

Adaptive IMEX and Crank–Nicolson methods for nonlocal quenching: Invariant regions, convergence, and mesh grading

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Abstract. We develop and analyze adaptive-in-time IMEX Euler and adaptive-in-time Crank–Nicolson-type schemes on nonuniform spatial meshes for a singular nonlocal parabolic problem. Midpoint cell quadrature yields a discrete operator whose sign structure gives comparison and invariant-region properties. Under an explicit adaptive step restriction, both schemes preserve $0 \leq U^n < 1$, and entry into a sufficiently high discrete level implies finite numerical quenching. We prove convergence before quenching and convergence of discrete level times. Numerical tests compare uniform and graded meshes, locate the quenching region, and yield mutually consistent Richardson extrapolations with observed order close to two.

Keywords: Singular source, midpoint quadrature, M -matrix, level-time approximation, homogeneous exterior data.
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1 Introduction

Nonlocal diffusion operators describe interactions in which the value at one point is influenced by an integral average over a surrounding region. They arise in phase-transition models, population dispersal and image processing [2, 4]. The monograph of Andreu-Vaillio et al. gives a systematic account of existence, comparison and boundary conditions for such equations [2]. Integral diffusion models related to heat equations with nonlocal boundary interactions were developed by Cortázar et al. [8, 9]. Numerical convergence for nonlocal evolution equations has been studied, among others, by Pérez-Llanos and Rossi [17].

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Nonuniform meshes are useful when a solution develops steep gradients in a localized region. Their use requires care because accuracy and stability may depend on the structure of the discrete matrix, not only on the local truncation error. General analyses of finite-difference schemes on nonuniform grids are given in [6, 7, 12, 13, 19, 20]. For quenching problems, spatial grading must also be coordinated with adaptive time stepping and positivity restrictions [5]. Kouakou and Diabaté recently developed nonuniform-grid IMEX and Crank–Nicolson schemes for boundary quenching in a Christov–Deng type problem [16]. The present paper extends that nonuniform-grid strategy to a nonlocal diffusion operator with homogeneous exterior data and establishes the matrix properties required for positivity and comparison.

The present work is concerned with finite-time quenching. Let $\Omega = (-1, 1)$. We consider the singular nonlocal problem

$$\begin{cases} u_t(x, t) = \int_{\Omega} J(x-y)(u(y, t) - u(x, t)) dy - u(x, t) \int_{\mathbb{R} \setminus \Omega} J(x-y) dy \\ \quad + \lambda a(x)(1 - u(x, t))^{-p}, & x \in \Omega, t > 0, \\ u(x, t) = 0, & x \in \mathbb{R} \setminus \Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (1)$$

Here $p > 0$, $\lambda > 0$, $J \geq 0$ is an even interaction kernel and a is a positive weight. A solution quenches at a finite time T^* if

$$0 \leq u(x, t) < 1 \quad (0 < t < T^*), \quad \lim_{t \rightarrow T^*-} \|u(\cdot, t)\|_{\infty} = 1. \quad (2)$$

This singular behavior is closely related to touchdown in electrostatic MEMS models and to classical Kawarada-type reaction-diffusion problems; see, for instance, [11, 14, 15, 18]. Nonuniform-grid computations for quenching-type reaction-diffusion equations were also considered in [5].

The purpose of this paper is to analyze adaptive time discretizations combined with nonuniform spatial approximations for the nonlocal quenching problem. The individual numerical ingredients—midpoint cell quadrature, IMEX Euler discretization, trapezoidal diffusion, and adaptive time stepping—are standard. The contribution lies in their structure-preserving combination and analysis for the singular nonlocal Dirichlet problem (1). The spatial mesh may concentrate cells near an expected quenching point, while the time step decreases as $\|U^n\|_{\infty}$ approaches one. The analysis is organized around preservation of the invariant interval $[0, 1)$, because crossing the singular level would make the discrete source undefined.

More precisely, the cell quadrature produces a nonlocal matrix with the sign and M -matrix properties needed for comparison. These properties are then coupled with an explicit singular-source restriction to prove positivity and upper invariance for both adaptive IMEX Euler and Crank–Nicolson-type schemes. We further establish finite numerical quenching after entry into a sufficiently high level, convergence before quenching, and convergence of discrete level times. The numerical experiments compare uniform and graded meshes and show that the two time discretizations yield mutually consistent quenching-time approximations under the coupled space–time refinement.

The paper is organized as follows. Section 2 states the continuous assumptions. Section 3 constructs the nonuniform nonlocal operator. Section 4 introduces the adaptive schemes and proves invariant-region properties. Section 5 proves finite numerical quenching and convergence of discrete level times. Section 6 presents the numerical experiments.

2 Continuous setting

We impose the following assumptions throughout the paper.

(H1) $J \in W^{1,\infty}(\mathbb{R})$, $J \geq 0$, $J(z) = J(-z)$ and $\int_{\mathbb{R}} J(z) dz = 1$.

(H2) $a \in C(\overline{\Omega})$ and there exist $a_0, a_1 > 0$ such that $a_0 \leq a(x) \leq a_1$ on $\overline{\Omega}$.

(H3) $u_0 \in C(\overline{\Omega})$ and $0 \leq u_0(x) \leq 1 - \rho_0$ on $\overline{\Omega}$ for some $\rho_0 \in (0, 1)$.

For v extended by zero outside Ω , define

$$(L_J v)(x) = \int_{\Omega} J(x-y)(v(y) - v(x)) dy - b(x)v(x), \quad b(x) = \int_{\mathbb{R} \setminus \Omega} J(x-y) dy. \quad (3)$$

With L_J defined by (3), problem (1) is $u_t = L_J u + \lambda a(x)(1-u)^{-p}$ in Ω .

Proposition 1 (Local well-posedness). *Assume (H1)–(H3). Then problem (1) admits a unique maximal solution*

$$u \in C^1([0, T_{\max}); C(\overline{\Omega}))$$

with $0 \leq u(x, t) < 1$ on $\Omega \times [0, T_{\max})$. Moreover, if $T_{\max} < \infty$, then

$$\limsup_{t \rightarrow T_{\max}^-} \|u(\cdot, t)\|_{\infty} = 1,$$

that is, loss of continuation can occur only when the singular level is approached.

Proof. Step 1. Let $X = C(\overline{\Omega})$ with the supremum norm. The operator $L_J : X \rightarrow X$ is bounded and linear because $J \in L^1(\mathbb{R})$ and $b \in L^{\infty}(\Omega)$.

Step 2. For each $\delta \in (0, 1)$, the map

$$G(v)(x) = \lambda a(x)(1 - v(x))^{-p}$$

is locally Lipschitz on the closed set $\{v \in X : \|v\|_{\infty} \leq 1 - \delta\}$. Hence the abstract equation

$$u'(t) = L_J u(t) + G(u(t)), \quad u(0) = u_0,$$

has a unique local solution by the Banach-space fixed-point theorem for semilinear evolution equations; see Amann [1] and the nonlocal diffusion framework in [2].

Step 3. Since the right-hand side is locally Lipschitz as long as $\|u(\cdot, t)\|_{\infty} < 1$, the local solution extends uniquely up to a maximal time $T_{\max}$. To verify positivity, suppose that a first negative minimum occurs at (x_0, t_0) . At that point, $u(y, t_0) - u(x_0, t_0) \geq 0$, $-b(x_0)u(x_0, t_0) > 0$, and the singular source is positive, which contradicts $u_t(x_0, t_0) \leq 0$. Hence $u \geq 0$. The only obstruction to continuation at a finite maximal time is therefore approach to the singular level $u = 1$. $\square$

Proposition 1 shows that finite-time loss of continuation is equivalent to approach to the singular level. We therefore use the following terminology.

Definition 1. *Let u be the maximal solution of (1) on $[0, T_{\max})$. The time $T^* = T_{\max}$ is called the quenching time if $T^* < \infty$ and (2) holds.*

We approximate the quenching time below using continuous and discrete level times. The following elementary estimates are used later to compare them.

Lemma 1. Assume (H1)–(H3). Let u be a solution of (1) on $[0, T]$ and suppose that $\|u(\cdot, t)\|_\infty \leq 1 - \delta$ on $[0, T]$ for some $\delta > 0$. Then

$$\|u_t(\cdot, t)\|_\infty \leq 2\|J\|_{L^1(\mathbb{R})}\|u(\cdot, t)\|_\infty + \lambda a_1 \delta^{-p} \quad (0 \leq t \leq T). \quad (4)$$

Moreover, if u quenches at T^* , then for every $\eta \in (0, 1)$ the level time

$$T_\eta = \inf\{t > 0 : \|u(\cdot, t)\|_\infty \geq 1 - \eta\} \quad (5)$$

satisfies $T_\eta \uparrow T^*$ as $\eta \downarrow 0$.

Proof. Step 1. Since $J \geq 0$ and $\int_{\mathbb{R}} J = 1$,

$$|(L_J u)(x, t)| \leq \int_{\Omega} J(x-y)|u(y, t)| dy + |u(x, t)| \int_{\mathbb{R}} J(x-y) dy \leq 2\|u(\cdot, t)\|_\infty.$$

The singular source is bounded by $\lambda a_1 \delta^{-p}$. This proves (4).

Step 2. The function $t \mapsto \|u(\cdot, t)\|_\infty$ is continuous on compact subintervals of $[0, T^*)$. The definition of quenching gives $T_\eta < T^*$ for every sufficiently small $\eta > 0$, and T_η is nondecreasing as $\eta \downarrow 0$. Let $L = \lim_{\eta \downarrow 0} T_\eta \leq T^*$. If $L < T^*$, continuity on $[0, (L + T^*)/2]$ would give $\|u(\cdot, t)\|_\infty \leq 1 - \varepsilon$ there for some $\varepsilon > 0$, contradicting $T_\eta \leq L$ for $0 < \eta < \varepsilon$. Hence $L = T^*$. $\square$

3 Nonuniform discretization of the nonlocal operator

Let $I \geq 2$. We divide Ω into nonuniform cells

$$-1 = x_0 < x_1 < \cdots < x_I = 1, \quad C_i = (x_{i-1}, x_i), \quad h_i = x_i - x_{i-1},$$

and define the cell center

$$\xi_i = \frac{x_{i-1} + x_i}{2}, \quad 1 \leq i \leq I.$$

The unknown $U_i(t)$ approximates $u(\xi_i, t)$ and h_i is the quadrature weight of C_i . We write

$$h_{\max} = \max_{1 \leq i \leq I} h_i, \quad h_{\min} = \min_{1 \leq i \leq I} h_i.$$

Thus $h_{\max}$ is the global refinement parameter; the local quantities remain the cell lengths h_i . We use mesh families satisfying $h_{\max} \rightarrow 0$ and, when a mesh-uniform regularity condition is needed, $C_m^{-1} \leq h_{i+1}/h_i \leq C_m$ with C_m independent of i and $h_{\max}$. This assumption excludes abrupt isolated jumps but still allows graded meshes near the expected quenching point.

The discretization is cell-based rather than a pointwise extension of uniform-grid formulas. On nonuniform grids, two formulas with the same order on a uniform mesh may have different matrix stability properties. The present operator is built so that the off-diagonal coefficients remain nonnegative and the row sums have the correct sign.

For $1 \leq i, j \leq I$, set

$$B_{ij} = h_j J(\xi_i - \xi_j), \quad b_i = \int_{\mathbb{R} \setminus \Omega} J(\xi_i - y) dy.$$

The discrete nonlocal operator is

$$(L_{J,h}U)_i = \sum_{j=1}^I B_{ij}(U_j - U_i) - b_i U_i.$$

Equivalently, $L_{J,h}U = A_h U$, where

$$(A_h)_{ij} = B_{ij} \quad (i \neq j), \quad (A_h)_{ii} = -\sum_{j \neq i} B_{ij} - b_i. \quad (6)$$

Let

$$r_i = \sum_{j \neq i} B_{ij} + b_i, \quad r_h = \max_{1 \leq i \leq I} r_i.$$

Lemma 2. *The matrix A_h has the following properties.*

- (1) $(A_h)_{ij} \geq 0$ for $i \neq j$ and $(A_h)_{ii} \leq 0$.
- (2) $A_h \mathbf{1} \leq 0$ componentwise.
- (3) For every $\tau > 0$, $I - \tau A_h$ is a nonsingular M -matrix and $(I - \tau A_h)^{-1} \geq 0$.
- (4) If $0 \leq V \leq c\mathbf{1}$, then $(I - \tau A_h)^{-1}V \leq c\mathbf{1}$.

Proof. Step 1. The signs of the off-diagonal and diagonal entries follow from $J \geq 0$, $h_j > 0$ and $b_i \geq 0$.

Step 2. By (6),

$$(A_h \mathbf{1})_i = \sum_{j \neq i} B_{ij} - \sum_{j \neq i} B_{ij} - b_i = -b_i \leq 0.$$

Step 3. More explicitly, by (6),

$$(I - \tau A_h)_{ii} = 1 + \tau r_i > 0, \quad (I - \tau A_h)_{ij} = -\tau B_{ij} \leq 0 \quad (i \neq j).$$

Moreover,

$$1 + \tau r_i = 1 + \tau b_i + \tau \sum_{j \neq i} B_{ij} > \tau \sum_{j \neq i} B_{ij},$$

so $I - \tau A_h$ is strictly row diagonally dominant and therefore a nonsingular M -matrix. The strict dominance is the sufficient criterion used here to obtain nonsingularity and hence a nonnegative inverse. If only weak row diagonal dominance were known, this argument alone would not exclude singularity and would not justify the positive inverse required in the invariant-region proof; additional M -matrix hypotheses would then be needed.

Step 4. Since $(I - \tau A_h)^{-1} \geq 0$, $V \geq 0$ implies $(I - \tau A_h)^{-1}V \geq 0$. Moreover

$$(I - \tau A_h)c\mathbf{1} = c\mathbf{1} - \tau c A_h \mathbf{1} \geq c\mathbf{1}.$$

Applying the positive inverse gives $(I - \tau A_h)^{-1}c\mathbf{1} \leq c\mathbf{1}$. □

The next estimates isolate the spatial discretization error of the nonlocal operator. They are the nonlocal analogue of truncation-error estimates for local finite differences on nonuniform meshes.

Lemma 3 (Cell quadrature error). *Let $P_h \phi = (\phi(\xi_1), \dots, \phi(\xi_I))^T$.*

(1) If $J \in W^{1,\infty}(\mathbb{R})$ and $\phi \in C^1(\overline{\Omega})$, then

$$\max_i |(L_{J,h}P_h\phi)_i - (L_J\phi)(\xi_i)| \leq Ch_{\max}. \quad (7)$$

(2) If $J \in W^{2,\infty}(\mathbb{R})$ and $\phi \in C^2(\overline{\Omega})$, then the midpoint cell quadrature satisfies

$$\max_i |(L_{J,h}P_h\phi)_i - (L_J\phi)(\xi_i)| \leq Ch_{\max}^2. \quad (8)$$

The constant C depends on J , ϕ and Ω , but not on the mesh.

Proof. Step 1. Fix i and set

$$F_i(y) = J(\xi_i - y)(\phi(y) - \phi(\xi_i)).$$

Then

$$(L_J\phi)(\xi_i) = \sum_{j=1}^I \int_{C_j} F_i(y) dy - b_i\phi(\xi_i), \quad (L_{J,h}P_h\phi)_i = \sum_{j=1}^I h_j F_i(\xi_j) - b_i\phi(\xi_i).$$

The exterior loss term is therefore represented exactly.

Step 2. Under the assumptions of the first part, F_i is uniformly Lipschitz in y , independently of i . Hence

$$\left| \int_{C_j} F_i(y) dy - h_j F_i(\xi_j) \right| \leq Ch_j h_{\max}.$$

Summing over all cells gives (7).

Step 3. Under the assumptions of the second part, F_i'' is uniformly bounded in $L^\infty(C_j)$, independently of i and j . Since ξ_j is the midpoint of C_j , the midpoint quadrature formula gives

$$\left| \int_{C_j} F_i(y) dy - h_j F_i(\xi_j) \right| \leq Ch_j^3 \|F_i''\|_{L^\infty(C_j)}.$$

Therefore

$$\sum_{j=1}^I \left| \int_{C_j} F_i(y) dy - h_j F_i(\xi_j) \right| \leq C \sum_{j=1}^I h_j^3 \leq Ch_{\max}^2 \sum_{j=1}^I h_j \leq Ch_{\max}^2.$$

This estimate is cellwise: the midpoint rule is second order on each interval C_j independently of the other cell widths. Equality of the h_j is never used; the global estimate follows from $\sum_j h_j^3 \leq h_{\max}^2 \sum_j h_j$. This proves (8). $\square$

4 Adaptive IMEX and Crank–Nicolson schemes

Let $U_i^0 = u_0(\xi_i)$. For a vector $V < \mathbf{1}$, define the singular source

$$F_h(V)_i = \lambda a_i (1 - V_i)^{-p}, \quad a_i = a(\xi_i).$$

Let

$$\delta_n = 1 - \|U^n\|_\infty, \quad a_h^* = \max_i a_i.$$

We fix parameters $\beta > 0$, $0 < \theta \leq 1$ and $\Delta t_{\max} > 0$. The time rule is written in terms of the global refinement parameter $h_{\max}$. We assume $h_{\max} \leq 1$, set $\mu_h = \beta h_{\max}^2$, and require $0 < \mu_h < 1$ on the meshes under consideration. The adaptive time step is

$$\Delta t_n = \min \left\{ \Delta t_{\max}, \beta h_{\max}^2 \frac{\delta_n^{p+1}}{\lambda a_h^*}, \theta \frac{h_{\max}^2}{r_h} \right\}. \quad (9)$$

If $r_h = 0$, we interpret the third component of (9) as $+\infty$. In this case $A_h = 0$, so no diffusion restriction is needed. The second component prevents the explicit singular source from crossing the singular level $u = 1$. The third component is obtained from the row bound of the integral-diffusion matrix: since $r_i \leq r_h$,

$$\Delta t_n r_i \leq \theta h_{\max}^2 \leq \theta \leq 1.$$

This is the bound used to preserve the nonnegativity of the Crank–Nicolson right-hand matrix and to couple the temporal refinement to the global spatial scale. The positivity of the left resolvent $(I - \tau A_h)^{-1}$ itself does not require a CFL-type restriction, since Lemma 2 shows that $I - \tau A_h$ is a nonsingular M -matrix for every $\tau > 0$. Thus $h_{\max}^2/r_h$, rather than a local quotient involving $h_{\min}^2$, is a sufficient restriction adapted to the present cell-based integral operator. For an explicit local Laplacian on a nonuniform mesh, by contrast, the largest coefficient is usually generated by the smallest cell and leads to a restriction involving $h_{\min}^2$. Here $h_{\max}^2$ supplies the global consistency scale, whereas the complete mesh dependence of the nonlocal diffusion is retained in r_h . Related h^2 -scaled adaptive restrictions on nonuniform grids have been used in local quenching computations [5].

The source restriction has a separate role: it controls the explicit singular reaction. The matrix restriction alone cannot provide this control. If $\beta h_{\max}^2 \delta_n^{p+1}/(\lambda a_h^*)$ were removed, the increment $\Delta t_n F_h(U^n)$ could exceed the remaining distance δ_n to the singular level. The upper estimate in Proposition 2 would then fail, and the invariant-region conclusions of Theorems 1 and 2 would no longer follow.

The adaptive IMEX Euler scheme is

$$(I - \Delta t_n A_h) U^{n+1} = U^n + \Delta t_n F_h(U^n). \quad (10)$$

It follows the usual implicit-explicit splitting principle: the nonlocal diffusion is treated implicitly, while the singular source is evaluated explicitly; see, for example, the IMEX framework in [3]. The adaptive Crank–Nicolson-type scheme is

$$\left(I - \frac{\Delta t_n}{2} A_h \right) U^{n+1} = \left(I + \frac{\Delta t_n}{2} A_h \right) U^n + \Delta t_n F_h(U^n). \quad (11)$$

It uses the classical trapezoidal treatment of the diffusion term introduced by Crank and Nicolson [10], combined with the same explicit singular-source restriction. The source is kept explicit. This avoids solving a nonlinear singular system at each time level and makes the invariant-region proof transparent. The corresponding invariant-region statements are Theorems 1 and 2 below.

4.1 Algorithmic implementation

Let $W = \text{diag}(h_1, \dots, h_I)$. The quadrature identity $h_i(A_h)_{ij} = h_j(A_h)_{ji}$ shows that A_h is self-adjoint in the W -weighted inner product. Hence

$$S_h = W^{1/2} A_h W^{-1/2}$$

is symmetric. In floating-point arithmetic we replace S_h by $(S_h + S_h^T)/2$ and compute once

$$S_h = Q\Lambda Q^T, \quad Q^T Q = I.$$

This is an eigenvalue decomposition, not an LU or Cholesky factorization. For a fixed mesh and a fixed run, it is computed once and reused at every time level; no decomposition is recomputed when Δt_n changes. The spectral representation above is used for both schemes. In the accompanying code, the IMEX and Crank–Nicolson computations are performed separately, and each run constructs this decomposition once. Thus,

$$(I - cA_h)^{-1}b = W^{-1/2}Q(I - c\Lambda)^{-1}Q^TW^{1/2}b,$$

with $c = \Delta t_n$ for IMEX Euler and $c = \Delta t_n/2$ for the Crank–Nicolson-type scheme.

The computational procedure used in Section 6 is as follows.

- A1.** Generate the cell edges, centers ξ_i and weights h_i ; assemble A_h , r_h , a_i and the spectral data (Q, Λ) above.
- A2.** Set $U^0 = P_h u_0$, $t_0 = 0$ and $n = 0$.
- A3.** Compute $\delta_n = 1 - \|U^n\|_\infty$ and Δt_n from (9). If $\Delta t_n \leq \Delta t_{\text{floor}}$, stop. In the reported runs, $\Delta t_{\text{floor}} = 10^{-16}$ and no additional threshold on δ_n is imposed.
- A4.** Form $F_h(U^n)$. For IMEX Euler, solve (10) using the spectral representation above. For the Crank–Nicolson-type scheme, first evaluate $A_h U^n$ with these spectral data and then solve (11). Set $t_{n+1} = t_n + \Delta t_n$, increase n , and return to A3.
- A5.** At termination, choose $q \in \arg\max_i U_i^n$ and report the corrected time

$$T_{h,\text{corr}} = t_n + \frac{\delta_n^{p+1}}{(p+1)\lambda a(x_{q,h})}, \quad x_{q,h} = \xi_q. \quad (12)$$

The correction in (12) is not part of the stopping test. It follows from the leading scalar balance near quenching. If the maximizing cell remains fixed over the final steps and the bounded diffusion contribution is lower order, then

$$\delta'(t) \sim -\lambda a(x_{q,h})\delta(t)^{-p}.$$

Integrating from the terminal value δ_n to zero gives the remaining time $\delta_n^{p+1}/((p+1)\lambda a(x_{q,h}))$. The accompanying code implements A1–A5 exactly.

Proposition 2. Assume $0 \leq U^n \leq (1 - \delta_n)\mathbf{1}$ with $\delta_n > 0$. If (9) holds, then

$$0 \leq U^n + \Delta t_n F_h(U^n) \leq (1 - (1 - \mu_h)\delta_n)\mathbf{1}. \quad (13)$$

Proof. Step 1. The lower bound is immediate because $U^n \geq 0$, $a_i > 0$ and $U_i^n < 1$.

Step 2. For each i ,

$$U_i^n + \Delta t_n \lambda a_i (1 - U_i^n)^{-p} \leq 1 - \delta_n + \beta h_{\max}^2 \frac{\delta_n^{p+1}}{\lambda a_h^*} \lambda a_h^* \delta_n^{-p} = 1 - (1 - \mu_h)\delta_n.$$

Since $0 < \mu_h < 1$, this proves (13). □

Theorem 1 (Invariant region for IMEX Euler). *Assume (H1)–(H3). Let U^n be generated by (10) and (9). If $0 \leq U^0 < \mathbf{1}$, then*

$$0 \leq U^n < \mathbf{1} \quad \text{for all computed levels } n. \quad (14)$$

Proof. Step 1. Suppose $0 \leq U^n < \mathbf{1}$. By Proposition 2, the right-hand side of (10) is nonnegative and bounded above by $(1 - (1 - \mu_h)\delta_n)\mathbf{1}$.

Step 2. Lemma 2 gives $(I - \Delta t_n A_h)^{-1} \geq 0$ and the inverse is order preserving below constants. Hence

$$0 \leq U^{n+1} \leq (1 - (1 - \mu_h)\delta_n)\mathbf{1} < \mathbf{1}.$$

Step 3. Induction gives (14). □

Theorem 2 (Invariant region for the Crank–Nicolson-type scheme). *Assume (H1)–(H3). Let U^n be generated by (11) and (9). If $0 \leq U^0 < \mathbf{1}$, then*

$$0 \leq U^n < \mathbf{1} \quad \text{for all computed levels } n.$$

Proof. Step 1. Since $\Delta t_n r_i \leq \Delta t_n r_h \leq \theta h_{\max}^2 \leq 1$, the entries of $M_n = I + (\Delta t_n/2)A_h$ satisfy

$$(M_n)_{ii} = 1 - \frac{\Delta t_n}{2} r_i \geq \frac{1}{2} > 0, \quad (M_n)_{ij} = \frac{\Delta t_n}{2} B_{ij} \geq 0 \quad (i \neq j).$$

Hence M_n is entrywise nonnegative. In addition,

$$M_n \mathbf{1} = \mathbf{1} + \frac{\Delta t_n}{2} A_h \mathbf{1} \leq \mathbf{1}.$$

Therefore $0 \leq U^n \leq (1 - \delta_n)\mathbf{1}$ gives, by entrywise monotonicity,

$$0 \leq M_n U^n \leq (1 - \delta_n)M_n \mathbf{1} \leq (1 - \delta_n)\mathbf{1}.$$

Thus the negative diagonal entries of A_h cause no difficulty: after multiplication by $\Delta t_n/2$ and addition of the identity, the whole right-hand matrix is nonnegative and has row sums at most one.

Step 2. The source contribution satisfies $\Delta t_n F_h(U^n) \leq \mu_h \delta_n \mathbf{1}$. Consequently, the right-hand side of (11) is bounded by $(1 - (1 - \mu_h)\delta_n)\mathbf{1}$.

Step 3. The matrix $I - (\Delta t_n/2)A_h$ is a nonsingular M -matrix. Lemma 2 gives the required lower and upper bounds after inversion. □

5 Finite numerical quenching and convergence of discrete level times

Define the numerical quenching time by

$$T_h^* = \sum_{n=0}^{\infty} \Delta t_n,$$

whenever the infinite computation is defined and $\delta_n \downarrow 0$. In the computations of Section 6, termination is based exclusively on $\Delta t_n \leq \Delta t_{\text{floor}}$; no additional threshold on δ_n is used. The tail correction is applied only after termination.

Lemma 4. Assume that $0 \leq U^n < 1$ and put

$$M_n = \|U^n\|_\infty, \quad \delta_n = 1 - M_n, \quad a_{0,h} = \min_i a_i.$$

If

$$\lambda a_{0,h} \delta_n^{-p} \geq r_h M_n, \quad (15)$$

then the solution generated by either (10) or (11) satisfies

$$M_{n+1} - M_n \geq \frac{\Delta t_n}{1 + \Delta t_n r_h} (\lambda a_{0,h} \delta_n^{-p} - r_h M_n). \quad (16)$$

Consequently, $M_{n+1} \geq M_n$ whenever (15) holds.

Proof. Step 1. Choose i_n such that $U_{i_n}^n = M_n$. For every nonnegative vector V ,

$$(A_h V)_{i_n} = \sum_{j \neq i_n} B_{i_n j} V_j - r_{i_n} V_{i_n} \geq -r_h V_{i_n}.$$

Step 2. For the IMEX scheme, the component i_n of (10) gives

$$U_{i_n}^{n+1} = M_n + \Delta t_n (A_h U^{n+1})_{i_n} + \Delta t_n \lambda a_{i_n} \delta_n^{-p}.$$

Therefore

$$(1 + \Delta t_n r_h) U_{i_n}^{n+1} \geq M_n + \Delta t_n \lambda a_{0,h} \delta_n^{-p}.$$

Since $M_{n+1} \geq U_{i_n}^{n+1}$, inequality (16) follows.

Step 3. For the Crank–Nicolson-type scheme,

$$U_{i_n}^{n+1} - M_n = \frac{\Delta t_n}{2} ((A_h U^{n+1})_{i_n} + (A_h U^n)_{i_n}) + \Delta t_n \lambda a_{i_n} \delta_n^{-p}.$$

At the maximum of U^n , $(A_h U^n)_{i_n} \geq -r_h M_n$, while $(A_h U^{n+1})_{i_n} \geq -r_h U_{i_n}^{n+1}$. The index i_n remains the index selected from U^n ; it is not assumed to maximize U^{n+1} . We use only $M_{n+1} \geq U_{i_n}^{n+1}$. Hence

$$M_{n+1} - M_n \geq \frac{\Delta t_n}{1 + \Delta t_n r_h / 2} (\lambda a_{0,h} \delta_n^{-p} - r_h M_n).$$

Under (15), this lower bound is not smaller than the right-hand side of (16). $\square$

Theorem 3 (Finite numerical quenching). Assume $a_{0,h} > 0$. Let U^n be generated by (10) or (11) with (9). There exists $\delta_* > 0$ such that, if

$$\|U^N\|_\infty \geq 1 - \delta_*$$

for some N , then

$$T_h^* < \infty, \quad \lim_{n \rightarrow \infty} \|U^n\|_\infty = 1.$$

Proof. Step 1. Choose $\delta_* > 0$ so that

$$\lambda a_{0,h} \delta_*^{-p} \geq 2r_h.$$

For $n \geq N$, Lemma 4 gives $M_{n+1} \geq M_n$ and therefore $\delta_{n+1} \leq \delta_n \leq \delta_*$. Moreover,

$$M_{n+1} - M_n \geq \frac{\lambda a_{0,h}}{2(1 + \Delta t_{\max} r_h)} \Delta t_n \delta_n^{-p}. \quad (17)$$

Step 2. Since M_n is increasing and bounded above by one, it has a limit $M_\infty \leq 1$. If $M_\infty < 1$, then $\delta_n \geq \delta_\infty > 0$, and every component in the minimum (9) is bounded below by a positive constant. Inequality (17) would then give $M_{n+1} - M_n \geq c > 0$, a contradiction. Hence $M_\infty = 1$.

Step 3. Put

$$K_h = \frac{\beta h_{\max}^2}{\lambda a_h^*}.$$

Since $\delta_n \rightarrow 0$, there exists $N_1 \geq N$ such that

$$\Delta t_n = K_h \delta_n^{p+1} \quad (n \geq N_1).$$

Combining this identity with (17) gives

$$\delta_{n+1} \leq (1 - c_h) \delta_n, \quad c_h = \min \left\{ \frac{1}{2}, \frac{K_h \lambda a_{0,h}}{2(1 + \Delta t_{\max} r_h)} \right\} \in (0, 1).$$

Thus δ_n decreases geometrically for $n \geq N_1$, and

$$\sum_{n=N_1}^{\infty} \Delta t_n = K_h \sum_{n=N_1}^{\infty} \delta_n^{p+1} < \infty.$$

Adding the finite partial sum proves $T_h^* < \infty$. □

To approximate the continuous quenching time, we use discrete level times. The results below establish convergence at each strictly crossed level and a diagonal approximation of T^* ; they do not assert convergence of the infinite-sum times T_h^* as $h_{\max} \rightarrow 0$. Let

$$T_{h,\eta} = \inf\{t_n : \|U^n\|_\infty \geq 1 - \eta\}.$$

Proposition 3 (Local discretization error before quenching). *Let $T < T^*$ and assume that, on $[0, T]$, the continuous solution is sufficiently smooth, with u_t and u_{tt} bounded in $C(\overline{\Omega})$ and $u(\cdot, t) \in C^2(\overline{\Omega})$ uniformly in t . Assume also that $J \in W^{2,\infty}(\mathbb{R})$. Define the residual obtained by inserting $P_h u(t_n)$ into the IMEX Euler scheme or into the Crank–Nicolson-type scheme. Then there exists $C_T > 0$, independent of h and n , such that*

$$\max_i |R_i^n| \leq C_T (\Delta t_n + h_{\max}^2), \quad t_n, t_{n+1} \leq T. \quad (18)$$

If only the first-order quadrature estimate (7) is available, the term $h_{\max}^2$ in (18) is replaced by $h_{\max}$.

Proof. Step 1. For the IMEX Euler scheme, the residual can be written as

$$R_i^n = \frac{u(\xi_i, t_{n+1}) - u(\xi_i, t_n)}{\Delta t_n} - (L_{J,h} P_h u(t_{n+1}))_i - \lambda a_i (1 - u(\xi_i, t_n))^{-p}.$$

Since u satisfies (1), the time difference gives an $O(\Delta t_n)$ error, and the replacement of $L_J u(\cdot, t_{n+1})$ by $L_J u(\cdot, t_n)$ is also $O(\Delta t_n)$ on $[0, T]$.

Step 2. By Lemma 3,

$$\max_i |(L_{J,h} P_h u(t_{n+1}))_i - (L_J u)(\xi_i, t_{n+1})| \leq C_T h_{\max}^2.$$

The singular source is evaluated at t_n and is Lipschitz on $\{u \leq 1 - \delta_T\}$, where $\delta_T > 0$ because $T < T^*$. This gives (18) for the IMEX Euler scheme.

Step 3. For the Crank–Nicolson-type scheme, the diffusion residual contains

$$\frac{1}{2} L_{J,h} P_h u(t_{n+1}) + \frac{1}{2} L_{J,h} P_h u(t_n).$$

The trapezoidal diffusion part is at least first order in time, and the explicit singular source remains first order. The temporal constants depend only on the pre-quenching regularity bounds, while Lemma 3 provides a mesh-independent spatial constant. The midpoint quadrature estimate therefore gives (18) with C_T independent of h and n . $\square$

Because (9) gives $\max_n \Delta t_n \leq \beta h_{\max}^2 / (\lambda a_0) = O(h_{\max}^2)$, the temporal contribution in (18) is asymptotically no larger than the spatial midpoint-quadrature contribution. Hence $C_T(\Delta t_n + h_{\max}^2) = O(h_{\max}^2)$ along the coupled refinement sequence. The two contributions need not be comparable at every time level: when the singular-source restriction becomes active near quenching, Δt_n may be much smaller than $h_{\max}^2$, in which case the spatial term dominates the displayed upper bound. This comparison concerns bounds, not the actual error contributions. All estimates here hold for a fixed $T < T^*$; the constants may increase as T approaches T^* . Consequently, the near-second-order Richardson behavior observed under mesh refinement in Section 6 is consistent with the coupled refinement rule, but it should not be interpreted as second-order temporal accuracy of either scheme or as a consequence of Proposition 3 alone.

Theorem 4 (Convergence before quenching). *Let u be the solution of (1) on $[0, T]$ and assume that $T < T^*$. Let U^n be generated by (10) or (11). Consider a refinement sequence such that $h_{\max} \rightarrow 0$, with the parameters in (9) fixed. Under the local residual estimate of Proposition 3 and the adaptive restriction (9),*

$$\max_{t_n \leq T} \|U^n - P_h u(\cdot, t_n)\|_{\infty} \rightarrow 0 \quad \text{as } h_{\max} \rightarrow 0. \quad (19)$$

Proof. Step 1. Since $T < T^*$, there exists $\delta_T > 0$ such that $u(x, t) \leq 1 - \delta_T$ on $\bar{\Omega} \times [0, T]$. The source $s \mapsto (1 - s)^{-p}$ is Lipschitz on $(-\infty, 1 - \delta_T/2]$.

Step 2. Let $e^n = U^n - P_h u(\cdot, t_n)$. On the set where $\|e^k\|_{\infty} \leq \delta_T/2$, the source is Lipschitz with the mesh-independent constant

$$L_T = p \lambda a_1 (\delta_T/2)^{-(p+1)}.$$

For IMEX Euler, set $P_n = (I - \Delta t_n A_h)^{-1}$. Lemma 2 gives $P_n \geq 0$ and $P_n \mathbf{1} \leq \mathbf{1}$; hence $\|P_n\|_{\infty} \leq 1$, uniformly in h and n .

For the Crank–Nicolson-type scheme, set

$$P_n = \left(I - \frac{\Delta t_n}{2} A_h \right)^{-1}, \quad M_n = I + \frac{\Delta t_n}{2} A_h.$$

The adaptive restriction (9) implies $\Delta t_n r_i \leq \theta h_{\max}^2 \leq 1$. Therefore $M_n \geq 0$ entrywise and $M_n \mathbf{1} \leq \mathbf{1}$, as in Theorem 2. Lemma 2 also gives $P_n \geq 0$ and $P_n \mathbf{1} \leq \mathbf{1}$. Consequently,

$$P_n M_n \geq 0, \quad P_n M_n \mathbf{1} \leq P_n \mathbf{1} \leq \mathbf{1},$$

so that

$$\|P_n M_n\|_\infty \leq 1, \quad \|P_n\|_\infty \leq 1.$$

These stability bounds are independent of the mesh size. Subtracting the exact-grid relation, including the residual of Proposition 3, now gives for either scheme

$$\|e^{n+1}\|_\infty \leq (1 + L_T \Delta t_n) \|e^n\|_\infty + C_T \Delta t_n (\Delta t_n + h_{\max}^2),$$

where C_T is independent of h and n .

Step 3. Discrete Gronwall and $\sum_{t_{n+1} \leq T} \Delta t_n \leq T$ give

$$\max_{t_n \leq T} \|e^n\|_\infty \leq C'_T \left(\|e^0\|_\infty + h_{\max}^2 + \max_{t_n \leq T} \Delta t_n \right),$$

where C'_T depends on T, p, λ, a_1 and the pre-quenching regularity bounds, but not on the mesh size. Here $e^0 = 0$. In addition, (9) gives

$$\max_n \Delta t_n \leq \frac{\beta h_{\max}^2}{\lambda a_h^*} \leq \frac{\beta h_{\max}^2}{\lambda a_0},$$

which tends to zero.

Step 4. For sufficiently small $h_{\max}$, the right-hand side in Step 3 is smaller than $\delta_T/2$. A first-exit argument therefore closes the estimate on the whole interval $[0, T]$. On this interval, the adaptive time steps have a positive lower bound for each fixed mesh, so the discrete time grid reaches T . This proves (19). Under only first-order spatial consistency, this proof gives the corresponding bound with $h_{\max}$ instead of $h_{\max}^2$. $\square$

Theorem 5 (Convergence of discrete level times). *Assume that u quenches at T^* and that the hypotheses of Theorem 4 hold on every compact interval $[0, T]$ with $T < T^*$. Fix $\eta \in (0, \rho_0)$. Assume that the level $1 - \eta$ is crossed strictly: for every sufficiently small $\varepsilon > 0$ there exists $t_+ \in (T_\eta, T_\eta + \varepsilon)$ such that*

$$\|u(\cdot, t_+)\|_\infty > 1 - \eta.$$

Then

$$T_{h,\eta} \rightarrow T_\eta \quad \text{as } h_{\max} \rightarrow 0, \tag{20}$$

where T_η is defined by (5).

Proof. Step 1. Fix $\varepsilon > 0$ such that $0 < \varepsilon < T_\eta$ and $T_\eta + \varepsilon < T^*$. By the definition of T_η and continuity,

$$\max_{0 \leq t \leq T_\eta - \varepsilon} \|u(\cdot, t)\|_\infty \leq 1 - \eta - \sigma_-$$

for some $\sigma_- > 0$. By the strict-crossing assumption, choose $t_+ \in (T_\eta, T_\eta + \varepsilon)$ and $\sigma_+ > 0$ such that

$$\|u(\cdot, t_+)\|_\infty \geq 1 - \eta + 2\sigma_+.$$

Step 2. The adaptive rule implies $\max_n \Delta t_n = O(h_{\max}^2)$. Hence, for all sufficiently small h , there is a discrete time t_{n_+} such that $|t_{n_+} - t_+| \leq \max_n \Delta t_n$. Choose $x_+ \in \overline{\Omega}$ with $u(x_+, t_+) = \|u(\cdot, t_+)\|_\infty$. By continuity of u on the compact set $\overline{\Omega} \times [0, T_\eta + \varepsilon]$, there exists $h_1 > 0$ such that

$$|u(x, t) - u(x_+, t_+)| \leq \frac{\sigma_+}{2}$$

whenever $|x - x_+| + |t - t_+| \leq h_1$. The cell centers are dense as $h_{\max} \rightarrow 0$, so one can choose a center $\xi_{i(h)}$ with $|\xi_{i(h)} - x_+| \leq h_{\max}$. Since also $|t_{n_+} - t_+| \leq \max_n \Delta t_n \rightarrow 0$, for sufficiently small h ,

$$\begin{aligned} \|P_h u(\cdot, t_{n_+})\|_\infty &\geq u(\xi_{i(h)}, t_{n_+}) \\ &\geq u(x_+, t_+) - \sigma_+ \geq 1 - \eta + \sigma_+. \end{aligned}$$

This explicitly accounts for both the spatial sampling error and the time-grid error.

Step 3. Theorem 4 gives uniform convergence at all discrete times not exceeding $T_\eta + \varepsilon$. Therefore, for sufficiently small h ,

$$\|U^n\|_\infty < 1 - \eta \quad \text{whenever } t_n \leq T_\eta - \varepsilon,$$

and

$$\|U^{n_+}\|_\infty > 1 - \eta.$$

Thus

$$T_\eta - \varepsilon < T_{h,\eta} < T_\eta + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, (20) follows. $\square$

Corollary 1. Assume that there exists a sequence of strictly crossed levels $\eta_k \downarrow 0$. Then one can choose a piecewise constant function $\eta(h) \downarrow 0$ such that

$$T_{h,\eta(h)} \rightarrow T^* \quad \text{as } h_{\max} \rightarrow 0. \quad (21)$$

Proof. Step 1. By Theorem 5, for every k there exists $h_k > 0$ such that

$$|T_{h,\eta_k} - T_{\eta_k}| \leq \frac{1}{k} \quad \text{whenever } h_{\max} \leq h_k.$$

Choose $h_k \downarrow 0$ and define $\eta(h) = \eta_k$ for $h_{k+1} < h_{\max} \leq h_k$.

Step 2. Then

$$|T_{h,\eta(h)} - T^*| \leq |T_{h,\eta_k} - T_{\eta_k}| + |T_{\eta_k} - T^*|.$$

The first term is at most $1/k$, while the second tends to zero by Lemma 1. This proves (21). $\square$

6 Numerical experiments

This section presents a sequence of separate numerical tests. All computations are reproducible and use the cell-based nonlocal matrix A_h introduced in Section 3; no local Laplacian is used.

6.1 Common numerical setting

The model is solved on $\Omega = (-1, 1)$ with

$$p = 2, \quad \lambda = 18,$$

and

$$J(z) = \frac{1}{\sqrt{\pi\rho}} \exp\left(-\frac{z^2}{\rho^2}\right), \quad \rho = 0.22, \quad a(x) = 1 + 0.5 \exp(-30x^2).$$

The initial function is

$$u_0(x) = 0.05(1 + 0.10 \exp(-20x^2)).$$

Since $0 \leq u_0(x) \leq 0.055$, assumption (H3) is satisfied, for example, with $\rho_0 = 0.94$ (indeed $0.055 < 1 - 0.94 = 0.06$). The value $\rho_0 = 0.95$ would not satisfy (H3) for this initial datum. The weight a is maximal at $x = 0$; hence the expected quenching point is located near the origin. The nonuniform mesh is generated from uniform computational edges $\eta_i = -1 + 2i/I$ by

$$x_i = \text{sign}(\eta_i) |\eta_i|^\gamma.$$

For $\gamma > 1$, the cells are concentrated near $x = 0$. Unless otherwise stated, the default value is $\gamma = 1.55$.

The two time discretizations considered in the numerical experiments are the adaptive IMEX Euler scheme (10) and the adaptive Crank–Nicolson-type scheme (11). Both use the adaptive time rule (9). The numerical parameters are

$$\beta = 20, \quad \theta = 0.80, \quad \Delta t_{\max} = 2 \times 10^{-3}, \quad \Delta t_{\text{floor}} = 10^{-16}.$$

These values were not optimized. The choice $\beta = 20$ gives $\mu_h = \beta h_{\max}^2 < 1$ even on the coarsest mesh used below, as required in Proposition 2. The value $\theta = 0.80$ leaves a safety margin below the theoretical threshold $\theta \leq 1$ in the matrix-positivity estimate. The computation is terminated when $\Delta t_n \leq 10^{-16}$; no separate δ_n threshold is used. The floor 10^{-16} is only a practical termination criterion; it is not the device that prevents a crossing of $U = 1$. At every accepted step, the singular-source restriction in (9), together with Theorems 1 and 2, preserves $0 \leq U^n < 1$. The tail correction is the quantity derived in (12):

$$T_{h,\text{corr}} = t_n + \frac{(1 - \|U^n\|_\infty)^{p+1}}{(p+1)\lambda a(x_{q,h})}.$$

In Tables 1–2, the largest terminal gap is $\delta_{\text{stop}} = 1.54 \times 10^{-4}$. Since $p = 2$, $\lambda = 18$, and $a(x) \geq 1$, the correction is bounded by

$$\frac{(1.54 \times 10^{-4})^3}{3 \times 18} < 6.8 \times 10^{-14},$$

which is far below the 10^{-10} precision displayed for the quenching times (and is less than 7×10^{-12} of a time of order 10^{-2}). Hence T_h and $T_{h,\text{corr}}$ coincide at the displayed precision.

Table 1: Adaptive IMEX Euler scheme on graded meshes with $\gamma = 1.55$

I	T_h	$T_{h,\text{corr}}$	N_{step}	δ_{stop}	CPU (s)	$x_{q,h}$	s_h
16	0.0126612625	0.0126612625	15	5.91×10^{-6}	0.001	0.01992	–
32	0.0118080108	0.0118080108	55	2.37×10^{-5}	0.001	-0.00680	–
64	0.0109211907	0.0109211907	214	3.87×10^{-5}	0.004	-0.00232	–
128	0.0105466682	0.0105466682	825	6.09×10^{-5}	0.018	0.00079	1.244
256	0.0104548543	0.0104548543	3140	9.73×10^{-5}	0.107	-0.00027	2.028
512	0.0104319601	0.0104319601	11915	1.54×10^{-4}	0.951	0.00009	2.004

N_{step} is the number of accepted time steps, $\delta_{\text{stop}} = 1 - \|U^n\|_\infty$ at termination, CPU is a representative single-run wall-clock time in seconds, $x_{q,h}$ is a maximizing cell center, and s_h is the three-grid Richardson order. CPU values may vary with the hardware and software environment.

To keep the time notation unambiguous in this section, $T_h^* = \sum_{n \geq 0} \Delta t_n$ denotes the theoretical numerical quenching time defined in Section 5, whereas $T_h := t_n$ denotes the finite-run accumulated terminal time when the stopping criterion is met. The quantity $T_{h,\text{corr}}$ is the tail-corrected value in (12). The symbol T_{ref} is reserved for the Richardson reference value defined below in the subsection “Richardson reference value.” By contrast, $T_{h,\eta}$ denotes the discrete level time defined before Theorem 5; it is a theoretical level-crossing quantity and is not the terminal time reported in Tables 1–4.

The dense matrix is assembled once in $O(I^2)$ operations, and the symmetric eigenvalue decomposition costs $O(I^3)$ once. Each subsequent matrix action or solve uses two dense orthogonal transforms and therefore costs $O(I^2)$ per time step. For the reported sizes $I \leq 512$, this cost is manageable; the CPU columns give wall-clock seconds from single runs of the accompanying Python implementation and are reported only as indicative, environment-dependent timings; they are not used as benchmark data or to support the convergence conclusions. No singular-level crossing occurred and no post-step projection was applied. The observed order is the Richardson value

$$s_h = \frac{\log((T_{4h} - T_{2h}) / (T_{2h} - T_h))}{\log 2},$$

whenever the quotient is positive. For the coarsest mesh, $\beta h_{\text{max}}^2 = 0.6990729 < 1$, so the source restriction is compatible with Proposition 2.

6.2 IMEX Euler quenching times

Table 1 reports the quenching times obtained with the adaptive IMEX Euler scheme. This is the main numerical test, because the proof of invariance and comparison is most directly tied to the IMEX structure.

The computed terminal time decreases and stabilizes as the mesh is refined. The detected point $x_{q,h}$ approaches zero, in agreement with the location of the maximum of the weight a . The first coarse levels are pre-asymptotic, while the refined levels give Richardson values close to two. Along the coupled refinement sequence, this behavior is consistent with the $O(h_{\text{max}}^2)$ midpoint-quadrature error together with $\max_n \Delta t_n = O(h_{\text{max}}^2)$. It should not be interpreted as second-order temporal accuracy of the IMEX Euler method.

Table 2: Adaptive Crank–Nicolson-type scheme on graded meshes with $\gamma = 1.55$

I	T_h	$T_{h,\text{corr}}$	N_{step}	δ_{stop}	CPU (s)	$x_{q,h}$	s_h
16	0.0126593998	0.0126593998	15	5.90×10^{-6}	0.001	0.01992	–
32	0.0118064866	0.0118064866	55	2.37×10^{-5}	0.001	-0.00680	–
64	0.0109205559	0.0109205559	214	3.87×10^{-5}	0.005	-0.00232	–
128	0.0105465106	0.0105465106	825	6.09×10^{-5}	0.023	0.00079	1.244
256	0.0104548150	0.0104548150	3140	9.73×10^{-5}	0.167	-0.00027	2.028
512	0.0104319503	0.0104319503	11915	1.54×10^{-4}	1.808	0.00009	2.004

The notation follows Table 1; s_h is computed from three successive corrected quenching times, and CPU values are representative wall-clock measurements.

Table 3: Richardson reference values from the three finest levels

Mesh	Scheme	γ	s_{ref}	T_{ext}
Uniform	IMEX	1.00	2.0046	0.0104243353
Graded	IMEX	1.55	2.0037	0.0104243550
Graded	CN	1.55	2.0037	0.0104243550

s_{ref} is the three-grid order computed from the levels $I = 128, 256, 512$, and T_{ext} is the corresponding fine-grid Richardson extrapolation.

6.3 Crank–Nicolson-type verification

Table 2 repeats the experiment with the Crank–Nicolson-type scheme. This second test is performed separately from the IMEX computation and assesses the effect of replacing backward Euler diffusion by trapezoidal diffusion. Both schemes retain the explicit singular source and are first order in time. Their agreement does not exclude a temporal bias shared by the two schemes.

The Crank–Nicolson-type values remain close to the IMEX Euler values across the refinement sequence. At $I = 512$, the displayed terminal times differ by 9.8×10^{-9} , compared with 1.8627×10^{-6} at $I = 16$.

6.4 Richardson reference value

No closed analytical quenching time is available for this test. We therefore use the last three levels $I = 128, 256, 512$ and compute

$$T_{\text{ext}} = T_h + \frac{T_h - T_{2h}}{2^{s_h} - 1}.$$

Table 3 gives separate extrapolations from a uniform IMEX sequence and from the two graded sequences. Their spread is less than 2.0×10^{-8} , so the subsequent numerical comparisons use

$$T_{\text{ref}} = 0.01042435.$$

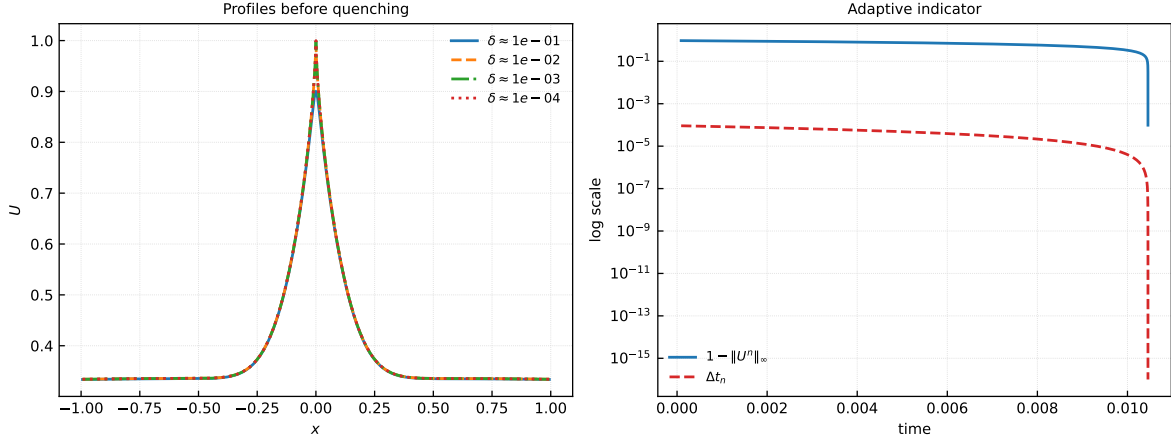


Figure 1: Adaptive IMEX computation with $I = 256$ and $\gamma = 1.55$. Left: profiles recorded when $\delta_n = 1 - \|U^n\|_\infty$ passes approximately 10^{-1} , 10^{-2} , 10^{-3} and 10^{-4} ; the peak sharpens and remains centered near the maximum of $a(x)$. Right: δ_n and Δt_n on logarithmic scale; their simultaneous decay confirms that the singular-source component of (9) controls the final stages without crossing the singular level $u = 1$.

Table 4: Uniform-mesh and graded-mesh comparison for the IMEX scheme with $I = 256$; the row $\gamma = 1$ is the uniform mesh.

γ	$h_{\min}$	$h_{\max}$	$T_{h,\text{corr}}$	$ T_{h,\text{corr}} - T_{\text{ref}} $	N_{step}	CPU (s)	$ x_{q,h} $
1.00	7.812×10^{-3}	7.812×10^{-3}	0.0104387075	1.436×10^{-5}	7279	0.229	3.906×10^{-3}
1.20	2.960×10^{-3}	9.368×10^{-3}	0.0104428896	1.854×10^{-5}	5131	0.147	1.480×10^{-3}
1.55	5.418×10^{-4}	1.208×10^{-2}	0.0104548543	3.050×10^{-5}	3140	0.094	2.709×10^{-4}
2.00	6.104×10^{-5}	1.556×10^{-2}	0.0104750289	5.068×10^{-5}	1926	0.064	3.052×10^{-5}

All CPU values in this table are single-run wall-clock measurements; they are not averages. They are reported only as indicative, environment-dependent timings and are not used as benchmark data or to support the convergence conclusions.

6.5 Profiles and adaptive indicator

Figure 1 reports diagnostics for the IMEX computation with $I = 256$. The left panel shows solution profiles before quenching, while the right panel shows the decay of $1 - \|U^n\|_\infty$ together with the adaptive time step. This figure verifies that the time step decreases before the singular level is reached. The curves combine color with different line styles so that the figure remains readable in black-and-white printing.

6.6 Influence of the grading parameter

The last test studies the effect of the grading parameter. Here the scheme is fixed to IMEX Euler, the number of cells is fixed to $I = 256$, and only γ is changed. Each mesh is used with the adaptive rule (9).

Table 4 gives a direct uniform-versus-graded comparison at equal dimension. In this experiment, the uniform mesh gives the smallest discrepancy from the Richardson reference value. When $\gamma > 1$, cells are concentrated near $x = 0$ and the outer cells become wider, so $h_{\max}$ increases at fixed I . The spatial consistency bound is controlled by $h_{\max}^2$, and the adaptive time restriction also depends on this quantity. The observed time discrepancies therefore combine spatial and temporal effects; this comparison does

Table 5: Practical advantages and limitations of the numerical choices used in this study

Numerical choice	Main advantages in the present problem	Main limitations / cautions
Adaptive IMEX Euler	Linear diffusion solve; explicit singular source; direct invariant-region argument; one linear solve per accepted time step.	First-order temporal consistency because the source is explicit; the time step decreases as the singular level is approached.
Crank–Nicolson-type	Trapezoidal treatment of the diffusion term; uses the spectral solver described above; allows comparison with IMEX under a different diffusion treatment.	The singular source remains explicit, so the source-based adaptive restriction is still required; the scheme is not second order in time as a whole.
Uniform mesh	Smallest discrepancy from the Richardson reference value at fixed I in Table 4; no prior quenching location is required.	Coarser local sampling of the peak and larger localization error at an equal number of cells.
Graded mesh	Improves localization near a known quenching region and reduces the number of accepted time steps in the numerical experiments.	Changes both spatial resolution and the adaptive time restriction at fixed I ; fixed grading requires prior information about the quenching region.

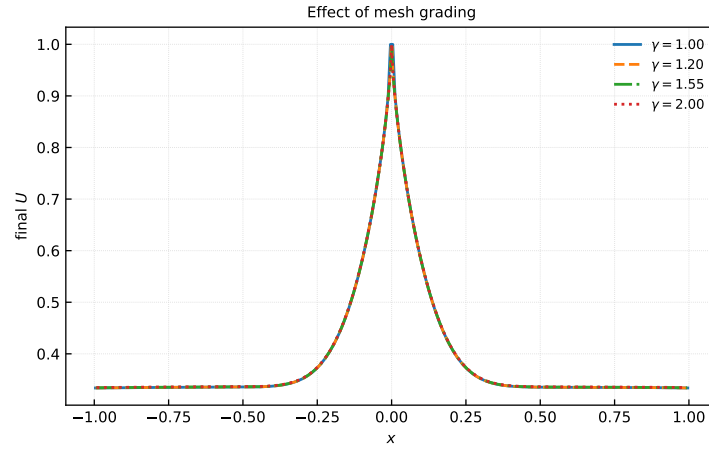


Figure 2: Final IMEX profiles for $I = 256$ and $\gamma = 1, 1.20, 1.55, 2.00$. Increasing γ concentrates cell centers near $x = 0$, sharpens the sampled peak and decreases $|x_{q,h}|$. This comparison also shows the trade-off identified in Table 4: stronger grading improves localization but enlarges the global mesh width $h_{\max}$.

not determine their separate contributions. Grading improves local sampling of the quenching point by more than one order of magnitude and reduces the number of accepted time steps under the modified adaptive restriction. The reported CPU times also decrease, although they are only indicative single-run measurements. These gains do not establish a general reduction of the global time error. If the quenching point is not known a priori, a uniform mesh is the safer default because it does not prescribe a preferred location. If additional localization is required, solution-driven spatial refinement would be preferable to fixed grading. Dynamic spatial refinement is not considered here and is left for future work. Figure 2 shows the corresponding final profiles.

Conclusion

We developed adaptive IMEX Euler and Crank–Nicolson-type approximations on nonuniform meshes for a singular nonlocal problem with homogeneous exterior data. The matrix structure and adaptive restriction

preserve the invariant region $0 \leq U^n < 1$, and the analysis establishes finite numerical quenching after entry into a sufficiently high level, convergence before quenching, and convergence of discrete level times. The two time discretizations produce mutually consistent terminal-time approximations, and their Richardson values are close to two along the coupled space–time refinement; this observation does not imply second-order temporal accuracy. The mesh comparison indicates that grading mainly improves local resolution of the quenching point and can reduce the number of accepted time steps, whereas a uniform mesh may retain a smaller global quenching-time error at fixed dimension.

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Data availability statement

The Python source code, CSV tables, figure-generation files, dependency list and README used to reproduce the computations are supplied as supplementary material with this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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