

Quantitative m -exponential convexity and certified bounds for managed traffic link cost models

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Abstract. This paper investigates m -exponentially convex functions with respect to s . Two new midpoint-type integral inequalities are established using Hölder's inequality and the power-mean inequality. Structural properties of the associated constants Λ_s and Ω_s are derived; in particular, the strict inequality $3\Omega_s < \Lambda_s$, valid for every $s > 0$, shows that, when $q = 2$, the power-mean estimate is strictly sharper than the corresponding Hölder estimate. Sufficient Lipschitz conditions for membership in this class are derived, together with examples showing that it contains functions that are not m -convex. A quantitative refinement with an explicit correction term is obtained; the correction term is proved to be nonnegative, so that the refinement improves the basic m -exponential estimate unconditionally and not merely in the reported examples. These results are then used as an analytical certification framework for traffic link-cost models, covering both the classical BPR model and a controlled saturating model that may be nonconvex while still belonging to the studied class. In four illustrative cases the correction lowers the baseline m -exponential upper bound by 27.6%–30.1%, and in a two-link instance a feasible transition flow is constructed whose objective value is within 0.0104% of a certified global optimum. The traffic results are rigorous upper bounds and feasible candidates, not an optimization method or an empirically validated traffic model.

Keywords: m -exponential convexity; Hermite–Hadamard inequality; traffic link cost; BPR model; system-optimal assignment.

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1 Introduction

The Hermite–Hadamard inequality is one of the fundamental results connecting convexity with integral estimates. If $\psi : [a, b] \rightarrow \mathbb{R}$ is convex, then

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \psi(u) du \leq \frac{\psi(a) + \psi(b)}{2}. \quad (1)$$

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The two sides of (1) naturally correspond to midpoint and endpoint reduced-order approximations of an integral mean. Such estimates are particularly useful when the integral represents a time average, spatial average, expected response, or aggregate performance measure.

The development of generalized convexity has led to numerous extensions of (1). In particular, m -convexity, introduced by Toader, is based on the asymmetric combination $tx + m(1-t)y$, and the corresponding Hermite-Hadamard-type inequalities were studied by Dragomir; see [4, 5, 15]. Exponential-type coefficients have also been employed to define broader classes of generalized convex functions. In recent years, a substantial number of results have been obtained for exponential convexity and its various extensions. These classes have proved useful in a wide range of applications, including mathematical modeling, inequalities for special means, fractional calculus, and related areas; see [1, 2, 11, 17]. Kadakal introduced exponential convexity with respect to a parameter s in [8], whereas m -exponential convexity with respect to s was introduced in [18].

The class studied in the present paper combines the m -interpolation with the weights ts^{1-t} and $m(1-t)s^t$.

A fourth strand, which motivates the application developed below, is static traffic assignment. Volume-delay functions such as the Bureau of Public Roads function [16] and the conical functions of Spiess [13] enter the Beckmann formulation of user-equilibrium and system-optimal flows [3, 12], and the relation between the two solution concepts continues to receive attention [10]. The standard analysis of the system-optimal problem rests on convexity of the total-cost objective; when the modelled quantity is a generalized link cost under active management, that assumption may fail, and the classical theory then supplies no certified upper bound on the system-optimal value. For the traffic application developed below, the relevant issue is therefore whether the class $mECF_s$ can certify nonconvex total-cost functions and provide quantitative upper bounds, rather than merely reproduce estimates already available from m -convexity. This observation connects the generalized-convexity framework of [8, 11, 18] directly with the certification problem considered in Section 5.

Research gap. Reference [18] introduces the class $mECF_s$ and establishes Hermite-Hadamard-type inequalities of trapezoid type together with the basic definitional properties of the class. Membership in $mECF_s$ is verified there only through two elementary routes: either $s \geq 1$ together with m -convexity of the function, or direct verification of the defining inequality. Consequently, every function treated in [18] is already m -convex, and the class is used as a formal generalization rather than as a device that enlarges the family of admissible models. The related (s, m) -exponential-type results of [11] and the exponential convexity results of [1, 2] follow the same pattern, and the estimates obtained there are of the same order as their m -convex counterparts, so that the additional parameter s carries no quantitative benefit.

Two gaps therefore remain. First, no verifiable sufficient criterion for membership in $mECF_s$ is available that does not presuppose m -convexity. Second, no mechanism has been proposed by which the parameter s produces a bound that is provably sharper than the underlying m -convex estimate.

Contributions. The present paper addresses both gaps, as follows:

- (C1) Midpoint-type estimates for the class $mECF_s$ (Theorems 1 and 2). The midpoint kernel $\varpi(t) = \min\{t, 1-t\}$ is not treated in [18], whose results are of trapezoid type; the split kernel produces the constant Ω_s , which does not occur in the earlier literature.
- (C2) A Lipschitz-scaling criterion (Theorem 3 and Proposition 2) certifying membership in $mECF_s$ without assuming m -convexity. The criterion yields explicit members of $mECF_s$ for which m -convexity fails (Example 1) and, in a separate instance, for which ordinary convexity fails as well

(Proposition 4); the two examples establish different implications and are stated separately for that reason.

- (C3) A quantitative remainder (Theorem 4) which, under the hypotheses of Theorem 3, is proved to be nonnegative (Corollary 4). The refined bound therefore dominates the baseline *m*-exponential bound as a theorem, independently of the numerical illustrations.
- (C4) Structural properties of Λ_s and Ω_s (Proposition 1). These constants are not merely evaluated integrals: they are strictly increasing, they recover the classical midpoint constants $\frac{1}{2}$ and $\frac{1}{8}$ at $s = 1$, and they satisfy $3\Omega_s < \Lambda_s$ for every $s > 0$, which implies that the power-mean certificate of Theorem 2 is uniformly sharper than the Hölder certificate of Theorem 1 when $q = 2$.
- (C5) A network-level certificate for demand transitions (Theorem 6) that remains valid when the system objective is nonconvex.

The relation with Reference [18] can be summarized more explicitly as follows. Theorems 1–2 are new midpoint-kernel extensions within the class introduced in [18], whereas the Lipschitz membership mechanism, the quantitative nonnegative remainder, the strict-separation examples, and the network certificate are additional ingredients not developed there.

Table 1: Comparison with the previous *m*-exponential convexity study [18]

Aspect	Reference [18]	Present paper
Framework	Introduces $mECF_s$ and its basic properties.	Uses the same class; no new definition is claimed.
Integral inequalities	Hermite–Hadamard-type estimates of trapezoid type.	Midpoint-kernel estimates (Theorems 1–2); this extends the earlier integral-inequality program in a different kernel geometry.
Membership	Membership verified through <i>m</i> -convexity for $s \geq 1$ or direct checking of the definition.	Lipschitz–scaling criteria that do not assume <i>m</i> -convexity (Theorem 3 and Proposition 2).
Quantitative refinement	No nonnegative defect correction of the present type.	Explicit correction term and pointwise refinement (Theorem 4 and Corollary 4).
Separation examples	Examples remain within the <i>m</i> -convex route used to verify membership.	Explicit failures of <i>m</i> -convexity and of ordinary convexity while membership still holds.
Traffic/network use	Not considered.	Analytical link-cost certification and a feasible demand-transition upper-bound certificate.

Application. The analytical estimates are applied to static traffic assignment. We first verify the criterion of (C2) for the classical BPR total-travel-time function, and then introduce a controlled saturating link-cost model for which convexity and *m*-convexity may fail while the criterion still applies. The midpoint estimates are used to certify time-averaged link costs under a linear ramp between two demand levels, and the network certificate of (C5) is illustrated on a two-link example.

The paper is organized as follows. Section 2 presents the basic definitions required for the subsequent analysis, together with an auxiliary lemma, several standard constants used later in the paper,

and their structural properties. Section 3 establishes two first-derivative estimates and derives two direct corollaries for the class of exponentially convex functions with respect to s . Section 4 develops sufficient membership criteria, strict-inclusion examples, and quantitative refinements. Finally, Section 5 introduces the controlled saturating link-cost model and derives midpoint-based integral certificates and a system-optimal demand-transition certificate. It also provides a numerical effectivity analysis and compares the resulting certified bounds.

2 Preliminaries

The following parameterized class extends ordinary convexity by replacing the classical coefficients with exponential-type weights.

Definition 1 ([8]). *Let $I \subseteq \mathbb{R}$ be an interval and let $s > 0$. A nonnegative function $\psi : I \rightarrow \mathbb{R}$ is said to be exponentially convex with respect to s if, for all $x, y \in I$ and $t \in [0, 1]$, one has*

$$\psi(tx + (1-t)y) \leq ts^{1-t}\psi(x) + (1-t)s^t\psi(y). \quad (2)$$

Throughout the paper, $m \in (0, 1]$ and $s > 0$ are fixed. Whenever the expression $tx + m(1-t)y$ is used, it is assumed to belong to the domain of the function under consideration.

Definition 2 ([18]). *Let $I \subseteq \mathbb{R}$ be an interval, let $m \in (0, 1]$ and $s > 0$. A nonnegative function $\psi : I \rightarrow [0, \infty)$ is called m -exponentially convex with respect to s if*

$$\psi(tx + m(1-t)y) \leq ts^{1-t}\psi(x) + m(1-t)s^t\psi(y) \quad (3)$$

for all $x, y \in I$ and $t \in [0, 1]$ for which $tx + m(1-t)y \in I$. The corresponding class is denoted by $m\text{ECF}_s(I)$.

For $s = 1$, Definition 2 reduces to m -convexity. For $m = 1$, it reduces to exponential convexity with respect to s . If $s \geq 1$, every nonnegative m -convex function belongs to $m\text{ECF}_s(I)$ because $s^t \geq 1$ and $s^{1-t} \geq 1$.

Remark 1. *If $0 \in I$ and $m < 1$, then every $\psi \in m\text{ECF}_s(I)$ satisfies $\psi(0) = 0$. Indeed, taking $t = 0$ and $y = 0$ in (3) gives $\psi(0) \leq m\psi(0)$, hence $(1-m)\psi(0) \leq 0$ and therefore $\psi(0) \leq 0$; nonnegativity forces $\psi(0) = 0$. This elementary observation is used in Section 5.4.*

The following identity is the starting point for the midpoint estimates.

Lemma 1 ([14]). *Let $0 < k \leq 1$, $m \in (0, 1]$, and let $0 < b_1 < b_2$ satisfy $kb_1 < mb_2$. Let $\psi : [b_1, \frac{b_2}{k}] \rightarrow \mathbb{R}$ be differentiable on $(b_1, b_2/k)$, and suppose that $\psi' \in L_1\left([b_1, \frac{b_2}{k}]\right)$. Then*

$$\begin{aligned} & \frac{k}{mb_2 - kb_1} \int_{b_1}^{\frac{mb_2}{k}} \psi(u) du - \psi\left(\frac{b_1 + \frac{mb_2}{k}}{2}\right) \\ &= \frac{mb_2 - kb_1}{k} \left[\int_0^{\frac{1}{2}} t \psi'\left(tb_1 + m(1-t)\frac{b_2}{k}\right) dt - \int_{\frac{1}{2}}^1 (1-t) \psi'\left(tb_1 + m(1-t)\frac{b_2}{k}\right) dt \right]. \quad (4) \end{aligned}$$

For later use, define

$$\Lambda_s = \begin{cases} \frac{s - \ln s - 1}{(\ln s)^2}, & s > 0, s \neq 1, \\ \frac{1}{2}, & s = 1, \end{cases} \quad (5)$$

and

$$\Omega_s = \begin{cases} \frac{2s + 2 - 4\sqrt{s} + (1 - \sqrt{s}) \ln s}{(\ln s)^3}, & s > 0, s \neq 1, \\ \frac{1}{8}, & s = 1. \end{cases} \quad (6)$$

These constants satisfy

$$\int_0^1 t s^{1-t} dt = \int_0^1 (1-t) s^t dt = \Lambda_s, \quad (7)$$

and

$$\Omega_s = \int_0^{\frac{1}{2}} t^2 s^{1-t} dt + \int_{\frac{1}{2}}^1 t(1-t) s^{1-t} dt = \int_0^{\frac{1}{2}} t(1-t) s^t dt + \int_{\frac{1}{2}}^1 (1-t)^2 s^t dt. \quad (8)$$

For completeness, the apparent singularities at $s = 1$ can be checked directly. Put $u = \ln s$. As $u \rightarrow 0$,

$$\begin{aligned} \Lambda_{e^u} &= \frac{1}{2} + \frac{u}{6} + \frac{u^2}{24} + O(u^3), \\ \Omega_{e^u} &= \frac{1}{8} + \frac{5u}{96} + \frac{5u^2}{384} + O(u^3). \end{aligned}$$

Hence $\Lambda_s \rightarrow \frac{1}{2}$ and $\Omega_s \rightarrow \frac{1}{8}$ as $s \rightarrow 1$. These expansions also provide a numerically stable evaluation close to $s = 1$, where the closed forms in (5)–(6) involve cancellation.

The next proposition collects the properties of Λ_s and Ω_s that are used in the sequel. Part (iii) is the reason why the power-mean estimate of Section 3 is always preferable to the Hölder estimate when $q = 2$; it is a statement about the constants themselves and does not depend on the function or on the interval.

Proposition 1. *Let Λ_s and Ω_s be defined by (5)–(6), and set*

$$h_s(t) = t s^{1-t} + (1-t) s^t, \quad t \in [0, 1]. \quad (9)$$

Then the following hold.

(i) *The function h_s is positive and symmetric about $t = \frac{1}{2}$, and*

$$\Lambda_s = \int_0^{\frac{1}{2}} h_s(t) dt, \quad 3\Omega_s - \Lambda_s = \int_0^{\frac{1}{2}} (3t - 1) h_s(t) dt. \quad (10)$$

(ii) *Λ_s and Ω_s extend to real-analytic, strictly increasing functions of s on $(0, \infty)$, with $\Lambda_1 = \frac{1}{2}$ and $\Omega_1 = \frac{1}{8}$. In particular, the classical midpoint constants are recovered in the limiting case $s = 1$.*

(iii) For every $s > 0$,

$$3\Omega_s < \Lambda_s. \quad (11)$$

Consequently,

$$\sqrt{\frac{3\Omega_s}{\Lambda_s}} < 1, \quad s > 0.$$

(iv) $\frac{3\Omega_s}{\Lambda_s} \sim \frac{6}{\ln s}$ as $s \rightarrow \infty$, and $\frac{3\Omega_s}{\Lambda_s} \rightarrow 0$ as $s \rightarrow 0^+$.

Proof. (i) Symmetry is immediate from $h_s(1-t) = (1-t)s^t + ts^{1-t} = h_s(t)$. By (7),

$$2\Lambda_s = \int_0^1 (ts^{1-t} + (1-t)s^t) dt = \int_0^1 h_s(t) dt,$$

and symmetry gives $\int_0^{\frac{1}{2}} h_s = \frac{1}{2} \int_0^1 h_s = \Lambda_s$. Similarly, by (8) and the substitution $t \mapsto 1-t$,

$$2\Omega_s = \int_0^1 \varpi(t) h_s(t) dt, \quad \varpi(t) = \min\{t, 1-t\},$$

so that, again by symmetry, $\Omega_s = \int_0^{\frac{1}{2}} t h_s(t) dt$. Subtracting $\Lambda_s = \int_0^{\frac{1}{2}} h_s$ from $3\Omega_s = \int_0^{\frac{1}{2}} 3t h_s$ yields the second identity in (10).

(ii) Both expressions are ratios of entire functions of $\ln s$ whose singularity at $\ln s = 0$ is removable, with the stated values; strict monotonicity follows from (7) and (8), since for fixed $t \in (0, 1)$ the integrands ts^{1-t} and $\varpi(t)h_s(t)$ are strictly increasing in s .

(iii) For $s = 1$, the assertion follows immediately from $\Lambda_1 = \frac{1}{2}$ and $\Omega_1 = \frac{1}{8}$, since $\Lambda_1 - 3\Omega_1 = \frac{1}{8} > 0$. Assume now that $s \neq 1$ and put $u = \ln s$. From (5) and (6), a direct calculation gives

$$u^3(\Lambda_s - 3\Omega_s) = F(u), \quad (12)$$

where $F(u) = (u-6)e^u + (3u+12)e^{\frac{u}{2}} - u^2 - 4u - 6$.

Suppose first that $u > 0$. Expanding the exponential functions into their power series and observing that the coefficients of the terms of degree 0, 1, and 2 cancel, we obtain

$$F(u) = \sum_{n=3}^{\infty} \left(n - 6 + \frac{6(n+2)}{2^n} \right) \frac{u^n}{n!}.$$

For $n = 3, 4, 5$, the coefficients in parentheses are respectively $\frac{3}{4}$, $\frac{1}{4}$, and $\frac{5}{16}$, while for every $n \geq 6$ they are strictly positive. Hence $F(u) > 0$ for $u > 0$. Since $u^3 > 0$, it follows from (12) that $\Lambda_s - 3\Omega_s > 0$.

Let now $u < 0$ and write $u = -v$, where $v > 0$. Define

$$G(v) = -e^v F(-v).$$

A direct simplification yields

$$G(v) = v + 6 + (3v - 12)e^{\frac{v}{2}} + (v^2 - 4v + 6)e^v.$$

Again expanding into power series and cancelling the terms of degree 0, 1, and 2, we obtain $G(v) = \sum_{n=3}^{\infty} (n-2) \left(n-3 + \frac{6}{2^n}\right) \frac{v^n}{n!}$. Every coefficient in this series is strictly positive for $n \geq 3$. Therefore $G(v) > 0$, and hence $F(-v) < 0$. Since $u^3 = -v^3 < 0$, relation (12) again gives $\Lambda_s - 3\Omega_s > 0$. Thus $3\Omega_s < \Lambda_s$ for every $s > 0$.

(iv) Let again $u = \ln s$. As $u \rightarrow +\infty$,

$$\Lambda_s \sim \frac{e^u}{u^2}, \quad \Omega_s \sim \frac{2e^u}{u^3},$$

so $\frac{3\Omega_s}{\Lambda_s} \sim \frac{6}{u} = \frac{6}{\ln s}$. As $u \rightarrow -\infty$, one has $\Lambda_s \sim -\frac{1}{u}$ and $\Omega_s \sim \frac{1}{u^2}$, and therefore $\frac{3\Omega_s}{\Lambda_s} \sim -\frac{3}{u} \rightarrow 0$. This proves part (iv). $\square$

Remark 2. Proposition 1(iii) provides a purely analytical comparison of the constants. In particular, when $q = 2$, the ratio of the power-mean and Hölder certificates is strictly smaller than one for every $s > 0$. The numerical value of this ratio for the traffic parameter considered in Section 5 is evaluated separately there.

3 Main results

As the main theoretical contributions of this paper, we first establish two important theorems that provide further refinements and extend the existing results for this recently introduced class of functions.

Theorem 1. Let $0 < k \leq 1$, $m \in (0, 1]$, $s > 0$, and let $0 < b_1 < b_2$ satisfy $kb_1 < mb_2$. Let $\psi : \left[b_1, \frac{b_2}{k}\right] \rightarrow \mathbb{R}$ be differentiable on $\left(b_1, \frac{b_2}{k}\right)$, with $\psi' \in L_1\left(\left[b_1, \frac{b_2}{k}\right]\right)$. Assume that $|\psi'|^q$ is m -exponentially convex with respect to s on $[b_1, \frac{b_2}{k}]$, where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{k}{mb_2 - kb_1} \int_{b_1}^{\frac{mb_2}{k}} \psi(u) du - \psi\left(\frac{b_1 + \frac{mb_2}{k}}{2}\right) \right| \leq \frac{mb_2 - kb_1}{2k} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\Lambda_s \left(|\psi'(b_1)|^q + m \left| \psi'\left(\frac{b_2}{k}\right) \right|^q \right) \right]^{\frac{1}{q}}. \quad (13)$$

Proof. Let $z_t = tb_1 + m(1-t)\frac{b_2}{k}$ and define

$$\varpi(t) = \begin{cases} t, & 0 \leq t \leq \frac{1}{2}, \\ 1-t, & \frac{1}{2} < t \leq 1. \end{cases}$$

By Lemma 1 and the triangle inequality,

$$\left| \frac{k}{mb_2 - kb_1} \int_{b_1}^{\frac{mb_2}{k}} \psi(u) du - \psi\left(\frac{b_1 + \frac{mb_2}{k}}{2}\right) \right| \leq \frac{mb_2 - kb_1}{k} \int_0^1 \varpi(t) |\psi'(z_t)| dt. \quad (14)$$

Hölder's inequality gives

$$\int_0^1 \varpi(t) |\psi'(z_t)| dt \leq \left(\int_0^1 \varpi(t)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |\psi'(z_t)|^q dt \right)^{\frac{1}{q}}. \quad (15)$$

A direct calculation yields

$$\int_0^1 \varpi(t)^p dt = \frac{1}{2^p(p+1)}, \quad \text{hence} \quad \left(\int_0^1 \varpi(t)^p dt \right)^{\frac{1}{p}} = \frac{1}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}}.$$

Furthermore, the m -exponential convexity of $|\psi'|^q$ implies

$$|\psi'(z_t)|^q \leq ts^{1-t} |\psi'(b_1)|^q + m(1-t)s^t \left| \psi' \left(\frac{b_2}{k} \right) \right|^q.$$

Integration over $[0, 1]$ and (7) give

$$\int_0^1 |\psi'(z_t)|^q dt \leq \Lambda_s \left(|\psi'(b_1)|^q + m \left| \psi' \left(\frac{b_2}{k} \right) \right|^q \right).$$

Substitution into (15) and then into (14) proves the assertion. $\square$

Corollary 1. Let $0 < k \leq 1$, $s > 0$, and $0 < b_1 < b_2$. Let $\psi : [b_1, \frac{b_2}{k}] \rightarrow \mathbb{R}$ be differentiable on $(b_1, \frac{b_2}{k})$, with $\psi' \in L_1 \left([b_1, \frac{b_2}{k}] \right)$. Assume that $|\psi'|^q$ is exponentially convex with respect to s on $[b_1, \frac{b_2}{k}]$, where $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left| \frac{k}{b_2 - kb_1} \int_{b_1}^{\frac{b_2}{k}} \psi(u) du - \psi \left(\frac{b_1 + \frac{b_2}{k}}{2} \right) \right| \leq \frac{b_2 - kb_1}{2k} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\Lambda_s \left(|\psi'(b_1)|^q + \left| \psi' \left(\frac{b_2}{k} \right) \right|^q \right) \right]^{\frac{1}{q}}.$$

Theorem 2. Let $0 < k \leq 1$, $m \in (0, 1]$, $s > 0$, and let $0 < b_1 < b_2$ satisfy $kb_1 < mb_2$. Let $\psi : [b_1, \frac{b_2}{k}] \rightarrow \mathbb{R}$ be differentiable on $(b_1, \frac{b_2}{k})$, with $\psi' \in L_1 \left([b_1, \frac{b_2}{k}] \right)$. Assume that $|\psi'|^q$ is m -exponentially convex with respect to s on $[b_1, \frac{b_2}{k}]$ for some $q \geq 1$. Then

$$\begin{aligned} & \left| \frac{k}{mb_2 - kb_1} \int_{b_1}^{\frac{mb_2}{k}} \psi(u) du - \psi \left(\frac{b_1 + \frac{mb_2}{k}}{2} \right) \right| \\ & \leq \frac{mb_2 - kb_1}{k} \left(\frac{1}{4} \right)^{1-\frac{1}{q}} \left[\Omega_s \left(|\psi'(b_1)|^q + m \left| \psi' \left(\frac{b_2}{k} \right) \right|^q \right) \right]^{\frac{1}{q}}. \end{aligned} \quad (16)$$

Proof. Using the notation from the proof of Theorem 1, the power-mean inequality gives

$$\int_0^1 \varpi(t) |\psi'(z_t)| dt \leq \left(\int_0^1 \varpi(t) dt \right)^{1-\frac{1}{q}} \left(\int_0^1 \varpi(t) |\psi'(z_t)|^q dt \right)^{\frac{1}{q}}. \quad (17)$$

Here $\int_0^1 \varpi(t) dt = \frac{1}{4}$, so that the first factor equals $\left(\frac{1}{4} \right)^{1-\frac{1}{q}}$. Since

$$\int_0^1 \varpi(t) |\psi'(z_t)|^q dt \leq |\psi'(b_1)|^q \int_0^1 \varpi(t) ts^{1-t} dt + m \left| \psi' \left(\frac{b_2}{k} \right) \right|^q \int_0^1 \varpi(t) (1-t)s^t dt,$$

formula (8) yields

$$\int_0^1 \varpi(t) |\psi'(z_t)|^q dt \leq \Omega_s \left(|\psi'(b_1)|^q + m \left| \psi' \left(\frac{b_2}{k} \right) \right|^q \right).$$

Combining this estimate with Lemma 1 proves (16). $\square$

Corollary 2. Let $0 < k \leq 1$, $s > 0$, and $0 < b_1 < b_2$. Let $\psi : \left[b_1, \frac{b_2}{k}\right] \rightarrow \mathbb{R}$ be differentiable on $\left(b_1, \frac{b_2}{k}\right)$, with $\psi' \in L_1\left(\left[b_1, \frac{b_2}{k}\right]\right)$. Assume that $|\psi'|^q$ is exponentially convex with respect to s on $\left[b_1, \frac{b_2}{k}\right]$ for some $q \geq 1$. Then

$$\left| \frac{k}{b_2 - kb_1} \int_{b_1}^{\frac{b_2}{k}} \psi(u) du - \psi\left(\frac{b_1 + \frac{b_2}{k}}{2}\right) \right| \leq \frac{b_2 - kb_1}{k} \left(\frac{1}{4}\right)^{1-\frac{1}{q}} \left[\Omega_s \left(|\psi'(b_1)|^q + \left| \psi'\left(\frac{b_2}{k}\right) \right|^q \right) \right]^{\frac{1}{q}}.$$

Remark 3. The two estimates require different exponent ranges, and the difference is intrinsic to the two inequalities used. Theorem 1 relies on Hölder's inequality and therefore needs a finite conjugate exponent: $q > 1$ and $p = q/(q-1) \in (1, \infty)$, so that $\left(\int_0^1 \varpi^p\right)^{1/p} = \frac{1}{2}(p+1)^{-1/p}$ is well defined; the case $q = 1$ is excluded there. Theorem 2 relies on the power-mean inequality and is valid for every $q \geq 1$; for $q = 1$ the factor $\left(\frac{1}{4}\right)^{1-1/q}$ equals 1 and the estimate reduces to (14) combined with $\int_0^1 \varpi(t) |\psi'(z_t)| dt \leq \Omega_s(|\psi'(b_1)| + m|\psi'(b_2/k)|)$. The pair $p = q = 2$ used in Section 5 and in Corollary 3 is admissible in both theorems, which is what makes the two certificates directly comparable there.

Corollary 3 (Comparison of the two certificates). Let $q = 2$ and let the assumptions of Theorems 1 and 2 hold simultaneously. Denote by $\mathcal{B}_H$ and $\mathcal{B}_P$ the right-hand sides of (13) and (16), respectively. Then

$$\frac{\mathcal{B}_P}{\mathcal{B}_H} = \sqrt{\frac{3\Omega_s}{\Lambda_s}} < 1$$

for every $s > 0$, and this ratio does not depend on k , m , b_1 , b_2 or ψ . Hence, for $q = 2$, the power-mean certificate is strictly sharper than the corresponding Hölder certificate for every admissible $s > 0$.

Proof. For $q = p = 2$ the two right-hand sides are $\frac{mb_2 - kb_1}{2k} \cdot 3^{-\frac{1}{2}} [\Lambda_s(\cdot)]^{\frac{1}{2}}$ and $\frac{mb_2 - kb_1}{2k} [\Omega_s(\cdot)]^{\frac{1}{2}}$, with the same bracket. Their quotient is $\sqrt{\frac{3\Omega_s}{\Lambda_s}}$, and Proposition 1(iii) gives the conclusion. $\square$

At $s = 1$ the estimates are evaluated directly with $\Lambda_1 = \frac{1}{2}$ and $\Omega_1 = \frac{1}{8}$; thus no division by a vanishing power of $\ln s$ is involved in the limiting case.

4 Membership criteria and quantitative refinements for m -exponential convexity

The next result provides a convenient criterion for constructing functions in $m\text{ECF}_s$ without first proving m -convexity.

Theorem 3. Let $R > 0$ and let $\psi : [0, R] \rightarrow [0, \infty)$ be L -Lipschitz continuous. Assume that there exists $\kappa_0 > 0$ such that $\psi(x) \geq \kappa_0 x$, $x \in [0, R]$, and $\psi(mx) \leq m\psi(x)$, $x \in [0, R]$. If

$$\ln s \geq \frac{2L}{\kappa_0}, \tag{18}$$

then $\psi \in m\text{ECF}_s([0, R])$.

Proof. Fix $x, y \in [0, R]$ and $t \in [0, 1]$, and let $z = tx + (1-t)my$. Because $m \in (0, 1]$ and the arguments are nonnegative, $0 \leq z \leq [t + m(1-t)]R \leq R$, so every value used below lies in the domain. The endpoint cases $t = 0$ and $t = 1$ are included. The Lipschitz property gives

$$\psi(z) \leq \psi(x) + L(1-t)|x - my|$$

and

$$\psi(z) \leq \psi(my) + Lt|x - my|.$$

Multiplying the first inequality by t , the second by $1-t$, and adding, we obtain

$$\begin{aligned} \psi(z) &\leq t\psi(x) + (1-t)\psi(my) + 2Lt(1-t)|x - my| \\ &\leq t\psi(x) + m(1-t)\psi(y) + 2Lt(1-t)|x - my|. \end{aligned} \quad (19)$$

Since (18) implies $s \geq 1$ (indeed $\ln s \geq 0$), the inequality $e^u - 1 \geq u$ for $u \geq 0$ gives

$$\begin{aligned} t(s^{1-t} - 1)\psi(x) + m(1-t)(s^t - 1)\psi(y) &\geq t(1-t)\ln s [\psi(x) + m\psi(y)] \\ &\geq \kappa_0 t(1-t)\ln s (x + my) \\ &\geq 2Lt(1-t)|x - my|. \end{aligned}$$

In the last step we used $x \geq 0$, $my \geq 0$, and hence $x + my \geq |x - my|$. Adding this estimate to the first two terms on the right-hand side of (19) proves (3). $\square$

Remark 4. Theorem 3 provides a sufficient condition only. It is not a characterization of $m\text{ECF}_s$, and the threshold (18) is in general not sharp. For instance, let $\psi(x) = x$ on $[0, R]$. The hypotheses hold with $L = \kappa_0 = 1$, so the criterion requires $s \geq e^2$; yet ψ is nonnegative and m -convex, and therefore $\psi \in m\text{ECF}_s([0, R])$ already for every $s \geq 1$. Determining the exact threshold

$$s^*(\psi, m) = \inf\{s > 0 : \psi \in m\text{ECF}_s\}$$

for a given ψ and m , and identifying the functions for which (18) is attained, remain open problems.

The factor 2 in (18) comes from adding the two one-sided Lipschitz estimates in the proof: each contributes a term $Lt(1-t)|x - my|$. It is therefore a sufficient proof constant and is not claimed to be optimal. More generally, if additional structure yields a sharper defect estimate

$$\psi(tx + m(1-t)y) \leq t\psi(x) + m(1-t)\psi(y) + Dt(1-t)|x - my|$$

with some $D < 2L$, the same argument gives the reduced sufficient threshold $\ln s \geq D/\kappa_0$. No universal improvement below $2L$ is asserted under the Lipschitz assumptions alone.

Example 1. Let

$$\psi(x) = x \left(2 + x + \frac{\sin(4\pi x)}{4\pi} \right), \quad x \in [0, 1]. \quad (20)$$

Then $\psi \in m\text{ECF}_s([0, 1])$ for every $m \in (0, 1]$ and every $s \geq e^6$. However, for $m = \frac{1}{2}$, the function is not m -convex.

Proof. Let $\psi(x) = x\eta(x)$, where

$$\eta(x) = 2 + x + \frac{\sin(4\pi x)}{4\pi}.$$

Since $\eta'(x) = 1 + \cos(4\pi x) \geq 0$, the function η is nondecreasing. Consequently,

$$\psi(mx) = mx\eta(mx) \leq mx\eta(x) = m\psi(x).$$

Moreover, $\psi(x) \geq \left(2 - \frac{1}{4\pi}\right)x > \frac{19}{10}x$. A direct differentiation gives $\psi'(x) = 2 + 2x + \frac{\sin(4\pi x)}{4\pi} + x\cos(4\pi x)$, so $|\psi'(x)| < 51/10$ on $[0, 1]$. Hence, $2L/\kappa_0 < 6$; explicitly, the admissible choices $L = \frac{51}{10}$ and $\kappa_0 = \frac{19}{10}$ give

$$\frac{2L}{\kappa_0} = \frac{102}{19} \approx 5.368421 < 6$$

, and Theorem 3 implies that $\psi \in m\text{ECF}_s([0, 1])$ whenever $s \geq e^6$.

To show that ψ is not $\frac{1}{2}$ -convex, take $x = \frac{3}{4}, y = \frac{1}{2}, t = \frac{7}{8}, m = \frac{1}{2}$. Then $tx + m(1-t)y = \frac{11}{16}$, and

$$\psi\left(\frac{11}{16}\right) - \left[\frac{7}{8}\psi\left(\frac{3}{4}\right) + \frac{1}{2} \cdot \frac{1}{8}\psi\left(\frac{1}{2}\right)\right] = \frac{11\sqrt{2}}{128\pi} - \frac{9}{256} \approx 3.529217 \cdot 10^{-3} > 0.$$

Thus, the defining inequality of $\frac{1}{2}$ -convexity fails. $\square$

This example establishes specifically that membership in $m\text{ECF}_s$ does not imply m -convexity (here for $m = \frac{1}{2}$); no separate claim about ordinary convexity is needed from this example.

The next criterion covers the case $m = 1$ on a compact interval not necessarily contained in $[0, \infty)$.

Proposition 2. Let $I = [a, b]$, $\psi : I \rightarrow (0, \infty)$ be L -Lipschitz continuous, and suppose that $\psi(x) \geq \kappa_1 > 0$ on I . If

$$\ln s \geq \frac{L(b-a)}{\kappa_1},$$

then $\psi \in 1\text{ECF}_s(I)$.

Proof. For $z = tx + (1-t)y$, the same weighted Lipschitz argument as in the proof of Theorem 3 gives

$$\psi(z) \leq t\psi(x) + (1-t)\psi(y) + 2Lt(1-t)|x-y|.$$

On the other hand, since $|x-y| \leq b-a$, the condition $\ln s \geq \frac{L(b-a)}{\kappa_1}$ implies $\kappa_1 \ln s \geq L|x-y|$.

$$t(s^{1-t} - 1)\psi(x) + (1-t)(s^t - 1)\psi(y) \geq 2\kappa_1 t(1-t) \ln s \geq 2Lt(1-t)|x-y|.$$

Combining the two estimates proves the claim. $\square$

The constants κ_0 and κ_1 are lower-bound certificates, not traffic or fitting parameters. When the indicated infima are positive, the largest admissible choices are

$$\kappa_0^{\max} = \inf_{x \in (0, R]} \frac{\psi(x)}{x}, \quad \kappa_1^{\max} = \min_{x \in I} \psi(x),$$

respectively. Any smaller positive value is also admissible, but it produces a more conservative sufficient threshold for s .

Example 2. The function $\chi(x) = 3 - x^2$ on $[-1, 1]$ is positive and strictly concave, hence not convex. It satisfies $\chi(x) \geq 2$ and is 2-Lipschitz. Therefore, Proposition 2 gives

$$\chi \in \text{IECF}_s([-1, 1]) \quad \text{for every } s \geq e^2.$$

Thus, even the case $m = 1$ contains positive nonconvex functions when $s > 1$.

In contrast with Example 1, this example establishes failure of ordinary convexity: membership in IECF_s for $s > 1$ does not imply classical convexity.

The proof of Theorem 3 also provides a quantitative estimate of the difference between the right-hand side of the defining inequality and the value at the corresponding m -interpolated point.

Theorem 4. Under the assumptions of Theorem 3, for every $x, y \in [0, R]$ and $t \in [0, 1]$,

$$\psi(tx + m(1-t)y) \leq ts^{1-t}\psi(x) + m(1-t)s^t\psi(y) - t(1-t)[\kappa_0 \ln s(x + my) - 2L|x - my|]. \quad (21)$$

Moreover,

$$\psi(tx + m(1-t)y) \leq ts^{1-t}\psi(x) + m(1-t)s^t\psi(y) - 4Lt(1-t)\min\{x, my\}. \quad (22)$$

Proof. As established in the proof of Theorem 3,

$$\psi(tx + m(1-t)y) \leq t\psi(x) + m(1-t)\psi(y) + 2Lt(1-t)|x - my|. \quad (23)$$

Furthermore,

$$t(s^{1-t} - 1)\psi(x) + m(1-t)(s^t - 1)\psi(y) \geq \kappa_0 t(1-t) \ln s(x + my).$$

Combining the last inequality with (23) proves (21).

Since $\kappa_0 \ln s \geq 2L$,

$$\begin{aligned} \kappa_0 \ln s(x + my) - 2L|x - my| &\geq 2L(x + my - |x - my|) \\ &= 4L\min\{x, my\}, \end{aligned}$$

which proves (22). $\square$

The correction term appearing in (21) is not merely smaller in the numerical examples reported later: under the hypotheses of Theorem 3 it is nonnegative for all admissible data, so the refinement is unconditional. We state this explicitly.

Corollary 4. Under the assumptions of Theorem 3, define the baseline m -exponential upper bound and the correction term by

$$\mathcal{U}_0(x, y, t) = ts^{1-t}\psi(x) + m(1-t)s^t\psi(y), \quad (24)$$

$$\mathcal{R}(x, y, t) = t(1-t)[\kappa_0 \ln s(x + my) - 2L|x - my|]. \quad (25)$$

Then, for all $x, y \in [0, R]$ and all $t \in [0, 1]$,

$$\mathcal{R}(x, y, t) \geq 4Lt(1-t)\min\{x, my\} \geq 0, \quad (26)$$

and consequently

$$\psi(tx + m(1-t)y) \leq \mathcal{U}_0(x, y, t) - \mathcal{R}(x, y, t) \leq \mathcal{U}_0(x, y, t). \quad (27)$$

The second inequality in (27) is strict whenever $t \in (0, 1)$, $L > 0$ and $\min\{x, my\} > 0$. Moreover, whenever $\mathcal{U}_0(x, y, t) > 0$, the relative improvement is well defined and admits the a priori lower bound

$$\frac{\mathcal{R}(x, y, t)}{\mathcal{U}_0(x, y, t)} \geq \frac{4Lt(1-t)\min\{x, my\}}{ts^{1-t}\psi(x) + m(1-t)s^t\psi(y)}, \quad (28)$$

which depends only on the data of Theorem 3 and not on any convexity assumption.

Proof. Since $x \geq 0$ and $my \geq 0$, one has $x + my - |x - my| = 2\min\{x, my\}$. Condition (18) gives $\kappa_0 \ln s \geq 2L$, hence

$$\kappa_0 \ln s (x + my) - 2L|x - my| \geq 2L(x + my - |x - my|) = 4L\min\{x, my\} \geq 0.$$

Multiplying by $t(1-t) \geq 0$ yields (26); combining it with (21) gives (27). Whenever $\mathcal{U}_0(x, y, t) > 0$, division of (26) by $\mathcal{U}_0(x, y, t)$ gives (28). Strictness in (27) is immediate, since all three factors $t(1-t)$, L and $\min\{x, my\}$ are then positive. $\square$

Remark 5. Corollary 4 identifies the precise sense in which the parameter s produces a genuine gain: the threshold (18) is exactly what converts the Lipschitz defect $2Lt(1-t)|x - my|$ in (19) into a surplus. The numerical tables of Section 5 therefore quantify a theorem rather than replace one.

Here and throughout the paper, “sharper” means pointwise smaller as an upper bound under the stated assumptions; no best-possible or optimal-error claim is intended. Moreover, (26) shows the precise relation between the two correction forms: the correction $\mathcal{R}(x, y, t)$ in (21) is at least $4Lt(1-t)\min\{x, my\}$. Hence subtracting $\mathcal{R}$ gives an upper bound no larger than the simpler $\min\{x, my\}$ version in (22).

Corollary 5. Under the assumptions of Theorem 3,

$$\psi\left(\frac{x + my}{2}\right) \leq \frac{\sqrt{s}}{2}(\psi(x) + m\psi(y)) - \frac{1}{4}[\kappa_0 \ln s (x + my) - 2L|x - my|].$$

In particular,

$$\psi\left(\frac{x + my}{2}\right) \leq \frac{\sqrt{s}}{2}(\psi(x) + m\psi(y)) - L\min\{x, my\}.$$

Proof. Take $t = \frac{1}{2}$ in Theorem 4. $\square$

5 Application to traffic congestion and controlled link-cost models

Volume–delay functions are fundamental components of static traffic assignment models. They relate the flow on a road link to its travel time and are used in the evaluation of both user-equilibrium and system-optimal traffic assignments. The Bureau of Public Roads (BPR) function remains one of the most widely used examples, while alternative functions, including conical volume–delay functions, have

been proposed to improve its analytical and empirical behavior; see [6, 13, 16]. In a system-optimal formulation, the principal objective is the total link cost

$$\sum_{e \in \mathcal{A}} x_e \tau_e(x_e),$$

where x_e is the flow and $\tau_e(x_e)$ is the corresponding link travel time; see [3, 10, 12].

The aim of this section is threefold. First, we show that the classical BPR total-travel-time function belongs to $m\text{ECF}_s$ under an explicit condition. Second, we introduce a controlled saturating link-cost model for which the ordinary convexity and m -convexity assumptions may fail, whereas the Lipschitz-scaling criterion still applies. Third, we derive numerical certificates and a rigorous upper-bound certificate for a specified feasible flow under attenuated transitions between traffic-demand scenarios; this is not an optimization algorithm or approximation-ratio result.

Remark 6. *The parameter values used throughout this section are illustrative. The controlled saturating model (35) introduced below has not been calibrated or validated against field traffic data, and the numerical results reported here are intended as an analytical illustration of the certificates rather than as an empirical assessment of link-cost models. The BPR function is used as a benchmark for the applicability of the membership criterion of Theorem 3; no claim of empirical superiority over the BPR function or over conical volume–delay functions is made or implied. The comparison drawn in Remark 8 is a comparison of admissible model classes, not of empirical fit. Finally, the roles of the two parameters are kept separate throughout: $m \in (0, 1]$ represents attenuation or partial realization in the interpolated flow/demand construction, whereas s is a mathematical certification parameter selected to satisfy sufficient inequalities such as (18), and is not an empirically calibrated traffic parameter.*

5.1 The classical BPR benchmark

Let $x \in [0, R]$ denote the flow on a road link, let $\tau_0 > 0$ be its free-flow travel time, let $c > 0$ be its practical capacity, and let $\alpha > 0$ and $\beta > 0$ be calibration parameters. The classical BPR volume–delay function is

$$\tau_{\text{BPR}}(x) = \tau_0 \left[1 + \alpha \left(\frac{x}{c} \right)^\beta \right]. \quad (29)$$

The total travel time accumulated by all vehicles on the link is therefore

$$\mathcal{C}_{\text{BPR}}(x) = x \tau_{\text{BPR}}(x) = \tau_0 x \left[1 + \alpha \left(\frac{x}{c} \right)^\beta \right]. \quad (30)$$

Proposition 3. *Let $R > 0$, $m \in (0, 1]$, and let $\mathcal{C}_{\text{BPR}}$ be defined by (30). If*

$$\ln s \geq 2 \left[1 + \alpha(\beta + 1) \left(\frac{R}{c} \right)^\beta \right], \quad (31)$$

then

$$\mathcal{C}_{\text{BPR}} \in m\text{ECF}_s([0, R]).$$

Moreover, for every $x, y \in [0, R]$ and $t \in [0, 1]$,

$$\mathcal{C}_{\text{BPR}}(tx + m(1-t)y) \leq ts^{1-t} \mathcal{C}_{\text{BPR}}(x) + m(1-t)s^t \mathcal{C}_{\text{BPR}}(y)$$

$$-t(1-t) \left[\tau_0 \ln s(x+my) - 2L_{\text{BPR}}|x-my| \right], \quad (32)$$

where

$$L_{\text{BPR}} = \tau_0 \left[1 + \alpha(\beta+1) \left(\frac{R}{c} \right)^\beta \right]. \quad (33)$$

Proof. Since $\mathcal{C}_{\text{BPR}}(x) \geq \tau_0 x$, the linear lower bound in Theorem 3 holds with $\kappa_0 = \tau_0$. Furthermore,

$$\mathcal{C}_{\text{BPR}}(mx) = \tau_0 mx \left[1 + \alpha m^\beta \left(\frac{x}{c} \right)^\beta \right] \leq m \tau_0 x \left[1 + \alpha \left(\frac{x}{c} \right)^\beta \right] = m \mathcal{C}_{\text{BPR}}(x).$$

A direct differentiation gives

$$\mathcal{C}'_{\text{BPR}}(x) = \tau_0 \left[1 + \alpha(\beta+1) \left(\frac{x}{c} \right)^\beta \right].$$

Hence $\mathcal{C}_{\text{BPR}}$ is L_{BPR} -Lipschitz continuous on $[0, R]$. Condition (31) is precisely

$$\ln s \geq \frac{2L_{\text{BPR}}}{\tau_0}.$$

The membership assertion follows from Theorem 3, while (32) follows from Theorem 4. $\square$

Remark 7. For $\beta > 0$, the function $\mathcal{C}_{\text{BPR}}$ is convex and, because $\mathcal{C}_{\text{BPR}}(0) = 0$, it is also *m*-convex. Consequently, Proposition 3 demonstrates applicability to a standard traffic model, but it does not yet exhibit any advantage of *m*-exponential convexity over *m*-convexity in terms of the admissible model class. This issue motivates the controlled link-cost model introduced next.

5.2 A controlled saturating link-cost model

The classical BPR model predicts an unbounded increase in travel time as the volume–capacity ratio increases. This behavior is appropriate for many static-assignment applications, but it may be overly rigid when the modeled quantity is a generalized link cost under active management, for example when ramp metering, demand management, rerouting, or a bounded operational penalty limits the additional cost assigned to a link. We therefore consider the following grey-box model, in the spirit of the grey-box formulations reviewed in [7, 9]. We distinguish explicitly between the saturating per-unit response and the total link cost:

$$\tau_\theta(x) = \tau_0 + \delta (1 - e^{-\gamma x}), \quad (34)$$

and the associated total cost is

$$\mathcal{C}_\theta(x) = x [\tau_0 + \delta (1 - e^{-\gamma x})], \quad \theta = (\tau_0, \delta, \gamma), \quad (35)$$

where $\tau_0 > 0, \delta > 0, \gamma > 0$. Here $\tau_0 x$ is the free-flow contribution, δ is the maximal additional cost per unit flow, and γ controls how rapidly the additional contribution approaches saturation. The saturation refers to τ_θ : one has $\tau_0 \leq \tau_\theta(x) < \tau_0 + \delta$. The total cost $\mathcal{C}_\theta(x) = x \tau_\theta(x)$ is not bounded; in fact $\mathcal{C}_\theta(x) \sim$

$(\tau_0 + \delta)x$ as $x \rightarrow \infty$. The precise form (35) is proposed in this paper; it should be viewed as a controlled or context-specific alternative rather than as a replacement for the BPR function in every traffic regime.

For $R > 0$, define

$$r_\gamma = \min \left\{ R, \frac{2}{\gamma} \right\} \quad \text{and} \quad L_{\theta,R} = \tau_0 + \delta [1 - e^{-\gamma r_\gamma} + \gamma r_\gamma e^{-\gamma r_\gamma}]. \quad (36)$$

Theorem 5. Let $\mathcal{C}_\theta$ be defined by (35) on $[0, R]$. For every $m \in (0, 1]$, if

$$\ln s \geq \frac{2L_{\theta,R}}{\tau_0}, \quad (37)$$

then

$$\mathcal{C}_\theta \in m\text{ECF}_s([0, R]).$$

More precisely, for all $x, y \in [0, R]$ and $t \in [0, 1]$,

$$\begin{aligned} \mathcal{C}_\theta(tx + m(1-t)y) &\leq ts^{1-t}\mathcal{C}_\theta(x) + m(1-t)s^t\mathcal{C}_\theta(y) \\ &\quad - t(1-t) \left[\tau_0 \ln s (x + my) - 2L_{\theta,R}|x - my| \right]. \end{aligned} \quad (38)$$

Proof. The lower estimate $\mathcal{C}_\theta(x) \geq \tau_0 x$ holds for every $x \in [0, R]$. Moreover, since the function $x \mapsto 1 - e^{-\gamma x}$ is increasing, with derivative $\gamma e^{-\gamma x} \geq 0$ for $x \geq 0$,

$$\mathcal{C}_\theta(mx) = mx [\tau_0 + \delta (1 - e^{-\gamma mx})] \leq mx [\tau_0 + \delta (1 - e^{-\gamma x})] = m\mathcal{C}_\theta(x).$$

Differentiating (35), we obtain

$$\mathcal{C}'_\theta(x) = \tau_0 + \delta [1 - e^{-\gamma x} + \gamma x e^{-\gamma x}]. \quad (39)$$

A second differentiation yields

$$\mathcal{C}''_\theta(x) = \delta \gamma e^{-\gamma x} (2 - \gamma x). \quad (40)$$

Thus, $\mathcal{C}'_\theta$ increases on $[0, \frac{2}{\gamma}]$ and decreases on $[\frac{2}{\gamma}, \infty)$. Consequently,

$$\max_{x \in [0, R]} \mathcal{C}'_\theta(x) = \mathcal{C}'_\theta(r_\gamma) = L_{\theta,R}.$$

Therefore, $\mathcal{C}_\theta$ is $L_{\theta,R}$ -Lipschitz continuous. The result now follows from Theorems 3 and 4 with $\kappa_0 = \tau_0$. $\square$

Proposition 4. If $R > \frac{2}{\gamma}$, then $\mathcal{C}_\theta$ is not convex on $[0, R]$. Nevertheless, Theorem 5 shows that it belongs to $m\text{ECF}_s([0, R])$ for every $m \in (0, 1]$ and every s satisfying (37).

For the concrete parameters

$$\tau_0 = 1, \quad \delta = 0.2, \quad \gamma = 8, \quad m = \frac{1}{2}, \quad (41)$$

the function is not even $\frac{1}{2}$ -convex on $[0, 1]$.

Proof. By (40),

$$\mathcal{C}_\theta''(x) < 0 \quad \text{for } x > \frac{2}{\gamma}.$$

Hence $\mathcal{C}_\theta$ cannot be convex on $[0, R]$ whenever $R > \frac{2}{\gamma}$.

For the parameters in (41), let $x = 1, y = \frac{1}{8}, t = \frac{5}{8}$. Then

$$tx + m(1-t)y = \frac{83}{128},$$

and direct evaluation gives

$$\mathcal{C}_\theta\left(\frac{83}{128}\right) - \frac{5}{8}\mathcal{C}_\theta(1) - \frac{1}{2} \cdot \frac{3}{8}\mathcal{C}_\theta\left(\frac{1}{8}\right) \approx 1.041939 \cdot 10^{-3} > 0.$$

Therefore, the defining inequality of $\frac{1}{2}$ -convexity fails. $\square$

Remark 8. The preceding proposition identifies the main structural advantage of the present class. Classical convexity cannot be applied on the entire interval $[0, 1]$, and the $\frac{1}{2}$ -convex inequality fails explicitly. In contrast, the Lipschitz–scaling criterion provides a computable value of s for which the refined estimate (38) remains valid. The advantage identified here is strictly one of analytical applicability: the admissible model class is larger.

5.3 Numerical assessment of the quantitative refinement

We consider

$$\tau_0 = 1, \quad \delta = 0.2, \quad \gamma = 8, \quad R = 1, \quad m = 0.8. \quad (42)$$

Since $\frac{2}{\gamma} = 0.25 < R$, formula (36) gives

$$L_{\theta,1} = 1 + 0.2(1 + e^{-2}) \approx 1.227067. \quad (43)$$

We choose the smallest value allowed by (37):

$$s = e^{2L_{\theta,1}} \approx 11.636353. \quad (44)$$

This is the smallest value delivered by the sufficient criterion (37) for the chosen constants; it is neither an empirically calibrated traffic parameter nor a claim about the minimal membership threshold.

For $z = tx + m(1-t)y$, define the unrefined and refined upper estimates

$$\mathcal{U}_0(x, y, t) = ts^{1-t}\mathcal{C}_\theta(x) + m(1-t)s^t\mathcal{C}_\theta(y), \quad (45)$$

$$\mathcal{U}_{\text{ref}}(x, y, t) = \mathcal{U}_0(x, y, t) - t(1-t) \left[\ln s(x + my) - 2L_{\theta,1}|x - my| \right]. \quad (46)$$

By Corollary 4 applied with $\kappa_0 = \tau_0 = 1$, the subtracted term in (46) is nonnegative, so $\mathcal{U}_{\text{ref}} \leq \mathcal{U}_0$ holds for all admissible data and not only for the entries of Table 2. The reduction percentage is defined by

$$\mathfrak{R} = 100 \frac{\mathcal{U}_0 - \mathcal{U}_{\text{ref}}}{\mathcal{U}_0}.$$

In these four illustrative cases, the correction lowers the original *m*-exponential bound by approximately 27.6%–30.1%. The improvement is not obtained by assuming convexity: the same parameterized link-cost function is nonconvex on $[0, 1]$ and fails the $\frac{1}{2}$ -convexity test in Proposition 4.

Table 2: Illustrative effect of the quantitative Lipschitz correction for the controlled link-cost model

x	y	t	z	$\mathcal{C}_\theta(z)$	$\mathcal{U}_0$	$\mathcal{U}_{\text{ref}}$	$\mathfrak{R}$
0.80	1.00	0.50	0.800	0.959734	3.274217	2.292563	29.98%
0.60	0.75	0.50	0.600	0.719012	2.453879	1.717639	30.00%
0.40	0.50	0.50	0.400	0.476739	1.629320	1.138493	30.12%
0.90	1.00	0.40	0.840	1.007797	3.420482	2.478095	27.55%

5.4 Midpoint certification of an average controlled link cost

We next apply the midpoint estimates from Theorems 1 and 2. Since $\mathcal{C}'_\theta(0) = \tau_0 > 0$, the function $|\mathcal{C}'_\theta|^2$ cannot belong to $m\text{ECF}_s([0, R])$ for $m < 1$ when the interval contains zero : by Remark 1, membership would force $|\mathcal{C}'_\theta(0)|^2 = 0$, whereas $|\mathcal{C}'_\theta(0)|^2 = \tau_0^2 > 0$. Accordingly, we work on a strictly positive interval and use the case $m = 1$ together with Proposition 2.

Set

$$\Gamma_\theta(x) = |\mathcal{C}'_\theta(x)|^2. \quad (47)$$

Since $\mathcal{C}_\theta \in C^2(I)$, the function Γ_θ belongs to $C^1(I)$ and

$$\Gamma'_\theta(x) = 2\mathcal{C}'_\theta(x)\mathcal{C}''_\theta(x).$$

Consequently $|\Gamma'_\theta| \leq 2\max_I |\mathcal{C}'_\theta| \max_I |\mathcal{C}''_\theta|$, which justifies the Lipschitz constant used below. On the interval $I = [0.3, 1]$, the numerical parameters (42) give

$$\max_{x \in I} \mathcal{C}'_\theta(x) = \mathcal{C}'_\theta(0.3) \approx 1.225401, \quad \max_{x \in I} |\mathcal{C}''_\theta(x)| = |\mathcal{C}''_\theta(\frac{3}{8})| \approx 0.079659.$$

Hence

$$L_\Gamma = 2\max_{x \in I} \mathcal{C}'_\theta(x) \max_{x \in I} |\mathcal{C}''_\theta(x)| \approx 0.195229 \quad (48)$$

is a Lipschitz constant for Γ_θ . Moreover,

$$\kappa_\Gamma = \min_{x \in I} \Gamma_\theta(x) = \Gamma_\theta(1) \approx 1.441127. \quad (49)$$

Therefore, Proposition 2 applies on every subinterval $[0.3, b] \subseteq I$ if

$$\ln s_M \geq \frac{L_\Gamma(1 - 0.3)}{\kappa_\Gamma}. \quad (50)$$

Let $s_M = e^{\frac{0.7L_\Gamma}{\kappa_\Gamma}} \approx 1.099471$. For this value, $\Lambda_{s_M} \approx 0.516187$, $\Omega_{s_M} \approx 0.130058$. For $0.3 < b \leq 1$, define

$$\begin{aligned} \overline{\mathcal{C}}_\theta(0.3, b) &= \frac{1}{b - 0.3} \int_{0.3}^b \mathcal{C}_\theta(u) du, \\ \widehat{\mathcal{C}}_\theta^M(0.3, b) &= \mathcal{C}_\theta\left(\frac{0.3 + b}{2}\right). \end{aligned}$$

An antiderivative of $\mathcal{C}_\theta$ is

$$\mathcal{J}_\theta(x) = \frac{\tau_0 + \delta}{2} x^2 - \frac{\delta}{\gamma^2} [1 - e^{-\gamma x} (1 + \gamma x)]. \quad (51)$$

Thus,

$$\overline{\mathcal{C}}_\theta(0.3, b) = \frac{\mathcal{J}_\theta(b) - \mathcal{J}_\theta(0.3)}{b - 0.3}.$$

Applying Theorem 1 with $k = m = 1$ and $p = q = 2$, and Theorem 2 with $k = m = 1$ and $q = 2$, both admissible by Remark 3, gives

$$\begin{aligned} \left| \overline{\mathcal{C}}_\theta(0.3, b) - \widehat{\mathcal{C}}_\theta^M(0.3, b) \right| &\leq \mathcal{B}_H^M(0.3, b), \\ \left| \overline{\mathcal{C}}_\theta(0.3, b) - \widehat{\mathcal{C}}_\theta^M(0.3, b) \right| &\leq \mathcal{B}_P^M(0.3, b), \end{aligned}$$

where, writing out the numerical prefactors of (13) and (16) explicitly for $k = m = 1$, $b_1 = 0.3$, $b_2 = b$ and $p = q = 2$,

$$\mathcal{B}_H^M(0.3, b) = \underbrace{\frac{b - 0.3}{2}}_{(mb_2 - kb_1)/(2k)} \cdot \underbrace{\left(\frac{1}{p+1}\right)^{\frac{1}{p}}}_{=3^{-1/2}} \cdot \sqrt{\Lambda_{sM}(\Gamma_\theta(0.3) + \Gamma_\theta(b))} = \frac{b - 0.3}{2\sqrt{3}} \sqrt{\Lambda_{sM}(\Gamma_\theta(0.3) + \Gamma_\theta(b))}, \quad (52)$$

$$\mathcal{B}_P^M(0.3, b) = \underbrace{(b - 0.3)}_{(mb_2 - kb_1)/k} \cdot \underbrace{\left(\frac{1}{4}\right)^{1 - \frac{1}{q}}}_{=1/2} \cdot \sqrt{\Omega_{sM}(\Gamma_\theta(0.3) + \Gamma_\theta(b))} = \frac{b - 0.3}{2} \sqrt{\Omega_{sM}(\Gamma_\theta(0.3) + \Gamma_\theta(b))}. \quad (53)$$

The factor $\left(\frac{1}{4}\right)^{1 - \frac{1}{q}} = \frac{1}{2}$ of Theorem 2 is therefore contained in the denominator 2 of (53); it is not an additional normalization. As a worked check, for $b = 0.5$ one has $\Gamma_\theta(0.3) = 1.501608$ and $\Gamma_\theta(0.5) = 1.466495$, so that $\Gamma_\theta(0.3) + \Gamma_\theta(0.5) = 2.968103$. Note that a *square root* is taken of the product $\Omega_{sM}(\Gamma_\theta(0.3) + \Gamma_\theta(b))$ (respectively $\Lambda_{sM}(\dots)$), as required by $q = 2$ in (13)–(16). Step by step,

$$\begin{aligned} \mathcal{B}_P^M(0.3, 0.5) &= 0.1 \cdot (0.130058 \cdot 2.968103)^{1/2} = 0.1 \cdot (0.386026)^{1/2} \\ &= 0.1 \cdot 0.621310 = 0.062131, \\ \mathcal{B}_H^M(0.3, 0.5) &= \frac{0.2}{2\sqrt{3}} \cdot (0.516187 \cdot 2.968103)^{1/2} = 0.057735 \cdot (1.532095)^{1/2} \\ &= 0.057735 \cdot 1.237778 = 0.071463, \end{aligned}$$

in agreement with the first row of Table 3. The remaining rows are obtained in exactly the same way from (52)–(53); the intermediate quantities are listed in Table 4.

Here, the last column reports the relative excess of the certified power-mean upper estimate over the exact integral mean,

$$100 \left[\frac{\widehat{\mathcal{C}}_\theta^M + \mathcal{B}_P^M}{\overline{\mathcal{C}}_\theta} - 1 \right],$$

which is a measure of the total conservatism of the certificate and should not be confused with the relative gain of the power-mean radius over the Hölder radius, discussed next.

Table 3: Certified upper estimates for the average controlled link cost. The columns $\mathcal{B}_H^M$ and $\mathcal{B}_P^M$ report the certified radii themselves, so that their ratio can be read off directly.

b	$\overline{\mathcal{C}}_\theta$	$\widehat{\mathcal{C}}_\theta^M$	$\mathcal{B}_H^M$	$\mathcal{B}_P^M$	$\widehat{\mathcal{C}}_\theta^M + \mathcal{B}_H^M$	$\widehat{\mathcal{C}}_\theta^M + \mathcal{B}_P^M$	Certified overhead
0.50	0.476612	0.476739	0.071463	0.062131	0.548202	0.538870	13.06%
0.70	0.597781	0.598168	0.142485	0.123878	0.740653	0.722046	20.79%
0.90	0.718425	0.719012	0.213511	0.185629	0.932524	0.904642	25.92%
1.00	0.778637	0.779283	0.249050	0.216527	1.028333	0.995810	27.89%

Table 4: Intermediate quantities for the radii (52)–(53), with $\Gamma_\theta(0.3) = 1.501608$, $\Lambda_{s_M} = 0.516187$ and $\Omega_{s_M} = 0.130058$. Here $S_b = \Gamma_\theta(0.3) + \Gamma_\theta(b)$, $\mathcal{B}_P^M = \frac{b-0.3}{2} \sqrt{\Omega_{s_M} S_b}$ and $\mathcal{B}_H^M = \frac{b-0.3}{2\sqrt{3}} \sqrt{\Lambda_{s_M} S_b}$. All entries were computed from unrounded data and then rounded to six decimals.

b	$\Gamma_\theta(b)$	S_b	$\sqrt{\Omega_{s_M} S_b}$	$\mathcal{B}_P^M$	$\sqrt{\Lambda_{s_M} S_b}$	$\mathcal{B}_H^M$
0.50	1.466495	2.968103	0.621310	0.062131	1.237778	0.071463
0.70	1.448176	2.949784	0.619390	0.123878	1.233953	0.142485
0.90	1.442223	2.943830	0.618764	0.185629	1.232707	0.213511
1.00	1.441127	2.942735	0.618649	0.216527	1.232477	0.249050

By Corollary 3, the power-mean radius is uniformly smaller than the Hölder radius, and the ratio is independent of b :

$$\frac{\mathcal{B}_P^M}{\mathcal{B}_H^M} = \sqrt{\frac{3\Omega_{s_M}}{\Lambda_{s_M}}} \approx 0.869413. \quad (54)$$

Explicitly, $3\Omega_{s_M}/\Lambda_{s_M} = 3 \cdot 0.130058/0.516187 = 0.755879$ and $\sqrt{0.755879} = 0.869413$, so that $100(1 - \sqrt{3\Omega_{s_M}/\Lambda_{s_M}}) = 13.0587\%$. Thus, for the present value of s_M , the power-mean result reduces the certified radius by approximately 13.06% for every interval in the table; dividing the two radius columns of Table 3 row by row gives the same value 0.869413. We note that the certified overhead in the first row of Table 3 (13.0627%) also rounds to 13.06%; this is a numerical coincidence between two different quantities.

Comparison with the classical C^2 midpoint estimate. The bounds in Table 3 are deliberately first-derivative/generalized-convexity certificates: they use only the endpoint values of $\Gamma_\theta = |\mathcal{C}'_\theta|^2$ together with membership in 1ECF_{s_M} , and no curvature information. Because the present illustrative cost function happens to be C^2 , it is informative to compare them with the standard curvature-based midpoint estimate (see, e.g., [4])

$$\left| \frac{1}{b-a} \int_a^b f(u) du - f\left(\frac{a+b}{2}\right) \right| \leq \frac{(b-a)^2}{24} \max_{u \in [a,b]} |f''(u)|. \quad (55)$$

For the present parameters and every $b \in \{0.5, 0.7, 0.9, 1\}$, the interval $[0.3, b]$ contains $\frac{3}{8}$, and

$$\max_{u \in [0.3, b]} |\mathcal{C}''_\theta(u)| = 1.6e^{-3} \approx 0.0796593.$$

In this smooth example the classical estimate turns out to be substantially tighter than either certificate, as the following table shows. This is not a contradiction, and no numerical superiority over curvature-based quadrature is claimed: Theorems 1–2 address a different information regime, in which a convenient

uniform second-derivative error estimate need not be available. The comparison is reported in order to make the conservatism of the present certificates explicit.

Table 5: Comparison with the classical second-derivative midpoint bound in the smooth illustrative example. The exact midpoint error $|\mathcal{C}_\theta - \hat{\mathcal{C}}_\theta^M|$ is computed from unrounded values.

b	exact midpoint error	$\mathcal{B}_2^M$	$\mathcal{B}_p^M$	$\mathcal{B}_H^M$
0.50	0.000128	0.000133	0.062131	0.071463
0.70	0.000387	0.000531	0.123878	0.142485
0.90	0.000587	0.001195	0.185629	0.213511
1.00	0.000646	0.001626	0.216527	0.249050

5.5 A Certified Demand-Transition Bound for System-Optimal Assignment

Let $G = (\mathcal{V}, \mathcal{A})$ be a directed network and let B be its node–arc incidence matrix. For a demand vector d , define

$$\mathcal{F}(d) = \left\{ x \in \mathbb{R}_+^{|\mathcal{A}|} : Bx = d, \quad 0 \leq x_e \leq R_e \right\}. \quad (56)$$

Let each link $e \in \mathcal{A}$ have a nonnegative total-cost function $\mathcal{C}_e$ satisfying the assumptions of Theorem 3 with constants κ_e, L_e, s_e . The system-optimal value is

$$V(d) = \min_{x \in \mathcal{F}(d)} \sum_{e \in \mathcal{A}} \mathcal{C}_e(x_e). \quad (57)$$

Theorem 6. Let $x \in \mathcal{F}(d^{(1)})$, $y \in \mathcal{F}(d^{(2)})$, and let $m \in (0, 1]$ and $t \in [0, 1]$. Define

$$\begin{aligned} z &= tx + m(1-t)y, \\ d_t &= td^{(1)} + m(1-t)d^{(2)}. \end{aligned}$$

Then $z \in \mathcal{F}(d_t)$ and

$$V(d_t) \leq \sum_{e \in \mathcal{A}} \left[ts_e^{1-t} \mathcal{C}_e(x_e) + m(1-t)s_e^t \mathcal{C}_e(y_e) \right] - t(1-t) \sum_{e \in \mathcal{A}} \left[\kappa_e \ln s_e (x_e + my_e) - 2L_e |x_e - my_e| \right]. \quad (58)$$

Moreover, by Corollary 4 every summand of the second sum is nonnegative, so the right-hand side of (58) never exceeds the unrefined bound obtained by deleting that sum.

Proof. Linearity of the flow-conservation constraints gives

$$Bz = tBx + m(1-t)By = td^{(1)} + m(1-t)d^{(2)} = d_t.$$

The box constraints are preserved because

$$0 \leq tx_e + m(1-t)y_e \leq [t + m(1-t)]R_e \leq R_e.$$

Thus, $z \in \mathcal{F}(d_t)$, and consequently

$$V(d_t) \leq \sum_{e \in \mathcal{A}} \mathcal{C}_e(z_e).$$

Applying Theorem 4 to every link and summing the resulting inequalities proves (58). $\square$

Remark 9. The right-hand side of (58) is a valid upper bound for $V(d_t)$ for the following reason only: it bounds $\sum_e \mathcal{C}_e(z_e)$ for the single specified feasible flow z , and $z \in \mathcal{F}(d_t)$ implies $V(d_t) \leq \sum_e \mathcal{C}_e(z_e)$. The summation over links is legitimate because every linkwise inequality of Theorem 4 is evaluated at the same z and all the terms are finite; no interaction between links is neglected, since the objective is separable. Accordingly, (58) is a certificate, not an optimization result: it yields neither optimality of z , nor an algorithm, nor a guaranteed approximation ratio.

Remark 10. Theorem 6 remains applicable even when the system objective is nonconvex. It does not assert that the transition flow z is globally optimal. Instead, it provides a feasible flow for the attenuated demand scenario d_t , together with a computable upper certificate for the corresponding system-optimal value. Such a construction can be used for scenario screening, warm starting, and continuation between operating-demand regimes.

5.6 A two-link case study in system-optimal traffic assignment

Consider two parallel links with total-cost functions of the form (35). Their parameters are

$$(\tau_{0,1}, \delta_1, \gamma_1) = (1, 0.2, 8), \quad (\tau_{0,2}, \delta_2, \gamma_2) = (1.05, 0.15, 5). \quad (59)$$

For a scalar demand d , the system-optimal problem reduces to

$$V(d) = \min_{0 \leq u \leq d} \varphi_d(u), \quad \varphi_d(u) = \mathcal{C}_1(u) + \mathcal{C}_2(d - u). \quad (60)$$

The reduced objective φ_d is *not* unimodal, so a stationary point or the output of a local solver is not by itself evidence of global optimality. We therefore certify the global minimum by the following elementary but rigorous procedure, which uses the same Lipschitz data as the rest of the paper.

Remark 11 (Certified global minimization). By (36), $\mathcal{C}_1$ is L_1 -Lipschitz and $\mathcal{C}_2$ is L_2 -Lipschitz on $[0, 1]$, with

$$L_1 \approx 1.227067, L_2 \approx 1.220300, L_1 + L_2 \approx 2.447367.$$

Hence φ_d is $(L_1 + L_2)$ -Lipschitz on $[0, d]$, and for a uniform grid $u_i = ih$, $i = 0, \dots, N$, $h = \frac{d}{N}$, one has the two-sided enclosure

$$\min_{0 \leq i \leq N} \varphi_d(u_i) - \frac{(L_1 + L_2)h}{2} \leq V(d) \leq \max_{0 \leq i \leq N} \varphi_d(u_i). \quad (61)$$

With $N = 4 \cdot 10^6$ the width of (61) is below $3 \cdot 10^{-7}$. The enclosure (61) alone constitutes the certificate of the global minimum value: it requires only the Lipschitz constants and finitely many function evaluations, and does not depend on locating stationary points. The procedure described next serves only as an independent numerical check. All interior stationary points were located by detecting sign changes of $\varphi'_d(u) = \mathcal{C}'_1(u) - \mathcal{C}'_2(d - u)$ on a grid of $2 \cdot 10^5$ subintervals. Each sign-changing bracket was refined by bisection until its width was below 10^{-12} ; Newton iteration from the bracket midpoint was used as an independent check, with residual $|\varphi'_d(u)| < 10^{-14}$. The objective and boundary values were then evaluated in 30-digit arithmetic. The two procedures agree in all cases reported below.

For $d = 0.73$ and $d = 0.9$ the local maximizers separating the two local minima are $u = 0.379304$ and $u = 0.395132$, respectively; they are omitted from Table 6.

Table 6: Stationary points of φ_d and certified global minima. For $d = 0.73$ and $d = 0.9$ the objective has three interior stationary points, and in both cases the larger of the two local minimizers is not global. The last column gives the lower and upper bounds of (61) with $N = 4 \cdot 10^6$, rounded outward to seven decimals.

d	stationary and boundary points $(u, \varphi_d(u))$				certified enclosure of $V(d)$
0.70	(0, 0.836829)	(0.165853, 0.825655)	—	(0.7, 0.839482)	[0.8256544, 0.8256547]
0.73	(0, 0.873154)	(0.161732, 0.862157)	(0.461511, 0.863180)	(0.73, 0.875575)	[0.8621564, 0.8621568]
0.90	(0, 1.078500)	(0.143776, 1.068311)	(0.685485, 1.068422)	(0.9, 1.079866)	[1.0683107, 1.0683110]

Let $d^{(1)} = 0.9, d^{(2)} = 0.7, m = 0.8, t = \frac{1}{2}$. Certified global minimization as in Remark 11 gives

$$\begin{aligned} x^* &= (0.143776, 0.756224), & V(0.9) &\approx 1.068311, \\ y^* &= (0.165853, 0.534147), & V(0.7) &\approx 0.825655. \end{aligned}$$

The attenuated transition demand and the associated feasible flow are

$$d_t = 0.73, \quad \text{and} \quad z = \frac{1}{2}x^* + 0.4y^* = (0.138229, 0.591771).$$

For this candidate, $\mathcal{C}_1(z_1) + \mathcal{C}_2(z_2) = 0.8622460137$ (ten decimals). Certified global minimization at demand $d_t = 0.73$, i.e. (61) with $N = 4 \cdot 10^6$, gives

$$0.8621564792 \leq V(0.73) \leq 0.8621567025.$$

Since the relative gap is decreasing in $V(0.73)$, it is enclosed by

$$\begin{aligned} 0.010359\% &= 100 \left(\frac{0.8622460137}{0.8621567025} - 1 \right) \leq 100 \left(\frac{\mathcal{C}_1(z_1) + \mathcal{C}_2(z_2)}{V(0.73)} - 1 \right) \\ &\leq 100 \left(\frac{0.8622460137}{0.8621564792} - 1 \right) = 0.010385\%. \end{aligned}$$

Both bounds round to 0.0104%, so the transition flow is 0.0104% above the certified global optimum. (Computing the percentage from the six-digit values 0.862246 and 0.862157 instead gives 0.0103%; the difference is due only to that rounding.) This 0.0104% figure is specific to the present two-link data and is not implied by Theorem 6.

For the network certificate (58) we take $\kappa_e = \tau_{0,e}, L_e$ as in Remark 11, and the smallest values allowed by (37),

$$s_1 = e^{2L_1/\tau_{0,1}} \approx 11.636353, \quad s_2 = e^{2L_2/\tau_{0,2}} \approx 10.220357.$$

The unrefined sum of the two linkwise *m*-exponential bounds equals approximately 2.797295, whereas the quantitative Lipschitz correction lowers the network certificate to 2.113029. The reduction in the certified upper value is therefore approximately 24.46%.

This example illustrates two complementary benefits. First, the attenuated demand-transition rule generates a feasible flow whose objective value is close to the certified global optimum. Second, the quantitative remainder produces a strictly sharper rigorous certificate than the unrefined *m*-exponential estimate, in accordance with Corollary 4.

Remark 12. All numerical values reported in Section 5 were computed in 30-digit arithmetic, and every tabulated quantity is reported to a number of digits well within that accuracy. A script reproducing Tables 2–6, the constants Λ_s, Ω_s , the enclosures (61) and the counterexamples of Example 1 and Proposition 4 is provided as supplementary material.

6 Conclusion

In this paper, several new results have been established for the recently introduced class of m -exponentially convex functions with respect to a parameter s . Starting from a midpoint integral identity, we derived two first-derivative estimates based on Hölder's inequality and the power-mean inequality. The corresponding results for exponential convexity with respect to s were obtained as direct consequences of the case $m = 1$. In addition, the constants Λ_s and Ω_s generated by these estimates were shown to satisfy $3\Omega_s < \Lambda_s$ for every $s > 0$, so that for $q = 2$ the power-mean certificate is strictly sharper than the corresponding Hölder certificate for every admissible $s > 0$.

A Lipschitz-scaling criterion was then developed as a practical sufficient condition for membership in $mECF_s$; it is sufficient but not necessary, and the exact membership threshold remains open (Remark 4). The separation examples show that this criterion applies to functions for which m -convexity fails (Example 1) and, in a separate instance, to functions for which ordinary convexity fails (Proposition 4). Moreover, the quantitative version of the criterion yields an explicit correction term which was proved to be nonnegative under the hypotheses of the criterion (Corollary 4); the refinement therefore improves the basic m -exponential upper estimate unconditionally, and not merely in the reported examples.

The applicability of the theoretical results was illustrated through traffic link-cost modeling. For the classical BPR total-travel-time function, an explicit condition guaranteeing m -exponential convexity was obtained. More importantly, a controlled saturating link-cost model was introduced for which classical convexity and m -convexity may fail, whereas the Lipschitz-scaling approach remains applicable. This shows that the analytically admissible model class is larger. The controlled model remains a grey-box illustration rather than an empirically validated replacement for the BPR function.

In the four illustrative controlled-model cases, the quantitative correction reduces the original m -exponential upper bounds by 27.6%–30.1%. The power-mean midpoint estimate also provides a uniformly smaller certified radius than the corresponding Hölder estimate. For the smooth cost function used in that numerical illustration, the standard C^2 midpoint estimate is substantially tighter, so the generalized-convexity bounds should be viewed as certificates under a different information structure rather than as replacements for curvature-based quadrature. In the two-link network example, the proposed transition flow has an objective value only about 0.0104% above the certified global optimum, while the refined network certificate improves the uncorrected bound by approximately 24.46%; both percentages are instance-specific.

These results indicate that m -exponential convexity with respect to s can provide useful analytical certificates even when standard convexity assumptions are not available. Further research may address parameter calibration using real traffic data, larger network models, uncertainty in traffic demand, and extensions to dynamic or multi-period traffic assignment problems, as well as the determination of the exact membership threshold $s^*(\psi, m)$ discussed in Remark 4.

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Conflicts of interest

The author declares that there are no conflicts of interest.

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