteq \tilde{\varphi}(y)$ and $\lambda(y) \leq \lambda(x)$.
(ii) If $x \leq y \to z$, then $\tilde{\varphi}(z) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(y)$ and $\lambda(z) \leq \lambda(x) \vee \lambda(y)$.

Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Define a binary relation $\equiv_{\tilde{\varphi}_\lambda}$ on X as follows:

$$x \equiv_{\tilde{\varphi}_\lambda} y \text{ if and only if } \tilde{\varphi}(x \to y) = \tilde{\varphi}(y \to x) = \tilde{\varphi}(1),\ \lambda(x \to y) = \lambda(y \to x) = \lambda(1)$$

Theorem 1 ([12]). *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of KL-algebra L over U. Then $\equiv_{\tilde{\varphi}_\lambda}$ is a congruence relation on L.*

Let $x \in L$ and $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then we denote the equivalence x w.r.t $\equiv_{\tilde{\varphi}_\lambda}$ by $[x]_{\tilde{\varphi}_\lambda}$ ($[x]$ for short). Furthermore, we define quotient algebra $\frac{L}{\tilde{\varphi}_\lambda} := \{[x] \mid x \in L\}$ and the operation $\rightrightarrows$, for $[x], [y] \in \frac{L}{\tilde{\varphi}_\lambda}$ as follow:

$$[x] \rightrightarrows [y] := [x \to y]$$

The top element is $[1]$ and $[x] \leq [y]$ if and only if $[x] \rightrightarrows [y] = [1]$ if and only if $[x \to y] = [1]$ if and only if $x \to y \equiv_{\tilde{\varphi}_\lambda} 1$ if and only if $\tilde{\varphi}(1 \to (x \to y)) = \tilde{\varphi}((x \to y) \to 1) = \tilde{\varphi}(1)$ and $\lambda(1 \to (x \to y)) = \lambda((x \to y) \to 1) = \lambda(1)$ if and only if $\tilde{\varphi}(x \to y) = \tilde{\varphi}(1)$ and $\lambda(x \to y) = \lambda(1)$.

Theorem 2 ([12]). *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of KL-algebra L over U. Then $(\frac{L}{\tilde{\varphi}_\lambda}, \rightrightarrows, [1])$ is an L-algebra which is called quotient L-algebra respect to $\tilde{\varphi}_\lambda$.*

Note. From now on, in this paper we let $(L; \to, 1)$ be an L-algebra, unless otherwise is stated.

3 Hybrid positive implicative ideals of L-algebras

In this section, we introduce hybrid positive implicative ideals of L-algebras and present some related results.

Definition 5. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is called a* hybrid positive implicative ideal *of L over U if it satisfies the following for all $x, y, z \in L$:*
(HPI) $\tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to ((x \to y) \to x))$ *and* $\lambda(x) \leq \lambda(z) \vee \lambda(z \to ((x \to y) \to x))$.
The set of all hybrid positive implicative ideals of L is denoted by **HPII(L)**.

Example 1. Let $L = \{0, a, b, 1\}$ be a chain where $0 < a < b < 1$. Define the operation $\to$ on L as follows:

Table 1: Implication

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	a	1	1
1	0	a	b	1

Then $(L; \to, 0, 1)$ is a bounded L-algebra [9]. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in L over U which is given by Table. 2:

Table 2: Hybrid structure

L	$\tilde{\varphi}$	λ
0	$\{u_2, u_3\}$	0.5
a	$\{u_2, u_4\}$	0.5
b	U	0.3
1	U	0.3

Then by direct calculations we verify that $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.

Example 2. Let L be L-algebra Example 1 and $U = \{u_1, u_2, u_3, u_4, u_5\}$ be an initial universe set and $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in L over U which is given by Table. 3:

Table 3: Hybrid structure

L	$\tilde{\varphi}$	λ
0	$\{u_3, u_4\}$	0.5
a	$\{u_3, u_4\}$	0.5
b	$\{u_1, u_2, u_3, u_4\}$	0.4
1	U	0.3

Then, by direct calculations, we verify that $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U, but it is not a hybrid positive implicative ideal of L over U, because $\{u_1, u_2, u_3, u_4\} = \tilde{\varphi}(b) \not\supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to ((b \to a) \to b)) = \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to (a \to b)) = \tilde{\varphi}(1) \cap \tilde{\varphi}(1) = \tilde{\varphi}(1) = U$ $0.3 = \lambda(b) \not\leq \lambda(1) \vee \lambda(1 \to ((b \to a) \to b)) = \lambda(1) \vee \lambda(1 \to (a \to b)) = \lambda(1) \vee \lambda(1) = \lambda(1) = 0.4$

Theorem 3. *Let $\tilde{\varphi}_\lambda$ be a hybrid structure of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U if and only if for any $x, y, z \in L$, satisfies in the followings:*
(i) $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$ *and* $\lambda(1) \leq \lambda(x)$.
(ii) $\tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to ((x \to y) \to x))$ *and* $\lambda(x) \leq \lambda(z) \vee \lambda(z \to ((x \to y) \to x))$.
(iii) $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$ *and* $\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(z) \vee \lambda(z \to x)$.

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U. Then by $(HI0)$ and Definition 5, (i) and (ii) hold. Since $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U, by $(HI1)$ we get that $\tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$, $\lambda(x) \leq \lambda(z) \vee \lambda(z \to y)$. Now, by $(HI2)$ and $(HI3)$, we conclude that

$$\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$$

and

$$\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(x) \leq \lambda(z) \vee \lambda(z \to x).$$

Conversely, first of all, we show that $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U. By (i), we have $(HI0)$ and by (ii) and $(L1)$, for any $x, y \in L$, we have:

$$\begin{aligned} \tilde{\varphi}(y) &\supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to ((y \to 1) \to y)) \\ &= \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to (1 \to y)) \\ &= \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to y) \end{aligned}$$

and

$$\begin{aligned} \lambda(y) &\leq \lambda(x) \vee \lambda(x \to ((y \to 1) \to y)) \\ &\leq \lambda(x) \vee \lambda(x \to (1 \to y)) \\ &\leq \lambda(x) \vee \lambda(x \to y). \end{aligned}$$

Hence, we have $(HI1)$. Moreover, by (i) and (iii), for any $x, y \in L$ and $z = 1$, we have:

$$\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to x) = \tilde{\varphi}(1) \cap \tilde{\varphi}(x) = \tilde{\varphi}(x)$$

and

$$\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(1) \vee \lambda(1 \to x) = \lambda(1) \vee \lambda(x) = \lambda(x).$$

Hence, $\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(x)$, $\lambda((x \to y) \to y) \leq \lambda(x)$ and $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x)$, $\lambda(y \to x) \vee \lambda(y \to (x \to y)) \leq \lambda(x)$ and so $(HI2)$ and $(HI3)$ hold. Therefore, $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U and so by (ii), $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U. □

Proposition 5. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is hybrid positive implicative ideal of L over U if and only if for any $x, y \in L$, $\tilde{\varphi}(x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \leq \lambda((x \to y) \to x)$.*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U and $x, y \in L$. Then by Theorem 3 and $(L1)$, we get that

$$\begin{aligned} \tilde{\varphi}(x) &\supseteq \tilde{\varphi}(1 \to ((x \to y) \to x)) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}((x \to y) \to x) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}((x \to y) \to x) \end{aligned}$$

$$\lambda(x) \leq \lambda(1 \to ((x \to y) \to y)) \vee \lambda(1)$$

$$\begin{aligned} &\leq \lambda((x \to y) \to y) \vee \lambda(1) \\ &\leq \lambda((x \to y) \to y). \end{aligned}$$

Conversely, let $x, y, z \in L$. Then by $(HI1)$, $\tilde{\varphi}((x \to y) \to x) \supseteq \tilde{\varphi}(z \to ((x \to y) \to x)) \cap \tilde{\varphi}(z)$, $\lambda((x \to y) \to y) \leq \lambda(z \to ((x \to y) \to y)) \vee \lambda(z)$ and since $\tilde{\varphi}(x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \leq \lambda((x \to y) \to y)$, we conclude that

$$\tilde{\varphi}(x) \supseteq \tilde{\varphi}(z \to ((x \to y) \to x)) \cap \tilde{\varphi}(z)$$

$$\lambda(x) \leq \lambda(z \to ((x \to y) \to y)) \vee \lambda(z).$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.

□

Proposition 6. *Let $(L, \to, 1)$ be a bounded KL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U such that for any $x \in L$, $\tilde{\varphi}((x' \to x) \to x) = \tilde{\varphi}(1)$ and $\lambda((x' \to x) \to x) = \lambda(1)$. Then $\tilde{\varphi}_\lambda$ is hybrid positive implicative ideal of L over U.*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of bounded KL-algebra L over U such that $\tilde{\varphi}((x' \to x) \to x) = \tilde{\varphi}(1)$ and $\lambda((x' \to x) \to x) = \lambda(1)$ and $x, y \in L$. Then by Proposition 1 and Proposition 2, $0 \leq y$ and so $x \to 0 \leq x \to y$. Hence, $(x \to y) \to x \leq x' \to x$ and by Proposition 4(i), $\tilde{\varphi}(x' \to x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x' \to x) \leq \lambda((x \to y) \to x)$. Now, by $(HI1)$, we conclude that

$$\begin{aligned} \tilde{\varphi}(x) &\supseteq \tilde{\varphi}((x' \to x) \to x) \cap \tilde{\varphi}(x' \to x) \\ &= \tilde{\varphi}(1) \cap \tilde{\varphi}(x' \to x) \\ &= \tilde{\varphi}(x' \to x) \\ &\supseteq \tilde{\varphi}((x \to y) \to x) \end{aligned}$$

$$\begin{aligned} \lambda(x) &\leq \lambda((x' \to x) \to x) \vee \lambda(x' \to x) \\ &\leq \lambda(1) \vee \lambda(x' \to x) \\ &= \lambda(x' \to x) \\ &\leq \lambda((x \to y) \to y). \end{aligned}$$

Therefore, by Proposition 5, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U. □

Proposition 7. *Let $(L, \to, 1)$ be a bounded KL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U. Then for any $x, y \in L$, $\tilde{\varphi}(x \to y) \supseteq \tilde{\varphi}(y' \to (x \to y))$ and $\lambda(x \to y) \leq \lambda(y' \to (x \to y))$.*

Proof. Since L is a bounded KL-algebra, by Proposition 2, for any $x, y \in L$, we get that $y \leq x \to y$ and so $(x \to y) \to 0 \leq y \to 0$. Hence, $y' \to (x \to y) \leq (x \to y)' \to (x \to y)$. Now, by Proposition 4$(i)$, we have $\tilde{\varphi}((x \to y)' \to (x \to y)) \supseteq \tilde{\varphi}(y' \to (x \to y))$ and $\lambda((x \to y)' \to (x \to y)) \leq \lambda(y' \to (x \to y))$ and since $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U, by Proposition 5, we get that have $\tilde{\varphi}(x \to y) \supseteq \tilde{\varphi}(((x \to y) \to 0) \to (x \to y)) = \tilde{\varphi}((x \to y)' \to (x \to y))$ and $\lambda(x \to y) \leq \lambda(((x \to y) \to 0) \to (x \to y))) = \lambda((x \to y)' \to (x \to y))$. Therefore, $\tilde{\varphi}(x \to y) \supseteq \tilde{\varphi}(y' \to (x \to y))$ and $\lambda(x \to y) \leq \lambda(y' \to (x \to y))$. □

Theorem 4. *Let $(L, \to, 1)$ be a bounded CKL-algebra with double negation and let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then for any $x, y, z \in L$, the following statements are equivalent:*
(i) $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.
(ii) $\tilde{\varphi}(x \to z) \supseteq \tilde{\varphi}(y \to z) \cap \tilde{\varphi}(x \to (z' \to y))$ and $\lambda(x \to z) \leq \lambda(y \to z) \vee \lambda(x \to (z' \to y))$.
(iii) $\tilde{\varphi}(x \to y) \supseteq \tilde{\varphi}(y' \to (x \to y))$ and $\lambda(x \to y) \leq \lambda(y' \to (x \to y))$.
(iv) $\tilde{\varphi}(x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \leq \lambda((x \to y) \to x)$.
(v) $\tilde{\varphi}((x' \to x) \to x) = \tilde{\varphi}(1)$ and $\lambda((x' \to x) \to x) = \lambda(1)$.

Proof. By Proposition 5,(i) and (iv) are equivalent and by Proposition 6, we have $(v) \Rightarrow (i)$ and by Proposition 7, $(i) \Rightarrow (iii)$ holds.
$(iii) \Rightarrow (i)$: By Proposition 1 and 2, for $x, y, z \in L$, we have $x' \leq x \to y$ and $(x \to y) \to x \leq x' \to x$ and so $z \to ((x \to y) \to x) \leq z \to (x' \to x)$ and since $\tilde{\varphi}_\lambda$ is a hybrid ideal of a CKL-algebra L over U, by Proposition 4(i), we get that

$$\tilde{\varphi}(x' \to (z \to x)) = \tilde{\varphi}(z \to (x' \to x)) \supseteq \tilde{\varphi}(z \to ((x \to y) \to x))$$

$$\lambda(x' \to (z \to x)) = \lambda(z \to (x' \to x)) \leq \lambda(z \to ((x \to y) \to x))$$

Now, by (iii), we have $\tilde{\varphi}(z \to x) \supseteq \tilde{\varphi}(x' \to (z \to x))$ and $\lambda(z \to x) \leq \lambda(x' \to (z \to x))$. Hence,

$$\tilde{\varphi}(z \to x) \supseteq \tilde{\varphi}(z \to ((x \to y) \to x))$$

$$\lambda(z \to x) \leq \lambda(z \to ((x \to y) \to x))$$

And so by $(HI2)$, we conclude that

$$\begin{aligned} \tilde{\varphi}(x) &\supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x) \\ &\supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to ((x \to y) \to x)) \end{aligned}$$

$$\begin{aligned} \lambda(x) &\leq \lambda(z) \vee \lambda(z \to x) \\ &\leq \lambda(z) \vee \lambda(z \to ((x \to y) \to x)) \end{aligned}$$

Therefor, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.
$(ii) \Rightarrow (iii)$: Let $z = y$ in (ii). Then by condition (C), we have

$$\begin{aligned} \tilde{\varphi}(x \to y) &\supseteq \tilde{\varphi}(y \to y) \cap \tilde{\varphi}(x \to (y' \to y)) \\ &= \tilde{\varphi}(1) \cap \tilde{\varphi}(y' \to (x \to y)) \\ &= \tilde{\varphi}(y' \to (x \to y)) \end{aligned}$$

$$\begin{aligned} \lambda(x \to y) &\leq \lambda(y \to y) \vee \lambda(x \to (y' \to y)) \\ &= \lambda(1) \vee \lambda(y' \to (x \to y)) \\ &= \lambda(y' \to (x \to y)). \end{aligned}$$

$(iv) \Rightarrow (v)$: Suppose that $\alpha = (x' \to x) \to x$, for $x \in L$. Since L is a bounded CKL-algebra, we get that

$$x' \to \alpha = x' \to ((x' \to x) \to x) = (x' \to x) \to (x' \to x) = 1$$

and so $x' \leq \alpha$. Now, by Proposition 1 and 3(i), we have $(\alpha \to 0) \to x' \leq (\alpha \to 0) \to \alpha$ and since by Proposition 3(iii), $x \to \alpha \leq (\alpha \to 0) \to (x \to 0)$, we conclude that $x \to \alpha \leq (\alpha \to 0) \to \alpha$ and since $x \to \alpha = x \to ((x' \to x) \to x) = (x' \to x) \to (x' \to x) = 1$, we get that $(\alpha \to 0) \to \alpha = 1$. Now, by (iv), we have

$$\tilde{\varphi}(\alpha) \supseteq \tilde{\varphi}((\alpha \to 0) \to \alpha) = \tilde{\varphi}(1)$$

$$\lambda(\alpha) \leq \lambda((\alpha \to 0) \to \alpha) = \lambda(1)$$

Hence, by $(HI0)$, we conclude that $\tilde{\varphi}(\alpha) = \tilde{\varphi}(1)$ and $\lambda(\alpha) = \tilde{\varphi}(1)$ and so $\tilde{\varphi}((x' \to x) \to x) = \tilde{\varphi}(1)$ and $\lambda((x' \to x) \to x) = \lambda(1)$.
$(iii) \Rightarrow (ii)$: Let $x, y \in L$. Then by Proposition 3(iv) and (v) and condition (C), we have

$$\begin{aligned}
(y \to z) \to (z' \to (x \to z)) &= (z' \to y') \to (z' \to (x \to z)) \\
&\geq y' \to (x \to z) \\
&= y' \to (z' \to x') \\
&= z' \to (y' \to x') \\
&= z' \to (x \to y) \\
&= x \to (z' \to y)
\end{aligned}$$

Now, by Proposition 4(i), we have: $\tilde{\varphi}((y \to z) \to (z' \to (x \to z))) \supseteq \tilde{\varphi}(x \to (z' \to y))$ and $\lambda((y \to z) \to (z' \to (x \to z))) \leq \lambda(x \to (z' \to y))$ and by (iii) and $(HI2)$, we conclude that

$$\begin{aligned}
\tilde{\varphi}(x \to z) &\supseteq \tilde{\varphi}(z' \to (x \to z)) \\
&\supseteq \tilde{\varphi}(y \to z) \cap \tilde{\varphi}((y \to z) \to (z' \to (x \to z))) \\
&\supseteq \tilde{\varphi}(y \to z) \cap \tilde{\varphi}(x \to (z' \to y))
\end{aligned}$$

$$\begin{aligned}
\lambda(x \to z) &\leq \lambda(z' \to (x \to z)) \\
&\leq \lambda(y \to z) \vee \lambda((y \to z) \to (z' \to (x \to z))) \\
&\leq \lambda(y \to z) \vee \lambda(x \to (z' \to y)).
\end{aligned}$$

□

Theorem 5. *Let $(L, \to, 1)$ be a bounded CKL-algebra with double negation, and let $\tilde{\varphi}_\lambda$, $\tilde{\psi}_\nu$ be two hybrid ideals of L over U such that $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu$ and $\tilde{\psi}(1) = \tilde{\varphi}(1)$, $\lambda(1) = \nu(1)$. If $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U, then $\tilde{\psi}_\nu$ is, too.*

Proof. It follows from Theorem 4. □

4 Hybrid implicative ideals of L-algebras

In this section, we introduce hybrid implicative ideals of L-algebras and present some related results.

Definition 6. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is called a* hybrid implicative ideal *of L over U if it satisfies the following for all $x, y, z \in L$:*
(HII) $\tilde{\varphi}(x \to z) \supseteq \tilde{\varphi}(x \to y) \cap \tilde{\varphi}(x \to (y \to z))$ *and* $\lambda(x \to z) \leq \lambda(x \to y) \vee \lambda(x \to (y \to z))$.
The set of all hybrid implicative ideals of L is denoted by **HII(L)**.

Example 3. Let $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in Example 2. Then $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U.

Example 4. Let $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in Example 1. Then $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U, but it is not a hybrid implicative ideal of L over U, because $\{u_1, u_2, u_3, u_4\} = \tilde{\varphi}(b) = \tilde{\varphi}(a \to 0) \not\supseteq \tilde{\varphi}(a \to a) \cap \tilde{\varphi}(a \to (a \to 0)) = \tilde{\varphi}(1) \cap \tilde{\varphi}(a \to b) = \tilde{\varphi}(1) \cap \tilde{\varphi}(1) = \tilde{\varphi}(1) = U$ and $0.3 = \lambda(b) = \lambda(a \to 0) \not\leq \lambda(a \to a) \vee \lambda(a \to (a \to 0)) = \lambda(1) \vee \lambda(a \to b) = \lambda(1) \vee \lambda(1) = \lambda(1) = 0.4$

Theorem 6. *Let $\tilde{\varphi}_\lambda$ be a hybrid structure of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U if and only if for any $x, y, z \in L$, satisfies in the followings:*
(i) $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$ *and* $\lambda(1) \leq \lambda(x)$.
(ii) $\tilde{\varphi}(x \to z) \supseteq \tilde{\varphi}(x \to y) \cap \tilde{\varphi}(x \to (y \to z))$ *and* $\lambda(x \to z) \leq \lambda(x \to y) \vee \lambda(x \to (y \to z))$.
(iii) $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$ *and* $\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(z) \vee \lambda(z \to x)$.

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid implicative ideal of L over U. Then by $(HI0)$ and Definition 6, (i) and (ii) hold. Since $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U, by $(HI1)$, $(HI2)$ and $(HI3)$, we conclude that

$$\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$$

and

$$\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(x) \leq \lambda(z) \vee \lambda(z \to x).$$

Conversely, first of all, we show that $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U. By (i), we have $(HI0)$ and by (ii) and $(L1)$, for any $x, y \in L$, we have:

$$\begin{aligned} \tilde{\varphi}(y) &= \tilde{\varphi}(1 \to y) \\ &\supseteq \tilde{\varphi}(1 \to x) \cap \tilde{\varphi}(1 \to (x \to y)) \\ &= \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to y) \end{aligned}$$

and

$$\begin{aligned} \lambda(y) &= \lambda(1 \to y) \\ &\leq \lambda(1 \to x) \vee \lambda(1 \to (x \to y)) \\ &\leq \lambda(x) \vee \lambda(x \to y). \end{aligned}$$

Hence, we have $(HI1)$. Moreover, by (i) and (iii), for any $x, y \in L$ and $z = 1$, we have:

$$\tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to x) = \tilde{\varphi}(x)$$

and

$$\lambda(y \to x) \vee \lambda((x \to y) \to y) \vee \lambda(y \to (x \to y)) \leq \lambda(1) \vee \lambda(1 \to x) = \lambda(x).$$

Hence, $\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(x)$, $\lambda((x \to y) \to y) \leq \lambda(x)$ and $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x)$, $\lambda(y \to x) \vee \lambda(y \to (x \to y)) \leq \lambda(x)$ and so $(HI2)$ and $(HI3)$ hold. Therefore, $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U and so by (ii), $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U. □

Lemma 1. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then the following statements are equivalents for any $x, y \in L$:*
(i) $\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(y \to (y \to x))$ and $\lambda(y \to x) \leq \lambda(y \to (y \to x))$,
(ii) $\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$ and $\lambda(y \to x) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x)))$.

Proof. $(i) \Rightarrow (ii)$: Let $x, y, z \in L$. Then by $(HI1)$, $(L1)$ and (i), we have:

$$\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(y \to (y \to x)) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$$

$$\lambda(y \to x) \leq \lambda(y \to (y \to x)) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x)))$$

$(ii) \Rightarrow (i)$: By $(HI0)$ and (ii), for $x, y, z \in L$, we have:

$$\begin{aligned} \tilde{\varphi}(y \to x) &\supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to (y \to (y \to x))) \\ &= \tilde{\varphi}(1 \to (y \to (y \to x))) \\ &= \tilde{\varphi}(y \to (y \to x)) \end{aligned}$$

and

$$\begin{aligned} \lambda(y \to x) &\leq \lambda(1) \vee \lambda(1 \to (y \to (y \to x))) \\ &= \lambda(1 \to (y \to (y \to x))) \\ &= \lambda(y \to (y \to x)). \end{aligned}$$

□

Lemma 2. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U such that $\tilde{\varphi}((z \to y) \to (z \to x)) \supseteq \tilde{\varphi}(z \to (y \to x))$ and $\lambda((z \to y) \to (z \to x)) \leq \lambda(z \to (y \to x))$, for any $x, y, z \in L$. Then*

$$\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$$

$$\lambda(y \to x) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x))).$$

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L and $x, y, z \in L$. Then by $(HI1)$, we have

$$\tilde{\varphi}(y \to (y \to x)) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$$

$$\lambda(y \to (y \to x)) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x))).$$

Now, by $(L1)$ and hypothesis we have:

$$\begin{aligned} \tilde{\varphi}(y \to x) &= \tilde{\varphi}(1 \to (y \to x)) \\ &= \tilde{\varphi}((y \to y) \to (y \to x)) \\ &\supseteq \tilde{\varphi}(y \to (y \to x))) \end{aligned}$$

and

$$\begin{aligned}\lambda(y \to x) &= \lambda(1 \to (y \to x)) \\ &= \lambda((y \to y) \to (y \to x)) \\ &\leq \lambda(y \to (y \to x)).\end{aligned}$$

Hence,

$$\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$$
$$\lambda(y \to x) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x))).$$

□

Lemma 3. *Let $\tilde{\varphi}_\lambda$ be a hybrid implicative ideal of L over U. Then for any $x, y \in L$. Then*

$$\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(y \to (y \to x))$$
$$\lambda(y \to x) \leq \lambda(y \to (y \to x)).$$

Proof. Since $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U, by $(HI0)$ and $(L1)$ for $x, y \in L$, we have:

$$\begin{aligned}\tilde{\varphi}(y \to x) &\supseteq \tilde{\varphi}(y \to y) \cap \tilde{\varphi}(y \to (y \to x)) \\ &= \tilde{\varphi}(1) \cap \tilde{\varphi}(y \to (y \to x)) \\ &= \tilde{\varphi}(y \to (y \to x))\end{aligned}$$

and

$$\begin{aligned}\lambda(y \to x) &\leq \lambda(y \to y) \vee \lambda(y \to (y \to x)) \\ &= \lambda(1) \vee \lambda(y \to (y \to x)) \\ &= \lambda(y \to (y \to x)).\end{aligned}$$

□

Theorem 7. *Let L be a chain such that for any $x, y \in L$, $x \leq (x \to y) \to y$ and $\tilde{\varphi}_\lambda$ be a hybrid implicative ideal of L over U. If $\tilde{\varphi}((y \to x) \to x) \supseteq \tilde{\varphi}((x \to y) \to y)$ and $\lambda((y \to x) \to x) \leq \lambda((x \to y) \to y)$, for any $x, y \in L$. Then $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.*

Proof. Let $x, y \in L$. Since L is a chain, we get that $x \leq y$ or $y \leq x$. If $x \leq y$, then by $(L1)$, $(x \to y) \to x = 1 \to x = x$ and so $\tilde{\varphi}(x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \leq \lambda((x \to y) \to x)$. Hence, by Proposition 5, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U. Now, let $y \leq x$. Since $x \leq (x \to y) \to y$, by Proposition 1, we get that $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$ and so by Proposition 4(i),

$$\tilde{\varphi}((x \to y) \to ((x \to y) \to y)) \supseteq \tilde{\varphi}((x \to y) \to x)$$
$$\lambda((x \to y) \to ((x \to y) \to y)) \leq \lambda((x \to y) \to x).$$

Since $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U, by Lemma 3, we have

$$\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}((x \to y) \to ((x \to y) \to y))$$

$$\lambda((x \to y) \to y) \leq \lambda((x \to y) \to ((x \to y) \to y)).$$

Now, by hypothesis we have:

$$\begin{aligned} \tilde{\varphi}(x) &= \tilde{\varphi}(1 \to x) \\ &\supseteq \tilde{\varphi}((y \to x) \to x) \\ &\supseteq \tilde{\varphi}((x \to y) \to y) \\ &\supseteq \tilde{\varphi}((x \to y) \to ((x \to y) \to y)) \\ &\supseteq \tilde{\varphi}((x \to y) \to x) \end{aligned}$$

$$\begin{aligned} \lambda(x) &= \lambda(1 \to x) \\ &= \lambda((y \to x) \to x) \\ &\leq \lambda((x \to y) \to y) \\ &\leq \lambda((x \to y) \to ((x \to y) \to y)) \\ &\leq \lambda((x \to y) \to x). \end{aligned}$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U □

Theorem 8. *Let $(L; \to, 1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then for any $x, y, z \in L$, the following statements are equivalent:*
(i) $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U.
(ii) $\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(y \to (y \to x))$ and $\lambda(y \to x) \leq \lambda(y \to (y \to x))$.
(iii) $\tilde{\varphi}((z \to y) \to (z \to x)) \supseteq \tilde{\varphi}(z \to (y \to x))$ and $\lambda((z \to y) \to (z \to x)) \leq \lambda(z \to (y \to x))$.
(iv) $\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x)))$ and $\lambda(y \to x) \leq \lambda(z) \vee \lambda(z \to (y \to (y \to x)))$.

Proof. $(i) \Rightarrow (ii)$: It follows from Lemma 3.
$(ii) \Rightarrow (iii)$: Let $x, y, z \in L$. Then by Proposition 3(iv) and condition (C), we have $y \to x \leq (z \to y) \to (z \to x)$ and so $z \to (y \to x) \leq z \to ((z \to y) \to (z \to x)) = z \to (z \to ((z \to y) \to x))$ Hence, by (ii) and Proposition 4(i), we have:

$$\begin{aligned} \tilde{\varphi}((z \to y) \to (z \to x)) &= \tilde{\varphi}(z \to ((z \to y) \to x)) \\ &\supseteq \tilde{\varphi}(z \to (z \to ((z \to y) \to x))) \\ &\supseteq \tilde{\varphi}((z \to (y \to x)) \end{aligned}$$

$$\begin{aligned} \lambda((z \to y) \to (z \to x)) &= \lambda(z \to ((z \to y) \to x)) \\ &\leq \lambda(z \to (z \to ((z \to y) \to x))) \\ &\leq \lambda(z \to (y \to x)). \end{aligned}$$

$(iii) \Rightarrow (iv)$: Let $z = y$ in (iii). Then we have:

$$\begin{aligned} \tilde{\varphi}(y \to x)) &= \tilde{\varphi}((y \to y) \to (y \to x)) \\ &\supseteq \tilde{\varphi}(y \to (y \to x)) \\ &\supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to (y \to x))) \end{aligned}$$

$$\begin{aligned} \lambda(y \to x) &= \lambda((y \to y) \to (y \to x)) \\ &\leq \lambda(y \to (y \to x)) \\ &\leq \lambda(z) \vee \lambda(z \to (y \to (y \to x))). \end{aligned}$$

$(iv) \Rightarrow (i)$: By Proposition 3(iii) and (iv), for any $x, y, z \in L$, we have

$$x \to y \leq (y \to z) \to (x \to z) \leq (x \to (y \to z)) \to (x \to (x \to z))$$

And by Proposition 4(i), we get that

$$\tilde{\varphi}((x \to (y \to z)) \to (x \to (x \to z))) \supseteq \tilde{\varphi}(x \to y)$$

$$\lambda((x \to (y \to z)) \to (x \to (x \to z))) \leq \lambda(x \to y).$$

Now, by (iv), we conclude that

$$\begin{aligned} \tilde{\varphi}(x \to z) &\supseteq \tilde{\varphi}(x \to (y \to z)) \cap \tilde{\varphi}((x \to (y \to z)) \to (x \to (x \to z))) \\ &\supseteq \tilde{\varphi}(x \to (y \to z)) \cap \tilde{\varphi}(x \to y) \\ &= \tilde{\varphi}(x \to y) \cap \tilde{\varphi}(x \to (y \to z)) \end{aligned}$$

$$\begin{aligned} \lambda(x \to z) &\leq \lambda(x \to (y \to z)) \vee \lambda((x \to (y \to z)) \to (x \to (x \to z))) \\ &\leq \lambda(x \to (y \to z)) \vee \lambda(x \to y) \\ &= \lambda(x \to y) \vee \lambda(x \to (y \to z)). \end{aligned}$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U.

□

Theorem 9. *Let $(L; \to, 1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$, $\tilde{\psi}_\nu$ be two hybrid ideals of L over U such that $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu$ and $\tilde{\psi}(1) = \tilde{\varphi}(1)$, $\lambda(1) = \nu(1)$. If $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U, then $\tilde{\psi}_\nu$ is, too.*

Proof. Suppose that $\alpha := y \to (y \to x)$, for $x, y \in L$ and since L is a CKL-algebra, we have

$$y \to (y \to (\alpha \to x)) = y \to (\alpha \to (y \to x)) = \alpha \to (y \to (y \to x)) = \alpha \to \alpha = 1$$

Since $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U, by Theorem 8, we get that

$$\tilde{\varphi}(y \to (\alpha \to x)) \supseteq \tilde{\varphi}(y \to (y \to (\alpha \to x))) = \tilde{\varphi}(1)$$
$$\lambda(y \to (\alpha \to x)) \leq \lambda(y \to (y \to (\alpha \to x))) = \lambda(1).$$

Hence, by $(HI0)$, $\tilde{\varphi}(y \to (\alpha \to x)) = \tilde{\varphi}(1)$ and $\lambda(y \to (\alpha \to x)) = \lambda(1)$ and since $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu (\tilde{\varphi} \tilde{\subseteq} \tilde{\psi}$ and $\lambda \succeq \nu)$, we get that

$$\tilde{\psi}(1) = \tilde{\varphi}(1) = \tilde{\varphi}(y \to (\alpha \to x)) \subseteq \tilde{\psi}(y \to (\alpha \to x)) = \tilde{\psi}(\alpha \to (y \to x))$$

$$\nu(1) = \lambda(1) = \lambda(y \to (\alpha \to x)) \geq \nu(y \to (\alpha \to x)) = \nu(\alpha \to (y \to x))$$

And so by $(HI0)$, we have $\tilde{\psi}(\alpha \to (y \to x)) = \tilde{\psi}(1)$ and $\nu(\alpha \to (y \to x)) = \nu(1)$. Now, by $(HI1)$, we conclude that

$$\begin{aligned} \tilde{\psi}(y \to x) &\supseteq \tilde{\psi}(\alpha) \cap \tilde{\psi}(\alpha \to (y \to x)) \\ &= \tilde{\psi}(\alpha) \cap \tilde{\psi}(1) \\ &= \tilde{\psi}(\alpha) \\ &= \tilde{\psi}(y \to (y \to x)) \end{aligned}$$

$$\begin{aligned} \nu(y \to x) &\leq \nu(\alpha) \vee \nu(\alpha \to (y \to x)) \\ &\leq \nu(\alpha) \vee \nu(1) \\ &= \lambda(\alpha) \\ &= \nu(y \to (y \to x)). \end{aligned}$$

Therefore, by Theorem 8, $\tilde{\psi}_\nu$ is a hybrid implicative ideal of L over U.

□

Theorem 10. *Let $(L; \to, 1)$ be a bounded CKL-algebra with double negation and let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U.*

Proof. Since L is a bounded CKL-algebra with double negation, by Proposition 3(i) and (ii), we get that $y \to 0 \leq y \to x$ and $(y \to x)' \leq y'' = y$ and so $y \to (y \to x) \leq (y \to x)' \to (y \to x)$. Hence, by Proposition 4(i), we have

$$\tilde{\varphi}((y \to x)' \to (y \to x)) \supseteq \tilde{\varphi}(y \to (y \to x))$$

$$\lambda((y \to x)' \to (y \to x)) \leq \lambda(y \to (y \to x)).$$

Since $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U, by Theorem 4, we get that

$$\tilde{\varphi}(((y \to x)' \to (y \to x)) \to (y \to x)) = \tilde{\varphi}(1)$$

$$\lambda(((y \to x)' \to (y \to x)) \to (y \to x)) = \lambda(1).$$

And so by $(HI1)$, we conclude that

$$\begin{aligned}\tilde{\varphi}(y \to x) &\supseteq \tilde{\varphi}((y \to x)' \to (y \to x)) \cap \tilde{\varphi}(((y \to x)' \to (y \to x)) \to (y \to x)) \\ &= \tilde{\varphi}((y \to x)' \to (y \to x)) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}((y \to x)' \to (y \to x) \\ &\supseteq \tilde{\varphi}(y \to (y \to x))\end{aligned}$$

$$\begin{aligned}\lambda(y \to x) &\leq \lambda((y \to x)' \to (y \to x)) \vee \lambda(((y \to x)' \to (y \to x)) \to (y \to x)) \\ &= \lambda((y \to x)' \to (y \to x)) \vee \lambda(1) \\ &= \lambda((y \to x)' \to (y \to x)) \\ &\leq \lambda(y \to (y \to x)).\end{aligned}$$

Therefore, by Theorem 8, $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U.

□

Theorem 11. *Let $(L; \to, 1)$ be a bounded CKL-algebra with double negation and $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U if and only if $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U and for any $x, y \in L$, $\tilde{\varphi}((y \to x) \to x) \supseteq \tilde{\varphi}((x \to y) \to y)$ and $\lambda((y \to x) \to x) \leq \lambda((x \to y) \to y)$.*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U. Then by Theorem 10, $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U. Suppose that $x, y \in L$, then by Proposition 2 and 3(*iii*), $x \leq (y \to x) \to x$ and $((y \to x) \to x)' \leq x'$ and so $x' \to y \leq ((y \to x) \to x)' \to y$. Moreover, since $x' \leq x \to y$ and $(x \to y) \to y \leq x' \to y$. Hence, $(x \to y) \to y \leq ((y \to x) \to x)' \to y$. Now, by Proposition 4(*i*),

$$\tilde{\varphi}(((y \to x) \to x)' \to y) \supseteq \tilde{\varphi}((x \to y) \to y)$$

$$\lambda(((y \to x) \to x)' \to y) \leq \lambda((x \to y) \to y).$$

In addition, since $y \leq (y \to x) \to x$, we get that

$$((y \to x) \to x)' \to y \leq ((y \to x) \to x)' \to ((y \to x) \to x)$$

Hence, by Proposition 4(*i*),

$$\tilde{\varphi}(((y \to x) \to x)' \to ((y \to x) \to x)) \supseteq \tilde{\varphi}(((y \to x) \to x)' \to y)$$

$$\lambda(((y \to x) \to x)' \to ((y \to x) \to x)) \leq \lambda(((y \to x) \to x)' \to y).$$

and so

$$\tilde{\varphi}(((y \to x) \to x)' \to ((y \to x) \to x)) \supseteq \tilde{\varphi}((x \to y) \to y)$$

$$\lambda(((y \to x) \to x)' \to ((y \to x) \to x)) \leq \lambda((x \to y) \to y).$$

Since $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U. Then by Theorem 4, we have

$$\tilde{\varphi}((((y \to x) \to x)' \to ((y \to x) \to x)) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$$
$$\lambda((((y \to x) \to x)' \to ((y \to x) \to x)) \to ((y \to x) \to x)) = \lambda(1).$$

Now, by $(HI1)$, we conclude that

$$\begin{aligned}
\tilde{\varphi}((y \to x) \to x) &\supseteq \tilde{\varphi}((((y \to x) \to x)' \to ((y \to x) \to x)) \to ((y \to x) \to x)) \\
&\cap \tilde{\varphi}(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&= \tilde{\varphi}(1) \cap \tilde{\varphi}(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&= \tilde{\varphi}(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&\supseteq \tilde{\varphi}((x \to y) \to y)
\end{aligned}$$

$$\begin{aligned}
\lambda((y \to x) \to x) &\leq \lambda((((y \to x) \to x)' \to ((y \to x) \to x)) \to ((y \to x) \to x)) \\
&\vee \lambda(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&= \lambda(1) \vee \lambda(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&= \lambda(((y \to x) \to x)' \to ((y \to x) \to x)) \\
&\leq \lambda((x \to y) \to y).
\end{aligned}$$

Conversely, let $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U and $x, y \in L$. Then by Proposition 3(ii), by $0 \leq y$, we get that $x \to 0 \leq x \to y$ and so $(x \to y) \to x \leq x' \to x$ and since $x \leq x''$, we get that $x' \to x \leq x' \to x''$ and so $(x \to y) \to x \leq x' \to x''$. Hence, by Proposition 4(i), we have

$$\tilde{\varphi}(x' \to x'') \supseteq \tilde{\varphi}((x \to y) \to x)$$

$$\lambda(x' \to x'') \leq \lambda((x \to y) \to x)$$

Now, by hypothesis we have:

$$\begin{aligned}
\tilde{\varphi}(x' \to x'') &= \tilde{\varphi}(x' \to (x' \to 0)) \\
&\supseteq \tilde{\varphi}(x' \to 0) \\
&= \tilde{\varphi}((x' \to 0) \to 0) \\
&\supseteq \tilde{\varphi}((0 \to x) \to x) \\
&= \tilde{\varphi}(1 \to x) \\
&= \tilde{\varphi}(x)
\end{aligned}$$

$$\begin{aligned}
\lambda(x' \to x'') &= \lambda(x' \to (x' \to 0)) \\
&\geq \lambda(x' \to 0) \\
&= \lambda((x' \to 0) \to 0)
\end{aligned}$$

$$\begin{aligned} &\geq \lambda((0 \to x) \to x) \\ &= \lambda(1 \to x) \\ &= \lambda(x). \end{aligned}$$

Therefore, $\tilde{\varphi}(x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(x) \geq \lambda((x \to y) \to x)$ and so by Proposition 4, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.

□

5 Hybrid commutative ideals of L-algebras

In this section, we introduce hybrid commutative ideals of L-algebras and present some related results.

Definition 7. *Let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is called a* hybrid commutative ideal *of L over U if it satisfies the following for all $x, y \in L$:*
(HCI) $\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(y \to x)$ *and* $\lambda(((x \to y) \to y) \to x) \leq \lambda(y \to x)$.
The set of all hybrid commutative ideals of L is denoted by **HCI(L)**.

Example 5. Let $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in Example 2. Then $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.

Example 6. Let $\tilde{\varphi}_\lambda = (\tilde{\varphi}, \lambda)$ be a hybrid structure in Example 1. Then $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U, but is not a hybrid commutative ideal of L over U, because

$$\{u_1, u_2, u_3, u_4\} = \tilde{\varphi}(b) = \tilde{\varphi}(((b \to a) \to a) \to b) \not\supseteq \tilde{\varphi}(a \to b) = \tilde{\varphi}(1) = U$$

$$0.3 = \lambda(b) = \lambda(((b \to a) \to a) \to b) \not\leq \lambda(a \to b) = \lambda(1) = 0.4$$

Theorem 12. *Let $\tilde{\varphi}_\lambda$ be a hybrid structure of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U if and only if for any $x, y, z \in L$, satisfies in the followings:*
(i) $\tilde{\varphi}(1) \supseteq \tilde{\varphi}(x)$ *and* $\lambda(1) \leq \lambda(x)$.
(ii) $\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to x))$ *and* $\lambda(((x \to y) \to y) \to x) \leq \lambda(z) \vee \lambda(z \to (y \to x))$.
(iii) $\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$ *and* $\lambda((x \to y) \to y) \leq \lambda(z) \vee \lambda(z \to x)$.
(iv) $\tilde{\varphi}(y \to x) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$ *and* $\lambda(y \to x) \vee \lambda(y \to (x \to y)) \leq \lambda(z) \vee \lambda(z \to x)$.

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid commutative ideal of L over U. Then by Definition 4, (i) holds and for any $x, y, z \in L$, by $(HI1)$ and by Definition 7 we have:

$$\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to (y \to x))$$

$$\lambda(((x \to y) \to y) \to x) \leq \lambda(y \to x) \leq \lambda(z) \vee \lambda(z \to (y \to x))$$

Hence, (ii) holds. Moreover, by $(HI1)$, $(HI2)$ and $(HI3)$ we have:

$$\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$$

$$\lambda((x \to y) \to y) \leq \lambda(x) \leq \lambda(z) \vee \lambda(z \to x)$$

And

$$\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$$

$$\lambda(y \to x) \leq \lambda(x) \leq \lambda(z) \vee \lambda(z \to x)$$

And

$$\tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(x) \supseteq \tilde{\varphi}(z) \cap \tilde{\varphi}(z \to x)$$

$$\lambda(y \to (x \to y)) \leq \lambda(x) \leq \lambda(z) \vee \lambda(z \to x)$$

Hence, (iii) and (iv) hold.
Conversely, let $\tilde{\varphi}_\lambda$ be a hybrid structure of L over U and $x, y \in L$. Then by $(L1)$, $x \to (1 \to y) = x \to y$ and $y = ((y \to 1) \to 1) \to y$ and so by (ii) we have

$$\tilde{\varphi}(y) = \tilde{\varphi}(((y \to 1) \to 1) \to y) \supseteq \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to (1 \to y)) = \tilde{\varphi}(x) \cap \tilde{\varphi}(x \to y)$$

$$\lambda(y) = \lambda(((y \to 1) \to 1) \to y) \leq \lambda(x) \vee \lambda(x \to (1 \to y)) = \lambda(x) \vee \lambda(x \to y).$$

Hence, $\tilde{\varphi}_\lambda$ satisfies in $(HI1)$. By (iii), for $x, y \in L$, we have:

$$\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to x) = \tilde{\varphi}(1) \cap \tilde{\varphi}(x) = \tilde{\varphi}(x)$$

$$\lambda((x \to y) \to y) \leq \lambda(1) \vee \lambda(1 \to x) = \lambda(1) \vee \lambda(x) = \lambda(x)$$

and so $(HI2)$ holds. Moreover, by (iv), for $x, y \in L$, we have:

$$\tilde{\varphi}(y \to x) \cap \tilde{\varphi}(y \to (x \to y)) \supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to x) = \tilde{\varphi}(1) \cap \tilde{\varphi}(x) = \tilde{\varphi}(x)$$

$$\lambda(y \to x) \vee \lambda(y \to (x \to y)) \leq \lambda(1) \vee \lambda(1 \to x) = \lambda(1) \vee \lambda(x) = \lambda(x)$$

Hence, $\tilde{\varphi}_\lambda$ satisfies in $(HI3)$ and so $\tilde{\varphi}_\lambda$ is a hybrid ideal of L over U. Moreover, by considering $z =$ in (ii), we conclude that

$$\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(1) \cap \tilde{\varphi}(1 \to (y \to x)) = \tilde{\varphi}(y \to x)$$

$$\lambda(((x \to y) \to y) \to x) \leq \lambda(1) \vee \lambda(1 \to (y \to x)) = \lambda(y \to x)$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.

□

Theorem 13. *Let $(L; \to, 1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then for any $x, y, z \in L$, the following statements are equivalent:*
(i) $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.
(ii) $\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(y \to x)$ and $\lambda(((x \to y) \to y) \to x) \leq \lambda(y \to x)$.
(iii) $\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$ and $\lambda(((x \to y) \to y) \to ((y \to x) \to x)) = \lambda(1)$.

Proof. $(i) \Leftrightarrow (ii)$: It follows from Definition 7.
$(i) \Rightarrow (iii)$: Let $\tilde{\varphi}_\lambda$ be a hybrid commutative ideal of L over U and $x, y \in L$. Then by twice use of Proposition 3(iii) and $x \leq (y \to x) \to x$ we have $(x \to y) \to y \leq (((y \to x) \to x) \to y) \to y$ and so $((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x) \leq ((x \to y) \to y) \to ((y \to x) \to x)$. Hence, by Proposition 4(i), we have:

$$\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) \supseteq \tilde{\varphi}(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x))$$

$$\lambda(((x \to y) \to y) \to ((y \to x) \to x)) \leq \lambda(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x))$$

And since $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U, we conclude that

$$\tilde{\varphi}(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) \supseteq \tilde{\varphi}(y \to ((y \to x) \to x))$$

$$\lambda(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) \leq \lambda(y \to ((y \to x) \to x)).$$

Now, since by condition (C) and $(L1)$, $y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1$, by $(HI0)$, we conclude that

$$\tilde{\varphi}(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$$

$$\lambda(((((y \to x) \to x) \to y) \to y) \to ((y \to x) \to x)) = \lambda(1).$$

Hence,

$$\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) \supseteq \tilde{\varphi}(1)$$

$$\lambda(((x \to y) \to y) \to ((y \to x) \to x)) \leq \lambda(1).$$

Therefor, by $(HI0)$,

$$\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$$

$$\lambda(((x \to y) \to y) \to ((y \to x) \to x)) = \lambda(1).$$

$(iii) \Rightarrow (i)$: Let $x, y \in L$. Then by condition (C) and (iii), we have:

$$\tilde{\varphi}(1) = \tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}((y \to x) \to (((x \to y) \to y) \to x))$$

$$\lambda(1) = \lambda(((x \to y) \to y) \to ((y \to x) \to x)) = \lambda((y \to x) \to (((x \to y) \to y) \to x)).$$

Now, by $(HI1)$, we conclude that

$$\begin{aligned} \tilde{\varphi}((((x \to y) \to y) \to x)) &\supseteq \tilde{\varphi}(y \to x) \cap \tilde{\varphi}((y \to x) \to (((x \to y) \to y) \to x)) \\ &\supseteq \tilde{\varphi}(y \to x) \cap \tilde{\varphi}(1) \\ &= \tilde{\varphi}(y \to x) \end{aligned}$$

$$\begin{aligned} \lambda((((x \to y) \to y) \to x)) &= \lambda(y \to x) \vee \lambda((y \to x) \to (((x \to y) \to y) \to x)) \\ &= \lambda(y \to x)\lambda(1) \\ &= \lambda(y \to x). \end{aligned}$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.

□

Theorem 14. *Let $(L;\to,1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$, $\tilde{\psi}_\nu$ be two hybrid ideals of L over U such that $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu$ and $\tilde{\psi}(1)=\tilde{\varphi}(1)$, $\lambda(1)=\nu(1)$. If $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U, then $\tilde{\psi}_\nu$ is, too.*

Proof. Let $x,y\in L$, $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu$ such that $\tilde{\psi}(1)=\tilde{\varphi}(1)$, $\lambda(1)=\nu(1)$ and $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U. Then by Theorem 13,

$$\tilde{\varphi}(((x\to y)\to y)\to((y\to x)\to x))=\tilde{\varphi}(1)=\tilde{\psi}(1)$$

$$\lambda(((x\to y)\to y)\to((y\to x)\to x))=\lambda(1)=\nu(1).$$

Since, $\tilde{\varphi}_\lambda \ll \tilde{\psi}_\nu$, we get that

$$\tilde{\psi}(((x\to y)\to y)\to((y\to x)\to x))\supseteq\tilde{\varphi}(((x\to y)\to y)\to((y\to x)\to x))=\tilde{\psi}(1)$$

$$\nu(((x\to y)\to y)\to((y\to x)\to x))\leq\lambda(((x\to y)\to y)\to((y\to x)\to x))=\nu(1).$$

Hence, by $(HI0)$ we conclude that

$$\tilde{\psi}(((x\to y)\to y)\to((y\to x)\to x))=\tilde{\psi}(1)$$

$$\nu(((x\to y)\to y)\to((y\to x)\to x))=\nu(1).$$

Therefore, by Theorem 13, $\tilde{\psi}_\nu$ is a hybrid commutative ideal of L over U □

Definition 8. *[6] By a BCI-algebra, we mean an algebra $(X,*,0)$ of type $(2,0)$ satisfying the following axioms: for all $x,y,z\in X$:*
$(BCI1)$ $((x*y)*(x*z))*(z*y)=0$,
$(BCI2)$ $(x*(x*y))*y=0$,
$(BCI3)$ $x*x=0$,
$(BCI4)$ $x*y=y*x=0$ implies $x=y$.
On any BCI-algebra $(X,*,0)$ the natural order can be defined by putting $x\leq y$ if and only if $x*y=0$, for all $x,y\in X$. If in a BCI-algebra $(X,*,0)$ the condition $0*x=0$, for all $x\in X$ hold, then it is a BCK-algebra

Theorem 15. *[3] Every CKL-algebra is a BCK-algebra.*

Theorem 16. *Let $(L;\to,1)$ be a CKL-algebra and $\tilde{\varphi}_\lambda$ be a hybrid ideals of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U if and only if $(\frac{L}{\tilde{\varphi}_\lambda},\rightrightarrows,[1])$ is a commutative BCK-algebra.*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid commutative ideals of L over U. Then by Theorem 2, $(\frac{L}{\tilde{\varphi}_\lambda},\rightrightarrows,[1])$ is an L-algebra and for any $[x],[y],[z]\in\frac{L}{\tilde{\varphi}_\lambda}$ and since L is a CKL-algebra, we have

$$\begin{aligned}[x]\rightrightarrows([y]\rightrightarrows[z]) &= [x]\rightrightarrows[y\to z]\\ &= [x\to(y\to z)]\\ &= [y\to(z\to z)]\\ &= [y]\rightrightarrows[x\to z]\end{aligned}$$

$$= \quad [y] \rightrightarrows ([x] \rightrightarrows [z]).$$

Therefore, $(\frac{L}{\tilde{\varphi}_\lambda}, \rightrightarrows, [1])$ is a CKL-algebra and so by Theorem 15, it is a BCK-algebra and since $\tilde{\varphi}_\lambda$ is a hybrid commutative ideals of L over U, by Theorem 13, we get that $\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$ and $\lambda(((x \to y) \to y) \to ((y \to x) \to x)) = \lambda(1)$. Hence, $[(x \to y) \to y] \leq [(y \to x) \to x]$ and so $([x] \rightrightarrows [y]) \rightrightarrows [y] \leq ([y] \rightrightarrows [x]) \rightrightarrows [x]$ and by similar way we have $([y] \rightrightarrows [x]) \rightrightarrows [x] \leq ([x] \rightrightarrows [y]) \rightrightarrows [y]$ and so

$$([x] \rightrightarrows [y]) \rightrightarrows [y] = ([y] \rightrightarrows [x]) \rightrightarrows [x].$$

Thus, $(\frac{L}{\tilde{\varphi}_\lambda}, \rightrightarrows, [1])$ is a commutative BCK-algebra.
Conversely, if $(\frac{L}{\tilde{\varphi}_\lambda}, \rightrightarrows, [1])$ is a commutative BCK-algebra, then for any $[x], [y], [z] \in \frac{L}{\tilde{\varphi}_\lambda}$, $([x] \rightrightarrows [y]) \rightrightarrows [y] = ([y] \rightrightarrows [x]) \rightrightarrows [x]$ and so $[(x \to y) \to y] = [(y \to x) \to x]$. Hence, $(x \to y) \to y \equiv_{\tilde{\varphi}_\lambda} (y \to x) \to x$ and so

$$\tilde{\varphi}(((x \to y) \to y) \to ((y \to x) \to x)) = \tilde{\varphi}(1)$$

$$\lambda(((x \to y) \to y) \to ((y \to x) \to x)) = \lambda(1).$$

Therefore, by Theorem 13, $\tilde{\varphi}_\lambda$ is a hybrid commutative ideals of L over U. □

Theorem 17. *Let $(L; \to, 1)$ be a CKL-algebra and let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideals of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.*

Proof. Let $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideals of L over U and $x, y \in L$. Then by Proposition 3(ii) and $x \leq ((x \to y) \to y) \to x$, we have $(((x \to y) \to y) \to x) \to y \leq x \to y$ and $(x \to y) \to x \leq ((((x \to y) \to y) \to x) \to y) \to x$. Hence,

$$((x \to y) \to y) \to ((x \to y) \to x) \leq ((x \to y) \to y) \to (((((x \to y) \to y) \to x) \to y) \to x)$$

And since by condition (C): $((x \to y) \to y) \to (((((x \to y) \to y) \to x) \to y) \to x) = ((((x \to y) \to y) \to x) \to y) \to ((((x \to y) \to y) \to x)$.
We conclude that $((x \to y) \to y) \to ((x \to y) \to x) \leq (((((x \to y) \to y) \to x) \to y)) \to ((((x \to y) \to y) \to x)$. Now, by condition (C) and Proposition 3(iii), we have $(y \to x) \to (((x \to y) \to y) \to ((x \to y) \to x)) = ((x \to y) \to y) \to ((y \to x) \to ((x \to y) \to x)) = 1$.
Hence, $(y \to x) \leq (((x \to y) \to y) \to ((x \to y) \to x))$ and so

$$y \to x \leq (((((x \to y) \to y) \to x) \to y)) \to ((((x \to y) \to y) \to x)$$

And by Proposition 4(i),

$$\tilde{\varphi}(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \supseteq \tilde{\varphi}(y \to x)$$

$$\lambda(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)) \leq \lambda(y \to x).$$

Now, since $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideals of L over U, by Proposition 5, we get that

$$\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x))$$

$$\lambda(((x \to y) \to y) \to x) \leq \lambda(((((x \to y) \to y) \to x) \to y) \to (((x \to y) \to y) \to x)).$$

Hence,

$$\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(y \to x)$$
$$\lambda(((x \to y) \to y) \to x) \leq \lambda(y \to x).$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U.

□

Theorem 18. *Let $(L; \to, 1)$ be a bounded CKL-algebra with double negation and $\tilde{\varphi}_\lambda$ be a hybrid ideal of L over U. Then $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U if and only if $\tilde{\varphi}_\lambda$ is a hybrid implicative and hybrid commutative ideal of L over U.*

Proof. $\tilde{\varphi}_\lambda$ be a hybrid positive implicative ideal of L over U. Then by Theorems 10 and 17, $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U. Conversely, lmet $\tilde{\varphi}_\lambda$ be a hybrid implicative and hybrid commutative and of L over U and $x, y \in L$. Then by Proposition 3(iii), $(x \to y) \to x \leq (x \to y) \to ((x \to y) \to y)$ and so by Proposition 4(i),

$$\tilde{\varphi}((x \to y) \to ((x \to y) \to y)) \supseteq \tilde{\varphi}((x \to y) \to x)$$

$$\lambda((x \to y) \to ((x \to y) \to y)) \leq \lambda((x \to y) \to x).$$

And since $\tilde{\varphi}_\lambda$ is a hybrid implicative ideal of L over U, by Theorem 8, we conclude that

$$\tilde{\varphi}((x \to y) \to y) \supseteq \tilde{\varphi}((x \to y) \to ((x \to y) \to y))$$

$$\lambda((x \to y) \to y) \leq \lambda((x \to y) \to ((x \to y) \to y))$$

And since $\tilde{\varphi}_\lambda$ is a hybrid commutative ideal of L over U, we conclude that

$$\tilde{\varphi}(((x \to y) \to y) \to x) \supseteq \tilde{\varphi}(y \to x)$$

$$\lambda(((x \to y) \to y) \to x) \leq \lambda(y \to x)$$

Moreover, since $y \leq x \to y$, by Proposition 3(ii), we have $(x \to y) \to x \leq y \to x$ and so by Proposition 4(i), $\tilde{\varphi}(y \to x) \supseteq \tilde{\varphi}((x \to y) \to x)$ and $\lambda(y \to x) \leq \lambda((x \to y) \to x)$. Hence, by $(HI1)$, we conclude that

$$\begin{aligned}
\tilde{\varphi}(x) &\supseteq \tilde{\varphi}(((x \to y) \to y) \to x) \cap \tilde{\varphi}((x \to y) \to y) \\
&\supseteq \tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to y) \\
&\supseteq \tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to ((x \to y) \to y)) \\
&\supseteq \tilde{\varphi}(y \to x) \cap \tilde{\varphi}((x \to y) \to x) \\
&= \tilde{\varphi}((x \to y) \to x)
\end{aligned}$$

$$\begin{aligned}
\lambda(x) &\leq \lambda(((x \to y) \to y) \to x) \vee \lambda((x \to y) \to y) \\
&\leq \lambda(y \to x) \vee \lambda((x \to y) \to y)
\end{aligned}$$

$$\begin{aligned} &\leq \lambda(y \to x) \vee \lambda((x \to y) \to ((x \to y) \to y)) \\ &\leq \lambda(y \to x) \vee \lambda((x \to y) \to x) \\ &= \lambda((x \to y) \to x). \end{aligned}$$

Therefore, $\tilde{\varphi}_\lambda$ is a hybrid positive implicative ideal of L over U.

□

In the following diagrams, we summarize the relations between different types of hybrid ideals in a bounded CKL-algebra with double negation.

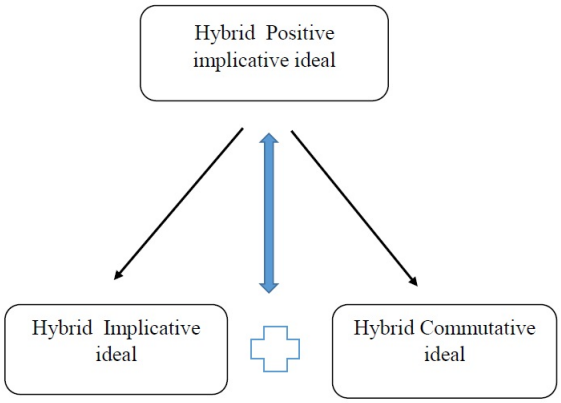

Figure1. Diagram of Hybrid ideals in CKL-algebra with double negation

6 Conclusion

The results of this paper focus on the study of hybrid positive implicative ideals, hybrid implicative ideals, and hybrid commutative ideals in L-algebras. By defining these notions, we establish various characterizations and investigate their related properties. Furthermore, we examine the relationship between hybrid implicative ideals and hybrid positive implicative ideals. It is also proved that the quotient CKL-algebra with respect to a hybrid ideal forms a commutative BCK-algebra if and only if the hybrid ideal is a hybrid commutative ideal. Several questions remain open for further research:
How can the notion of hybrid obstinate ideals in L-algebras be defined?
What is the relationship between hybrid obstinate ideals and other types of hybrid ideals in L-algebras?
These could be topics of further research.

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