Journal of Algebra and Related Topics
Vol. , No. -, 2027, pp. -. Research Article

A problem on partitions of a residue class ring with the same representation

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Abstract. Let $\mathbb{Z}_m$ denote the ring of residue classes modulo m. For $A \subseteq \mathbb{Z}_m$, the representation $R_A(\bar{n})$ counts the number of solutions of equation $a + b \equiv n \pmod{m}$ with $\bar{a}, \bar{b} \in A$. This paper investigates the conditions under which $A, B \subseteq \mathbb{Z}_m$ satisfy $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, under the constraint that $|(A \cup B) \setminus (A \cap B)| = m - 4$. We examine four structural cases based on the cardinality of the intersection $A \cap B$, building upon earlier results for the case where $|A \cap B| = 4$. We derive necessary structural conditions and establish nonexistence results for the remaining cases. The findings extend the classification of representation-equivalent subsets and contribute to ongoing studies of Sárközy's problem in modular arithmetic.

Keywords: Sárközy's problem, Partition, Representation function, Set of residue classes modulo m.
AMS Subject Classification 2020: 11B13.

1 Introduction and preliminary

Let X be an abelian semigroup under addition. For every subset $A \subseteq X$ and $n \in X$, the representation function $R_A(n)$ is defined as the number of ordered pairs $(a, b) \in A \times A$ satisfying $a + b = n$. Nathanson [1] introduced the concept of the representation function, which has since attracted significant interest among mathematicians. In 1985, Sárközy and Erdős [2] posed the question:

Are there subsets $A, B \subseteq \mathbb{N}$ with $|(A \cup B) \setminus (A \cap B)| = \infty$ such that $R_A(n) = R_B(n)$ for sufficiently large integers n?

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Received: 12 June 2025/ Revised: 12 August 2026/ Accepted: 22 August 2026
DOI: 10.22124/JART.2026.30956.1817

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Dombi [3] gave an affirmative answer by constructing sets with identical representation functions. These findings have led many researchers to extend the study of Sárközy's problem to modular arithmetic, specifically within the set of residue classes modulo m, denoted as $\mathbb{Z}_m$.

For a positive integer m, let $\mathbb{Z}_m$ denote the set of residue classes modulo m, defined as

$$\mathbb{Z}_m = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{m-1}\},$$

where $\bar{x} = \{a \in \mathbb{Z} : a \equiv x \pmod{m}\}$. We define addition on $\mathbb{Z}_m$ as follows:

$$\bar{a} \oplus \bar{b} = \overline{a+b}.$$

For $\bar{x} \in \mathbb{Z}_m$ and $A \subseteq \mathbb{Z}_m$, define

$$A \oplus \bar{x} = \{\bar{a} \oplus \bar{x} : \bar{a} \in A\}.$$

In 2012, Yang and Chen [4] explored Sárközy's problem in $\mathbb{Z}_m$ by studying the structure of two sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.

Theorem 1 ([4]). *Let $A, B \subseteq \mathbb{Z}_m$ be such that $B = \mathbb{Z}_m \setminus A$. The equality $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if m is an even integer and $|A| = \frac{m}{2}$.*

In 2017, Yang and Tang [5] considered the case of $|(A \cup B) \setminus (A \cap B)| = m - 1$ and proved the following result.

Theorem 2 ([5]). *Let $m \geq 3$ be an odd integer. Then, there do not exist sets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m - 1$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

In 2021, Chen and Yan [6] determined the necessary and sufficient conditions for the structure of subsets $A, B \subseteq \mathbb{Z}_m$ that satisfy $|(A \cup B) \setminus (A \cap B)| = m - 2$ and $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ where m is even. They further analyzed the three cases to consider:

Theorem 3 ([6]). *Let m be an even integer and let A, B be sets with $A \cup B = \mathbb{Z}_m$, $|A \cap B| = 2$. Then, $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if $B = A \oplus \overline{\frac{m}{2}}$.*

Theorem 4 ([6]). *Let m be an even integer and let A, B be sets with $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\}$ and $|A \cap B| = 1$ where $\bar{r}_1 \in \mathbb{Z}_m$.*

1. *If $m \equiv 0 \pmod 4$, then there do not exist two sets $A, B \subseteq \mathbb{Z}_m$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*
2. *If $m \equiv 2 \pmod 4$, then $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if there exist two sets C, D with $|C| = |D| = (m/2 - 1)/2$, $C \cup D = [\bar{0}, \overline{m/2 - 1}] \setminus \{\bar{r}_1, \bar{r}_1 \oplus \overline{\frac{m}{2}}\}$ and $C \cap D = \emptyset$ such that*

$$A = C \cup \left(C \oplus \overline{\frac{m}{2}}\right) \cup \left\{\bar{r}_1 \oplus \overline{\frac{m}{2}}\right\} \text{ and } B = D \cup \left(D \oplus \overline{\frac{m}{2}}\right) \cup \left\{\bar{r}_1 \oplus \overline{\frac{m}{2}}\right\}.$$

Theorem 5 ([6]). *Let m be an even integer and let $\bar{r}_1, \bar{r}_2 \in \mathbb{Z}_m$ be distinct. Let A, B satisfy $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2\}$, $|A \cap B| = 0$. Then, $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if $B = A \oplus \overline{\frac{m}{2}}$.*

In 2024, Tokrasae and Sawatraksa [7] considered the case of $|(A \cup B) \setminus (A \cap B)| = m-3$ and proved the following result.

Theorem 6 ([7]). *Let $m \geq 5$ be an odd integer. Then, there do not exist two subsets $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m-3$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

In this paper we determine the structure of $A, B \subseteq \mathbb{Z}_m$ with $|(A \cup B) \setminus (A \cap B)| = m-4$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B|$, and so $|(A \cup B) \setminus (A \cap B)| = m-4$ is even. Hence, we always assume that m is an even integer greater than 4. If $|(A \cup B) \setminus (A \cap B)| = m-4$, then there are five cases to consider:

1. $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$.
2. $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\}$, where $\bar{r}_1 \in \mathbb{Z}_m$ and $|A \cap B| = 3$.
3. $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2\}$, where $\bar{r}_1, \bar{r}_2 \in \mathbb{Z}_m$ and $|A \cap B| = 2$.
4. $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$, where $\bar{r}_1, \bar{r}_2, \bar{r}_3 \in \mathbb{Z}_m$ and $|A \cap B| = 1$.
5. $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4\}$, where $\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4 \in \mathbb{Z}_m$ and $|A \cap B| = 0$.

In the first case, Sun and Xiong [8] prove that if m is a positive integer with $4 \mid m$, then there exist two distinct sets $A, B \subseteq \mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or $m-4$, $B \neq A \oplus \overline{\frac{m}{2}}$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. If m is a positive integer with $m \equiv 2 \pmod 4$, $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or $m-4$, then $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if $B = A \oplus \overline{\frac{m}{2}}$, as follows:

Theorem 7 ([8]). *Let m be a positive integer with $m = 4k$ for some $k \in \mathbb{Z}$ and let*

$$A = \{\bar{0}, \bar{1}, \bar{2}, \ldots, \bar{k}\} \cup \{\overline{2k}, \overline{2k+1}, \overline{2k+2}, \ldots, \overline{3k}\} \text{ and}$$
$$B = \{\bar{k}, \overline{k+1}, \overline{k+2}, \ldots, \overline{2k}\} \cup \{\overline{3k}, \overline{3k+1}, \overline{3k+2}, \ldots, \overline{4k}\}.$$

Then, the following statements hold:

1. *$A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$.*
2. *$B \neq A \oplus \overline{\frac{m}{2}}$.*
3. *$R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

Theorem 8 ([8]). *Let m be a positive integer with $m \equiv 2 \pmod 4$. Let $A \cup B = \mathbb{Z}_m$ and $|A \cap B| = 4$ or $m-4$. Then, $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ if and only if $B = A \oplus \overline{\frac{m}{2}}$.*

This paper investigates the remaining four cases, beginning with the following fundamental concepts, which are introduced in Section 2.

Theorem 9. *Let $A, B \subseteq \mathbb{Z}_m$ be such that $|(A \cup B) \setminus (A \cap B)| = m-4$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B|$ and m is even.*

Proof. Assume that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then,

$$|A|^2 = \sum_{n=0}^{m-1} R_A(\bar{n}) = \sum_{n=0}^{m-1} R_B(\bar{n}) = |B|^2.$$

Thus, $|A| = |B|$. Since $|(A \cup B) \setminus (A \cap B)| = m - 4$, we have

$$m - 4 = |(A \cup B) \setminus (A \cap B)| = |A| + |B| - 2\,|A \cap B| = 2(|A| - |A \cap B|).$$

Therefore, $m - 4$ is even, and hence, m is also. □

Theorem 10. *Let m be even and let $A, B \subseteq \mathbb{Z}_m$. If $A = B \oplus \overline{\frac{m}{2}}$, then $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

Proof. Assume that $A = B \oplus \overline{\frac{m}{2}}$. Let $\bar{n} \in \mathbb{Z}_m$. Then,

$$\begin{aligned}
R_B(\bar{n}) &= |\{(\bar{x}, \bar{y}) \in B \times B : \bar{x} \oplus \bar{y} = \bar{n}\}| \\
&= \left|\left\{\left(\bar{a} \oplus \overline{\frac{m}{2}}, \bar{b} \oplus \overline{\frac{m}{2}}\right) : \bar{a}, \bar{b} \in A, \bar{a} \oplus \overline{\frac{m}{2}} \oplus \bar{b} \oplus \overline{\frac{m}{2}} = \bar{n}\right\}\right| \\
&= \left|\left\{\left(\bar{a}, \bar{b}\right) : \bar{a}, \bar{b} \in A, \bar{a} \oplus \bar{b} \oplus \bar{m} = \bar{n}\right\}\right| \\
&= \left|\left\{\left(\bar{a}, \bar{b}\right) : \bar{a}, \bar{b} \in A, \bar{a} \oplus \bar{b} = \bar{n}\right\}\right| \\
&= R_A(\bar{n}).
\end{aligned}$$

This completes the proof of Theorem 10. □

The characteristic function of $A \subseteq \mathbb{Z}_m$ is defined by

$$\chi_A(t) = \begin{cases} 0 & \text{if } \bar{t} \notin A, \\ 1 & \text{if } \bar{t} \in A. \end{cases}$$

Hence, $\chi_{\mathbb{Z}_m \setminus A}(t) = 1 - \chi_A(t)$ for all $t \in \mathbb{Z}$.

Lemma 1. *Let $A \subseteq \mathbb{Z}_m$. Then, $R_A(\bar{n}) = \sum_{i=0}^{m-1} \chi_A(i)\chi_A(n-i)$ for all $\bar{n} \in \mathbb{Z}_m$.*

Proof. Let $\bar{n} \in \mathbb{Z}_m$. Then,

$$\begin{aligned}
R_A(\bar{n}) &= \left|\left\{\left(\bar{a}, \bar{b}\right) \in A \times A : \bar{a} \oplus \bar{b} = \bar{n}\right\}\right| \\
&= |\{(\bar{a}, \overline{n-a}) : \bar{a}, \overline{n-a} \in A\}| \\
&= \sum_{i=0}^{m-1} \chi_A(i)\chi_A(n-i).
\end{aligned}$$

This completes the proof. □

For subsets $A, B \subseteq \mathbb{Z}_m$ and $\bar{n} \in \mathbb{Z}_m$, we define the mixed representation function $R_{A,B}$ by

$$R_{A,B}(\bar{n}) = |\{(a, b) \in A \times B : \bar{a} \oplus \bar{b} = \bar{n}\}|.$$

Then $R_{A,B}(\bar{n}) = R_{B,A}(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.

Lemma 2. *Let $A, B \subseteq \mathbb{Z}_m$ and $A \cap B = \emptyset$. Then,*

$$R_{A\cup B}(\bar{n}) = R_A(\bar{n}) + 2R_{A,B}(\bar{n}) + R_B(\bar{n}) \quad \textit{for all } \bar{n} \in \mathbb{Z}_m.$$

Proof. Let $\bar{n} \in \mathbb{Z}_m$. Then,

$$\begin{aligned}
R_{A\cup B}(\bar{n}) &= \left|\left\{(\bar{a},\bar{b}) \in (A\cup B)\times(A\cup B) : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&= \left|\left\{(\bar{a},\bar{b}) : \bar{a} \in A\cup B, \bar{b} \in A\cup B, \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&= \left|\left\{(\bar{a},\bar{b}) \in A\times A : \bar{a}\oplus\bar{b} = \bar{n}\right\} \cup \left\{(\bar{a},\bar{b}) \in B\times B : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right. \\
&\qquad \left. \cup \left\{(\bar{a},\bar{b}) \in A\times B : \bar{a}\oplus\bar{b} = \bar{n}\right\} \cup \left\{(\bar{a},\bar{b}) \in B\times A : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&= \left|\left\{(\bar{a},\bar{b}) \in A\times A : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| + \left|\left\{(\bar{a},\bar{b}) \in B\times B : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&\qquad + \left|\left\{(\bar{a},\bar{b}) \in A\times B : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| + \left|\left\{(\bar{a},\bar{b}) \in B\times A : \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&= R_A(\bar{n}) + R_{A,B}(\bar{n}) + R_{B,A}(\bar{n}) + R_B(\bar{n}) \\
&= R_A(\bar{n}) + 2R_{A,B}(\bar{n}) + R_B(\bar{n}),
\end{aligned}$$

which completes the proof. □

Lemma 3. *Let $A \subseteq \mathbb{Z}_m$ and $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq \mathbb{Z}_m$. Then,*

$$R_{A,\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_k\}}(\bar{n}) = \sum_{i=1}^{k} \chi_A(n - x_i) \quad \textit{for all } \bar{n} \in \mathbb{Z}_m.$$

Proof. Let $\bar{n} \in \mathbb{Z}_m$. Then,

$$\begin{aligned}
R_{A,\{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_k\}}(\bar{n}) &= \left|\left\{(\bar{a},\bar{b}) : \bar{a} \in A, \bar{b} \in \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_k\}, \bar{a}\oplus\bar{b} = \bar{n}\right\}\right| \\
&= \left|\left\{(\bar{n}-\bar{b},\bar{b}) : \bar{n}-\bar{b} \in A, \bar{b} \in \{\bar{x}_1,\bar{x}_2,\ldots,\bar{x}_k\}\right\}\right| \\
&= \left|\{(\bar{n}-\bar{x}_1,\bar{x}_1),(\bar{n}-\bar{x}_2,\bar{x}_2),\ldots,(\bar{n}-\bar{x}_k,\bar{x}_k) : \bar{n}-\bar{x}_1,\ldots,\bar{n}-\bar{x}_k \in A\}\right| \\
&= \chi_A(n-x_1) + \chi_A(n-x_2) + \cdots + \chi_A(n-x_k),
\end{aligned}$$

as asserted. □

For a property P, we define $\theta(P) = 1$ if P is true and $\theta(P) = 0$ otherwise.

Lemma 4. *Let $A = \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq \mathbb{Z}_m$. Then, for every $\bar{n} \in \mathbb{Z}_m$,*

$$R_A(\bar{n}) = \sum_{i=1}^{k} \theta(\bar{n} = 2\bar{x}_i) + 2 \sum_{1\le i<j\le k} \theta(\bar{n} = \bar{x}_i \oplus \bar{x}_j).$$

Proof. Every ordered pair in $A \times A$ is either of the form $(\bar{x}_i, \bar{x}_i)$, which contributes $\theta(\bar{n} = 2\bar{x}_i)$, or is associated with two distinct elements $\bar{x}_i, \bar{x}_j$, $i < j$. In the latter case, both ordered pairs $(\bar{x}_i, \bar{x}_j)$ and $(\bar{x}_j, \bar{x}_i)$ contribute whenever

$$\bar{n} = \bar{x}_i \oplus \bar{x}_j.$$

Hence,

$$R_A(\bar{n}) = \sum_{i=1}^{k} \theta(\bar{n} = 2\bar{x}_i) + 2 \sum_{1 \le i < j \le k} \theta(\bar{n} = \bar{x}_i \oplus \bar{x}_j).$$

This completes the proof. □

Lemma 5. ([6]) *Let $A \subseteq \mathbb{Z}_m$. Then, $R_{\mathbb{Z}_m \setminus A}(\bar{n}) = m - 2|A| + R_A(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

2 Main results

In this section, we present the main results concerning the existence of subsets $A, B \subseteq \mathbb{Z}_m$ whose representation functions are identical, that is,

$$R_A(\bar{n}) = R_B(\bar{n}) \text{ for all } \bar{n} \in \mathbb{Z}_m,$$

under the constraint that

$$|(A \cup B) \setminus (A \cap B)| = m - 4.$$

Building on prior work in which the case $|A \cap B| = 4$ was thoroughly analyzed, we now consider the remaining four structural cases corresponding to $|A \cap B| = 3, 2, 1$ and 0. For each case, we examine whether there exist subsets A and B satisfying the above condition and provide explicit constructions or proofs of nonexistence where appropriate. These results contribute to the broader understanding of representation-equivalent subsets in modular arithmetic.

We begin our investigation with Case 1, corresponding to the situation where

$$|A \cup B| = m - 1 \text{ and } |A \cap B| = 3.$$

Under this configuration, we examine the conditions under which subsets $A, B \subseteq \mathbb{Z}_m$ satisfy $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. The following theorem provides a characterization of such subsets.

Theorem 11. *Let $A, B \subseteq \mathbb{Z}_m$ and $\bar{r}_1 \in \mathbb{Z}_m$ be such that $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\}$ and $|A \cap B| = 3$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B| = \frac{m}{2} + 1$.*

Proof. Assume that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then, by Theorem 9, we have $|A| = |B|$. Since

$$A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\} \Rightarrow |A \cup B| = |\mathbb{Z}_m \setminus \{\bar{r}_1\}| = m - 1,$$

we get that

$$\begin{aligned} 2|A| &= |A| + |A| = |A| + |B| \\ &= |A \cup B| + |A \cap B| = m - 1 + 3 = m + 2. \end{aligned}$$

Hence, $|A| = |B| = \frac{m}{2} + 1$. □

Theorem 12. *Let $\bar{r}_1 \in \mathbb{Z}_m$. Then, there do not exist two subsets A and B of $\mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\}$ and $|A \cap B| = 3$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

Proof. Suppose that A and B are both subsets of $\mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1\}$ and $|A \cap B| = 3$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Let $\bar{r}_2, \bar{r}_3, \bar{r}_4 \in A \cap B$ be distinct. Then,

$$B = (\mathbb{Z}_m \setminus (A \cup \{\bar{r_1}\})\}) \cup \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}.$$

Let $\bar{n} \in \mathbb{Z}_m$. Then, by Lemma 2, we have

$$\begin{aligned} R_B(\bar{n}) &= R_{(\mathbb{Z}_m \setminus (A \cup \{\bar{r_1}\})\}) \cup \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}}(\bar{n}) \\ &= R_{\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})}(\bar{n}) + 2R_{[\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})], \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}}(\bar{n}) + R_{\{\bar{r}_2, \bar{r}_3, \bar{r}_4\}}(\bar{n}). \end{aligned}$$

By Lemma 4, we obtain that

$$\begin{aligned} R_B(\bar{n}) = R_{\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})}(\bar{n}) + 2R_{[\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})], \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}}(\bar{n}) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) \\ + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i). \end{aligned}$$

It follws from Lemma 3 that

$$\begin{aligned} R_B(\bar{n}) &= R_{\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})}(\bar{n}) + 2\left[\sum_{i=2}^{4} \chi_{\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})}(n - r_i)\right] + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) \\ &\quad + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i) \\ &= R_{\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1\})}(\bar{n}) + 2\sum_{i=2}^{4} \left(1 - \chi_{A \cup \{\bar{r}_1\}}(n - r_i)\right) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) \\ &\quad + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i). \end{aligned}$$

By Lemma 5, we get that

$$\begin{aligned} R_B(\bar{n}) &= \left[m - 2|A \cup \{\bar{r}_1\}| + R_{A \cup \{\bar{r}_1\}}(\bar{n})\right] + 2\sum_{i=2}^{4} \left(1 - \chi_{A \cup \{\bar{r}_1\}}(n - r_i)\right) \\ &\quad + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i) \\ &= m - 2|A \cup \{\bar{r}_1\}| + R_{A \cup \{\bar{r}_1\}}(\bar{n}) + 6 - 2\sum_{i=2}^{4} \chi_{A \cup \{\bar{r}_1\}}(n - r_i) \\ &\quad + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i). \end{aligned}$$

This follows Lemma 2, we have

$$R_B(\bar{n}) = m - 2|A \cup \{\bar{r}_1\}| + \left[R_A(\bar{n}) + 2R_{A,\{\bar{r}_1\}}(\bar{n}) + R_{\{\bar{r}_1\}}(\bar{n})\right] + 6 - 2\sum_{i=2}^{4} \chi_{A \cup \{\bar{r}_1\}}(n - r_i)$$

$$+ 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i).$$

Using Lemma 3 again,

$$R_B(\bar{n}) = m - 2|A \cup \{\bar{r}_1\}| + R_A(\bar{n}) + 2\chi_A(n - r_1) + R_{\{\bar{r}_1\}}(\bar{n}) + 6 - 2\sum_{i=2}^{4} \chi_{A\cup\{\bar{r}_1\}}(n - r_i)$$
$$+ 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i).$$

Applying Lemma 4 again,

$$\begin{aligned} R_B(\bar{n}) = {} & m - 2|A \cup \{\bar{r}_1\}| + R_A(\bar{n}) + 2\chi_A(n - r_1) + [\theta(\bar{n} = 2\bar{r}_1)] + 6 \\ & - 2\chi_{A\cup\{\bar{r}_1\}}(n - r_2) - 2\chi_{A\cup\{\bar{r}_1\}}(n - r_3) - 2\chi_{A\cup\{\bar{r}_1\}}(n - r_4) \\ & + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i). \end{aligned}$$

Therefore,

$$\begin{aligned} R_B(\bar{n}) = {} & m - 2|A \cup \{\bar{r}_1\}| + R_A(\bar{n}) + 2\chi_A(n - r_1) + [\theta(\bar{n} = 2\bar{r}_1)] + 6 \\ & - 2\left(\chi_A(n - r_2) + \chi_{\{\bar{r}_1\}}(n - r_2)\right) - 2\left(\chi_A(n - r_3) + \chi_{\{\bar{r}_1\}}(n - r_3)\right) \\ & - 2\left(\chi_A(n - r_4) + \chi_{\{\bar{r}_1\}}(n - r_4)\right) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) \\ & + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i) \\ = {} & m - 2|A \cup \{\bar{r}_1\}| + R_A(\bar{n}) + 2\chi_A(n - r_1) + \theta(\bar{n} = 2\bar{r}_1) + 6 \\ & - 2\chi_A(n - r_2) - 2\chi_A(n - r_3) - 2\chi_A(n - r_4) \\ & - 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_2) - 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_3) - 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_4) \\ & + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=2}^{4} \theta(\bar{n} = 2\bar{r}_i). \end{aligned}$$

Since $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$ and by Theorem 11, we have $|A| = |B| = \frac{m}{2} + 1$, and so

$$0 = m - 2\left(\frac{m}{2} + 2\right) + 2\chi_A(n - r_1) - 2\sum_{i=2}^{4} \chi_A(n - r_i) + 6 - 2\sum_{i=2}^{4} \theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_i)$$
$$+ 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=1}^{4} \theta(\bar{n} = 2\bar{r}_i)$$
$$= 2 + 2\chi_A(n - r_1) - 2\sum_{i=2}^{4} \chi_A(n - r_i) - 2\sum_{i=2}^{4} \theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_i) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3)$$

$$+ 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=1}^{4} \theta(\bar{n} = 2\bar{r}_i).$$

Hence,

$$2\sum_{i=2}^{4} \chi_A(n - r_i) + 2\sum_{i=2}^{4} \theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_i) = 2 + 2\chi_A(n - r_1) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=1}^{4} \theta(\bar{n} = 2\bar{r}_i) \quad (1)$$

for all $\bar{n} \in \mathbb{Z}_m$.

Next, we will show that $2\bar{r}_1 = 2\bar{r}$ for some $\bar{r} \in \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}$. Suppose that $2\bar{r}_1 \neq 2\bar{r}$ for all $\bar{r} \in \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}$. Taking $\bar{n} = 2\bar{r}_1$ in the equation (1), we have

$$2\sum_{i=2}^{4} \chi_A(2r_1 - r_i) + 2\sum_{i=2}^{4} \theta(2\bar{r}_1 = \bar{r}_1 \oplus \bar{r}_i) = 2 + 2\chi_A(r_1) + 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(2\bar{r}_1 = \bar{r}_3 \oplus \bar{r}_4) + \sum_{i=1}^{4} \theta(2\bar{r}_1 = 2\bar{r}_i).$$

This implies that

$$\begin{aligned} 2\sum_{i=2}^{4} \chi_A(2r_1 - r_i) &= 2 + 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_3) + 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(2\bar{r}_1 = \bar{r}_3 \oplus \bar{r}_4) + 1 \\ &= 2\left[1 + \theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_3) + \theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_4) + \theta(2\bar{r}_1 = \bar{r}_3 \oplus \bar{r}_4)\right] + 1, \end{aligned}$$

which is a contradiction. Hence, $2\bar{r}_1 = 2\bar{r}$ for some $\bar{r} \in \{\bar{r}_2, \bar{r}_3, \bar{r}_4\}$. Without loss of generality, suppose that $2\bar{r}_1 = 2\bar{r}_2$. Then,

$$\bar{r}_2 = \bar{r}_1 \oplus \overline{\frac{m}{2}}.$$

Now, to show that $2\bar{r}_3 = 2\bar{r}_4$, assume that $2\bar{r}_3 \neq 2\bar{r}_4$. Set $\bar{n} = 2\bar{r}_3$, we obtain that

$$\begin{aligned} 2\sum_{i=2}^{4} \chi_A(2r_3 - r_i) + 2\sum_{i=2}^{4} \theta(2\bar{r}_3 = \bar{r}_1 \oplus \bar{r}_i) &= 2 + 2\chi_A(2r_3 - r_1) + 2\theta(2\bar{r}_3 = \bar{r}_3 \oplus \bar{r}_4) \\ &\quad + 2\theta(2\bar{r}_3 = \bar{r}_2 \oplus \bar{r}_1) + 1 \\ &= 2\left[1 + \chi_A(n - r_1) + 2\theta(2\bar{r}_3 = \bar{r}_2 \oplus \bar{r}_1) + \theta(2\bar{r}_3 = \bar{r}_2 \oplus \bar{r}_4)\right] + 1, \end{aligned}$$

which is a contradiction. Hence, $2\bar{r}_3 = 2\bar{r}_4$, and so $\bar{r}_3 = \bar{r}_4 \oplus \overline{\frac{m}{2}}$.

Put $h = \frac{m}{2}$. We have already shown that, after relabeling if necessary,

$$\bar{r}_2 = \bar{r}_1 \oplus \bar{h} \quad \text{and} \quad \bar{r}_3 = \bar{r}_4 \oplus \bar{h}.$$

Substituting these relations into (1), we obtain

$$\begin{aligned}\chi_A(n-r_1-h)+\chi_A(n-r_4-h)+\chi_A(n-r_4)+\theta(\bar{n}=2\bar{r}_1\oplus\bar{h})\\ =1+\chi_A(n-r_1)+\theta(\bar{n}=2\bar{r}_4\oplus\bar{h})+\theta(\bar{n}=2\bar{r}_1)+\theta(\bar{n}=2\bar{r}_4)\end{aligned} \tag{2}$$

for every $\bar{n}\in\mathbb{Z}_m$.

Set $k=n-r_1-h$ and $l=n-r_4-h$. Then

$$\bar{l}-\bar{k}=\bar{r}_1-\bar{r}_4. \tag{3}$$

We divide the remainder of the proof into the following two cases.

Case 1. Suppose that $\bar{l}=\bar{k}$. By (3), we obtain $\bar{r}_1-\bar{r}_4=\bar{0}$, and hence $\bar{r}_1=\bar{r}_4$, which contradicts the fact that $\bar{r}_1,\bar{r}_2,\bar{r}_3,\bar{r}_4$ are distinct. Therefore, this case cannot occur.

Case 2. Suppose that $\bar{l}\neq\bar{k}$. In this case, by (3),

$$\bar{l}=\bar{k}\oplus(\bar{r}_1-\bar{r}_4),$$

where $\bar{r}_1-\bar{r}_4\neq\bar{0}$. We now compare equation (2) at $\bar{n}$ and at $\bar{n}\oplus\bar{h}$. Since $2\bar{h}=\bar{0}$, replacing $\bar{n}$ by $\bar{n}\oplus\bar{h}$ in (2) gives

$$\begin{aligned}\chi_A(n-r_1)+\chi_A(n-r_4)+\chi_A(n-r_4-h)+\theta(\bar{n}=2\bar{r}_1)\\ =1+\chi_A(n-r_1-h)+\theta(\bar{n}=2\bar{r}_4)+\theta(\bar{n}=2\bar{r}_1\oplus\bar{h})+\theta(\bar{n}=2\bar{r}_4\oplus\bar{h}).\end{aligned} \tag{4}$$

Subtracting (4) from (2), we obtain

$$2\big(\chi_A(n-r_1-h)-\chi_A(n-r_1)\big)=2\big(\theta(\bar{n}=2\bar{r}_1)-\theta(\bar{n}=2\bar{r}_1\oplus\bar{h})\big).$$

Hence,

$$\chi_A(n-r_1-h)-\chi_A(n-r_1)=\theta(\bar{n}=2\bar{r}_1)-\theta(\bar{n}=2\bar{r}_1\oplus\bar{h}).$$

Thus, whenever $\bar{n}\notin\{2\bar{r}_1,2\bar{r}_1\oplus\bar{h}\}$, we have $\chi_A(n-r_1-h)=\chi_A(n-r_1)$. Equivalently,

$$\chi_A(x+h)=\chi_A(x) \tag{5}$$

for every $\bar{x}\notin\{\bar{r}_1,\bar{r}_1\oplus\bar{h}\}$. Since $\bar{r}_1\notin A$ and $\bar{r}_1\oplus\bar{h}=\bar{r}_2\in A\cap B$, all elements of $A\setminus\{\bar{r}_2\}$ occur in pairs of the form $\{\bar{x},\bar{x}\oplus\bar{h}\}$. Therefore, $|A|-1$ is even. By Theorem 11,

$$|A|=\frac{m}{2}+1,$$

so $m/2$ is even. Hence $m\equiv 0 \pmod 4$. Since $m>4$, we have $m\geq 8$. Consider

$$S=\left\{2\bar{r}_1,\,2\bar{r}_1\oplus\bar{h},\,2\bar{r}_4,\,2\bar{r}_4\oplus\bar{h},\,\bar{r}_1\oplus\bar{r}_4,\,\bar{r}_1\oplus\bar{r}_4\oplus\bar{h}\right\}.$$

Since $|S|\leq 6<m$, there exists $\bar{n}\in\mathbb{Z}_m\setminus S$. For such $\bar{n}$, all the theta terms in (2) vanish. Moreover, by (5),

$$\chi_A(n-r_1-h)=\chi_A(n-r_1)$$

and

$$\chi_A(n - r_4 - h) = \chi_A(n - r_4).$$

Therefore, (2) reduces to

$$\chi_A(n - r_1) + 2\chi_A(n - r_4) = 1 + \chi_A(n - r_1).$$

Hence,

$$2\chi_A(n - r_4) = 1,$$

which is impossible since $\chi_A(n - r_4) \in \{0, 1\}$. This contradiction shows that Case 2 is also impossible. Therefore, no such subsets A and B exist. This completes the proof. □

We now proceed to Case 2, which corresponds to the situation where

$$|A \cup B| = m - 2 \text{ and } |A \cap B| = 2.$$

In this case, we analyze whether there exist subsets $A, B \subseteq \mathbb{Z}_m$ whose representation functions are identical. The following theorem addresses this configuration.

Theorem 13. *Let $\bar{r}_1, \bar{r}_2 \in \mathbb{Z}_m$ be distinct and let $A, B \subseteq \mathbb{Z}_m$ satisfy $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2\}$ and $|A \cap B| = 2$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B| = \frac{m}{2}$.*

Proof. The proof of Theorem 13 follows by an argument similar to that of Theorem 11. □

Theorem 14. *Let $\bar{r}_1, \bar{r}_2 \in \mathbb{Z}_m$ be distinct and let $A, B \subseteq \mathbb{Z}_m$ satisfy $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2\}$, $|A \cap B| = 2$ and $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then one of the following conditions holds:*

1. *If $\bar{r}_2 = \bar{r}_1 \oplus \overline{\frac{m}{2}}$, then there exist $r \in \mathbb{Z}_m$ and subsets $C, D, E \subseteq \mathbb{Z}_m$ with $|C| = |E|$ and $2|C| + |D| = \frac{m-4}{2}$ such that*

$$A = C \,\dot\cup\, \left(C \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, D \,\dot\cup\, \left\{\bar{r}, \bar{r} \oplus \overline{\frac{m}{2}}\right\}$$

and

$$B = E \,\dot\cup\, \left(E \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, \left(D \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, \left\{\bar{r}, \bar{r} \oplus \overline{\frac{m}{2}}\right\}.$$

2. *If $\bar{r}_2 \neq \bar{r}_1 \oplus \overline{\frac{m}{2}}$, then there exist subsets $C, D, E \subseteq \mathbb{Z}_m$ with $|C| = |E|$ and $2|C| + |D| = \frac{m-4}{2}$ such that*

$$A = C \,\dot\cup\, \left(C \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, D \,\dot\cup\, \left\{\bar{r}_1 \oplus \overline{\frac{m}{2}}, \bar{r}_2 \oplus \overline{\frac{m}{2}}\right\}$$

and

$$B = E \,\dot\cup\, \left(E \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, \left(D \oplus \overline{\frac{m}{2}}\right) \,\dot\cup\, \left\{\bar{r}_1 \oplus \overline{\frac{m}{2}}, \bar{r}_2 \oplus \overline{\frac{m}{2}}\right\}.$$

The symbol $\dot\cup$ denotes disjoint union.

Proof. Assume that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Let $\bar{r}_3, \bar{r}_4 \in A \cap B$ be distinct. Then,

$$B = (\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1, \bar{r}_2\})) \cup \{\bar{r}_3, \bar{r}_4\}.$$

By following the same approach as in Theorem 12 and applying Lemmas 1, 2, 3, 4, 5, we conclude that

$$\begin{aligned} 2\left[\sum_{i=3}^{4} \chi_A(n - r_i) - \sum_{i=1}^{2} \chi_A(n - r_i)\right] &= 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_2) - 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_3) \\ &\quad - 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_3) - 2\theta(\bar{n} = \bar{r}_1 \oplus \bar{r}_4) \\ &\quad - 2\theta(\bar{n} = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(\bar{n} = \bar{r}_3 \oplus \bar{r}_4) \\ &\quad + \sum_{i=1}^{4} \theta(\bar{n} = 2\bar{r}_i), \end{aligned} \tag{6}$$

for all $\bar{n} \in \mathbb{Z}_m$.

Case 1. Suppose that $\bar{r}_2 = \bar{r}_1 \oplus \frac{\overline{m}}{2}$. We first show that $2\bar{r}_3 = 2\bar{r}_4$. Suppose, to the contrary, that $2\bar{r}_3 \neq 2\bar{r}_4$. Substituting $\bar{n} = 2\bar{r}_3$ into (6), we obtain

$$\begin{aligned} 2\left[\sum_{i=3}^{4} \chi_A(2r_3 - r_i) - \sum_{i=1}^{2} \chi_A(2r_3 - r_i)\right] &= 2\theta(2\bar{r}_3 = \bar{r}_1 \oplus \bar{r}_2) - 2\theta(2\bar{r}_3 = \bar{r}_1 \oplus \bar{r}_3) \\ &\quad - 2\theta(2\bar{r}_3 = \bar{r}_2 \oplus \bar{r}_3) - 2\theta(2\bar{r}_3 = \bar{r}_1 \oplus \bar{r}_4) \\ &\quad - 2\theta(2\bar{r}_3 = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(2\bar{r}_3 = \bar{r}_3 \oplus \bar{r}_4) \\ &\quad + \sum_{i=1}^{4} \theta(2\bar{r}_3 = 2\bar{r}_i). \end{aligned}$$

Since $\bar{r}_2 = \bar{r}_1 \oplus \frac{\overline{m}}{2}$, we have $2\bar{r}_1 = 2\bar{r}_2$. Under the assumption $2\bar{r}_3 \neq 2\bar{r}_4$, the right-hand side is odd, whereas the left-hand side is even. This is a contradiction. Hence, $2\bar{r}_3 = 2\bar{r}_4$. Since $\bar{r}_3 \neq \bar{r}_4$, it follows that $\bar{r}_3 = \bar{r}_4 \oplus \frac{\overline{m}}{2}$. Therefore, $A \cap B = \left\{\bar{r}_4, \bar{r}_4 \oplus \frac{\overline{m}}{2}\right\}$. Put $h = \frac{\overline{m}}{2}$. For every $\bar{x} \in \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_1 \oplus h, \bar{r}_4, \bar{r}_4 \oplus h\}$, the pair $\{\bar{x}, \bar{x} \oplus h\}$ is contained in

$$(A \cup B) \setminus (A \cap B).$$

Hence, each such pair is of exactly one of the following types: both elements belong to $A \setminus B$, both elements belong to $B \setminus A$, or one element belongs to $A \setminus B$ and the other belongs to $B \setminus A$.

Choose one representative from every pair contained in $A \setminus B$ and collect these representatives in C. Choose one representative from every pair contained in $B \setminus A$ and collect these representatives in E. For every mixed pair, choose the element belonging to $A \setminus B$ and collect these elements in D.

Then

$$A = C \,\dot\cup \left(C \oplus \frac{\overline{m}}{2}\right) \dot\cup\, D \,\dot\cup \left\{\bar{r}_4, \bar{r}_4 \oplus \frac{\overline{m}}{2}\right\},$$

and

$$B = E \,\dot\cup \left(E \oplus \frac{\overline{m}}{2}\right) \dot\cup \left(D \oplus \frac{\overline{m}}{2}\right) \dot\cup \left\{\bar{r}_4, \bar{r}_4 \oplus \frac{\overline{m}}{2}\right\}.$$

By Theorem 13,

$$|A| = |B| = \frac{m}{2}.$$

Therefore,

$$2|C| + |D| + 2 = \frac{m}{2}$$

and

$$2|E| + |D| + 2 = \frac{m}{2}.$$

Consequently,

$$2|C| + |D| = \frac{m-4}{2}$$

and

$$|C| = |E|.$$

This completes the proof of Case 1.

Case 2. Suppose that $\bar{r}_2 \neq \bar{r}_1 \oplus \overline{\frac{m}{2}}$. We first show that, after interchanging $\bar{r}_3$ and $\bar{r}_4$ if necessary,

$$\bar{r}_3 = \bar{r}_1 \oplus \overline{\frac{m}{2}} \quad \text{and} \quad \bar{r}_4 = \bar{r}_2 \oplus \overline{\frac{m}{2}}.$$

Since $\bar{r}_2 \neq \bar{r}_1 \oplus \overline{\frac{m}{2}}$ and $\bar{r}_1 \neq \bar{r}_2$, we have $2\bar{r}_1 \neq 2\bar{r}_2$. Suppose, to the contrary, that

$$2\bar{r}_1 \neq 2\bar{r}_3 \quad \text{and} \quad 2\bar{r}_1 \neq 2\bar{r}_4.$$

Substituting $\bar{n} = 2\bar{r}_1$ into (6), we obtain

$$\begin{aligned} 2\left[\sum_{i=3}^{4} \chi_A(2r_1 - r_i) - \sum_{i=1}^{2} \chi_A(2r_1 - r_i)\right] &= 2\theta(2\bar{r}_1 = \bar{r}_1 \oplus \bar{r}_2) - 2\theta(2\bar{r}_1 = \bar{r}_1 \oplus \bar{r}_3) \\ &\quad - 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_3) - 2\theta(2\bar{r}_1 = \bar{r}_1 \oplus \bar{r}_4) \\ &\quad - 2\theta(2\bar{r}_1 = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(2\bar{r}_1 = \bar{r}_3 \oplus \bar{r}_4) \\ &\quad + \sum_{i=1}^{4} \theta(2\bar{r}_1 = 2\bar{r}_i). \end{aligned}$$

Under the above assumptions,

$$\sum_{i=1}^{4} \theta(2\bar{r}_1 = 2\bar{r}_i) = 1.$$

Hence, the right-hand side is odd, whereas the left-hand side is even, which is a contradiction. Therefore,

$$2\bar{r}_1 = 2\bar{r}_3 \quad \text{or} \quad 2\bar{r}_1 = 2\bar{r}_4.$$

Without loss of generality, assume that $2\bar{r}_1 = 2\bar{r}_3$. Since $\bar{r}_1 \neq \bar{r}_3$, it follows that

$$\bar{r}_3 = \bar{r}_1 \oplus \overline{\frac{m}{2}}. \tag{7}$$

We next show that $2\bar{r}_2 = 2\bar{r}_4$. Suppose, to the contrary, that $2\bar{r}_2 \neq 2\bar{r}_4$. Since $2\bar{r}_1 = 2\bar{r}_3$ and $2\bar{r}_1 \neq 2\bar{r}_2$, we have $2\bar{r}_2 \neq 2\bar{r}_1$ and $2\bar{r}_2 \neq 2\bar{r}_3$. Substituting $\bar{n} = 2\bar{r}_2$ into (6), we obtain

$$\begin{aligned} 2\left[\sum_{i=3}^{4}\chi_A(2r_2 - r_i) - \sum_{i=1}^{2}\chi_A(2r_2 - r_i)\right] &= 2\theta(2\bar{r}_2 = \bar{r}_1 \oplus \bar{r}_2) - 2\theta(2\bar{r}_2 = \bar{r}_1 \oplus \bar{r}_3) \\ &\quad - 2\theta(2\bar{r}_2 = \bar{r}_2 \oplus \bar{r}_3) - 2\theta(2\bar{r}_2 = \bar{r}_1 \oplus \bar{r}_4) \\ &\quad - 2\theta(2\bar{r}_2 = \bar{r}_2 \oplus \bar{r}_4) + 2\theta(2\bar{r}_2 = \bar{r}_3 \oplus \bar{r}_4) \\ &\quad + \sum_{i=1}^{4}\theta(2\bar{r}_2 = 2\bar{r}_i). \end{aligned}$$

Under the above assumptions,

$$\sum_{i=1}^{4}\theta(2\bar{r}_2 = 2\bar{r}_i) = 1.$$

Thus, the left-hand side is even whereas the right-hand side is odd, a contradiction. Hence,

$$2\bar{r}_2 = 2\bar{r}_4,$$

and consequently,

$$\bar{r}_4 = \bar{r}_2 \oplus \frac{\overline{m}}{2}. \tag{8}$$

It follows from (7) and (8) that

$$A \cap B = \left\{\bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\}.$$

Proceeding as in Case 1, by considering the pairs

$$\left\{\bar{x}, \bar{x} \oplus \frac{\overline{m}}{2}\right\}$$

for

$$\bar{x} \in \mathbb{Z}_m \setminus \left\{\bar{r}_1, \bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\},$$

there exist subsets $C, D, E \subseteq \mathbb{Z}_m$ such that

$$A = C \,\dot\cup \left(C \oplus \frac{\overline{m}}{2}\right) \dot\cup\, D \,\dot\cup \left\{\bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\},$$

and

$$B = E \,\dot\cup \left(E \oplus \frac{\overline{m}}{2}\right) \dot\cup \left(D \oplus \frac{\overline{m}}{2}\right) \dot\cup \left\{\bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\}.$$

Furthermore, as in Case 1,

$$|C| = |E| \quad \text{and} \quad 2|C| + |D| = \frac{m-4}{2}.$$

This completes the proof. □

The following examples illustrate the structural conditions in Theorem 14 and show that these conditions are necessary but not sufficient for the equality of the representation functions.

Example 1. For $m = 12$, we have $\mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$. Let

$$A = \{\bar{1}, \bar{2}, \bar{3}, \bar{7}, \bar{8}, \bar{9}\} \quad \text{and} \quad B = \{\bar{3}, \bar{4}, \bar{5}, \bar{9}, \bar{10}, \bar{11}\}.$$

Then, $A \cup B = \mathbb{Z}_{12} \setminus \{\bar{0}, \bar{6}\}$, $A \cap B = \{\bar{3}, \bar{9}\}$, $C = \{\bar{1}, \bar{2}\}$, $D = \emptyset$ and $E = \{\bar{4}, \bar{11}\}$. Then, the representation functions are computed as follows:

Table 1: Comparison of $R_A(\bar{n})$ and $R_B(\bar{n})$ for $\bar{n} \in \mathbb{Z}_{12}$

$\bar{n}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$
$R_A(\bar{n})$	2	0	2	4	6	4	2	0	2	4	6	4
$R_B(\bar{n})$	2	4	6	4	2	0	2	4	6	4	2	0

From Table 1, it is clear that the values of $R_A(\bar{n})$ and $R_B(\bar{n})$ differ for many $\bar{n} \in \mathbb{Z}_{12}$, such as $n = 1, 2, 4, 5, 7, 8, 10$ and 11. Therefore, the sets A and B in Example 1 are not representation-equivalent, i.e., $R_A(\bar{n}) \neq R_B(\bar{n})$ for some $\bar{n} \in \mathbb{Z}_{12}$.

Example 2. For $m = 12$, we have $\mathbb{Z}_{12} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}\}$. Let

$$A = \{\bar{0}, \bar{1}, \bar{4}, \bar{6}, \bar{7}, \bar{10}\} \quad \text{and} \quad B = \{\bar{0}, \bar{2}, \bar{5}, \bar{6}, \bar{8}, \bar{11}\}.$$

Then, $A \cup B = \mathbb{Z}_{12} \setminus \{\bar{3}, \bar{9}\}$, $A \cap B = \{\bar{0}, \bar{6}\}$, $C = \{\bar{1}, \bar{4}\}$, $D = \emptyset$ and $E = \{\bar{2}, \bar{5}\}$. The representation functions are computed as follows: As shown in Table 2, the values of the representation functions

Table 2: Comparison of $R_A(\bar{n})$ and $R_B(\bar{n})$ for $\bar{n} \in \mathbb{Z}_{12}$

$\bar{n}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$
$R_A(\bar{n})$	2	4	4	0	4	4	2	4	4	0	4	4
$R_B(\bar{n})$	2	4	4	0	4	4	2	4	4	0	4	4

$R_A(\bar{n})$ and $R_B(\bar{n})$ are identical for all $\bar{n} \in \mathbb{Z}_{12}$. This confirms that the sets A and B in Example 2 are representation-equivalent, i.e., $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_{12}$.

Example 3. Let $m = 18$ and consider the subsets

$$A = \{\bar{1}, \bar{2}, \bar{4}, \bar{8}, \bar{9}, \overline{10}, \overline{12}, \overline{13}, \overline{14}\}$$

and

$$B = \{\bar{5}, \bar{6}, \bar{7}, \bar{9}, \overline{11}, \overline{12}, \overline{15}, \overline{16}, \overline{17}\}$$

of $\mathbb{Z}_{18}$. Then, $A \cup B = \mathbb{Z}_{18} \setminus \{\bar{0}, \bar{3}\}$ and $A \cap B = \{\bar{9}, \overline{12}\}$. Taking $\bar{r}_1 = \bar{0}$ and $\bar{r}_2 = \bar{3}$, we have $\bar{r}_2 \neq \bar{r}_1 \oplus \frac{\overline{m}}{2}$, since $\bar{3} \neq \bar{0} \oplus \bar{9} = \bar{9}$. Moreover,

$$A \cap B = \left\{ \bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2} \right\} = \{\bar{9}, \overline{12}\}.$$

Let $C = \{\bar{1}, \bar{4}\}$, $D = \{\bar{2}, \bar{8}, \overline{14}\}$, and $E = \{\bar{6}, \bar{7}\}$. Then,

$$C \oplus \frac{\overline{m}}{2} = \{\overline{10}, \overline{13}\}, \quad D \oplus \frac{\overline{m}}{2} = \{\overline{11}, \overline{17}, \bar{5}\}, \quad E \oplus \frac{\overline{m}}{2} = \{\overline{15}, \overline{16}\}.$$

Hence,

$$A = C \dot{\cup} \left(C \oplus \frac{\overline{m}}{2}\right) \dot{\cup} D \dot{\cup} \left\{\bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\},$$

and

$$B = E \dot{\cup} \left(E \oplus \frac{\overline{m}}{2}\right) \dot{\cup} \left(D \oplus \frac{\overline{m}}{2}\right) \dot{\cup} \left\{\bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_2 \oplus \frac{\overline{m}}{2}\right\}.$$

Furthermore,

$$|C| = |E| = 2$$

and

$$2|C| + |D| = 2(2) + 3 = 7 = \frac{18 - 4}{2}.$$

Thus, the structural conditions in Theorem 14(2) are satisfied. The corresponding representation functions are given in Table 3.

Table 3: The representation functions of A and B in Example 3

$\bar{n}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\overline{10}$	$\overline{11}$	$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$	$\overline{16}$	$\overline{17}$
$R_A(\bar{n})$	5	2	4	6	7	6	5	2	4	4	5	4	4	4	6	4	5	4
$R_B(\bar{n})$	5	2	4	6	7	6	5	2	4	4	5	4	4	4	6	4	5	4

It follows from Table 3 that

$$R_A(\bar{n}) = R_B(\bar{n})$$

for every $\bar{n} \in \mathbb{Z}_{18}$. Therefore, this example satisfies the structural conditions in Theorem 14(2), with $D \neq \varnothing$, and has identical representation functions.

The three examples above illustrate the role of the structural conditions in Theorem 14. Example 1 satisfies the structural conditions in Theorem 14(1) but does not have identical representation functions, whereas Examples 2 and 3 satisfy the structural conditions in Theorem 14(1) and Theorem 14(2), respectively, and have identical representation functions. Thus, the structural conditions in Theorem 14 are necessary, but by themselves are not sufficient to guarantee the equality of the representation functions.

Next, we continue with Case 3, corresponding to the case where

$$|A \cup B| = m - 3 \text{ and } |A \cap B| = 1.$$

In this case, we aim to determine whether there exist subsets $A, B \subseteq \mathbb{Z}_m$ satisfying the condition

$$R_A(\bar{n}) = R_B(\bar{n}) \text{ for all } \bar{n} \in \mathbb{Z}_m$$

We will demonstrate that no such subsets exist under the given conditions. The nonexistence is formally established in the following theorem.

Theorem 15. *Let $\bar{r}_1, \bar{r}_2, \bar{r}_3 \in \mathbb{Z}_m$ be distinct and let $A, B \subseteq \mathbb{Z}_m$ satisfy $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ and $|A \cap B| = 1$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B| = \frac{m}{2} - 1$.*

Proof. The proof of Theorem 15 follows by an argument similar to that of Theorem 11. □

Theorem 16. *Let $\bar{r}_1, \bar{r}_2, \bar{r}_3 \in \mathbb{Z}_m$ be distinct. Then, there do not exist two subsets A and B of $\mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ and $|A \cap B| = 1$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$.*

Proof. Suppose that A and B are both subsets of $\mathbb{Z}_m$ with $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ and $|A \cap B| = 1$ such that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Let $A \cap B = \{r_4\}$. Then,

$$B = (\mathbb{Z}_m \setminus (A \cup \{\bar{r}_1, \bar{r}_2, \bar{r}_3\})) \cup \{\bar{r}_4\}.$$

By following the same approach as in Theorem 12 and applying Lemmas 1–5, we conclude that $\bar{r}_2 = \bar{r}_1 \oplus \frac{\overline{m}}{2}$, $\bar{r}_3 = \bar{r}_4 \oplus \frac{\overline{m}}{2}$ and

$$\chi_A(k) + \chi_A(l) + \chi_A\left(k + \frac{m}{2}\right) - \chi_A\left(l + \frac{m}{2}\right) = 1 - \theta\left(\bar{k} = \bar{r}_1\right) + \theta\left(\bar{l} = \bar{r}_4\right) - \theta\left(\bar{r}_1 = \bar{k} \oplus \frac{\bar{m}}{2}\right) - \theta\left(\bar{r}_4 = \bar{l} \oplus \frac{\bar{m}}{2}\right),$$

for all $\bar{k}, \bar{l} \in \mathbb{Z}_m$. Let $\bar{k} \in A \setminus \{\bar{r}_1, \bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_4, \bar{r}_4 \oplus \frac{\overline{m}}{2}\}$ and $\bar{l} = \bar{k} \oplus \frac{\overline{m}}{2}$. Then,

$$\bar{l} \in \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_1 \oplus \frac{\overline{m}}{2}, \bar{r}_4, \bar{r}_4 \oplus \frac{\overline{m}}{2}\}.$$

Therefore, $2\chi_A\left(k + \frac{m}{2}\right) = \chi_A\left(k + \frac{m}{2}\right) + \chi_A\left(k + \frac{m}{2}\right) = 1$, which is a contradiction. Hence, no such subsets A and B exist. □

Finally, we consider Case 4, where the subsets A and B are disjoint and their union satisfies

$$|A \cup B| = m - 4 \text{ and } |A \cap B| = 0.$$

In this setting, we determine whether it is possible to construct such subsets with equal representation functions. The following result summarizes our findings for this final case.

Theorem 17. *Let $\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4 \in \mathbb{Z}_m$ and $A, B \subseteq \mathbb{Z}_m$ be such that $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4\}$ and $|A \cap B| = 0$. If $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$, then $|A| = |B| = \frac{m}{2} - 2$.*

Proof. The proof of Theorem 17 follows by an argument similar to that of Theorem 11. □

Theorem 18. *Let $\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4 \in \mathbb{Z}_m$ be distinct, and let $A, B \subseteq \mathbb{Z}_m$ satisfy $A \cup B = \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4\}$, $A \cap B = \emptyset$, and $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then,*

$$\bar{r}_1 = \bar{r}_2 \oplus \frac{\overline{m}}{2}, \quad \bar{r}_3 = \bar{r}_4 \oplus \frac{\overline{m}}{2}$$

and there exist subsets $C, D, E \subseteq \mathbb{Z}_m$ with $|C| = |E|$ and $2|C| + |D| = \frac{m-4}{2}$ such that

$$A = C \dot{\cup} \left(C \oplus \frac{\overline{m}}{2}\right) \dot{\cup} D$$

and

$$B = E \dot{\cup} \left(E \oplus \frac{\overline{m}}{2}\right) \dot{\cup} \left(D \oplus \frac{\overline{m}}{2}\right).$$

Proof. Assume that $R_A(\bar{n}) = R_B(\bar{n})$ for all $\bar{n} \in \mathbb{Z}_m$. Then,

$$B = \mathbb{Z}_m \setminus (A \cup \{\bar{r}_1, \bar{r}_2, \bar{r}_3, \bar{r}_4\}).$$

Applying Lemmas 2–5 as in the proof of Theorem 12, we obtain, after a suitable relabeling,

$$2\bar{r}_1 = 2\bar{r}_2 \quad \text{and} \quad 2\bar{r}_3 = 2\bar{r}_4.$$

Since the four residue classes are distinct, it follows that

$$\bar{r}_2 = \bar{r}_1 \oplus \frac{\overline{m}}{2} \quad \text{and} \quad \bar{r}_4 = \bar{r}_3 \oplus \frac{\overline{m}}{2}.$$

Put $h = \frac{\overline{m}}{2}$. For every $\bar{x} \in \mathbb{Z}_m \setminus \{\bar{r}_1, \bar{r}_1 \oplus h, \bar{r}_3, \bar{r}_3 \oplus h\}$, the pair $\{\bar{x}, \bar{x} \oplus h\}$ is contained in $A \cup B$. Since $A \cap B = \varnothing$, each such pair is of exactly one of the following types: both elements belong to A, both elements belong to B, or one element belongs to A and the other belongs to B.

Choose one representative from every pair contained in A and collect these representatives in C. Choose one representative from every pair contained in B and collect these representatives in E. For every mixed pair, choose the element belonging to A and collect these elements in D. Then,

$$A = C \,\dot\cup \left(C \oplus \frac{\overline{m}}{2}\right) \dot\cup D$$

and

$$B = E \,\dot\cup \left(E \oplus \frac{\overline{m}}{2}\right) \dot\cup \left(D \oplus \frac{\overline{m}}{2}\right).$$

By Theorem 17,

$$|A| = |B| = \frac{m-4}{2}.$$

Therefore,

$$2|C| + |D| = \frac{m-4}{2}$$

and

$$2|E| + |D| = \frac{m-4}{2}.$$

Hence,

$$|C| = |E|.$$

This completes the proof. □

Acknowledgments

The authors would like to express their sincere gratitude to Nakhon Sawan Rajabhat University for the academic support that made this research possible.

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